Relatively hyperbolic groups with S^2 boundary

Bena Tshishiku

April 15, 2018

joint with Genevieve Walsh

Wall conjecture

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Conjecture (Wall, 1979).

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Relative version.

(G,P) *PD(n) pair* $\implies$ K(G,1) ~ compact manifold with aspherical boundary.

Duality and the boundary

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Warm-up. $G_3 = \langle\, a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \,\rangle$ is a PD(2) group.

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hyperbolic plane

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Examples:

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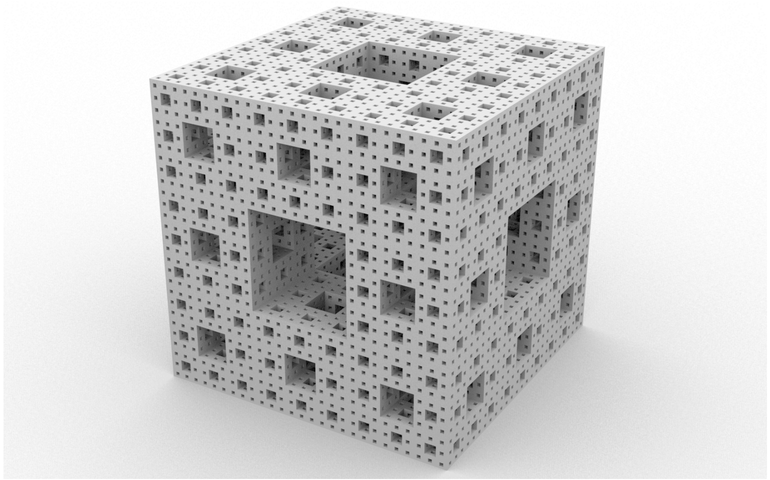

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Theorem (Bartels-Luck-Weinberger, 2010). n≥6.

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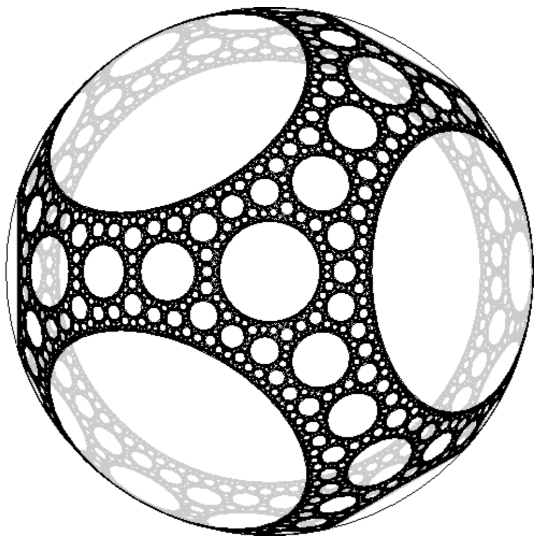

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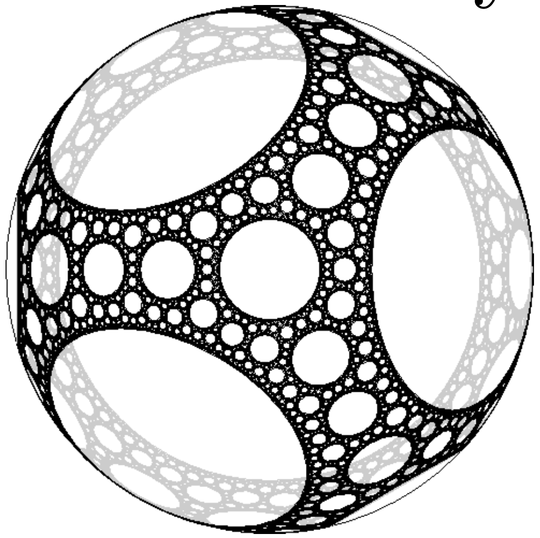
$\Longrightarrow$ K(G,1) ~ compact n-manifold with aspherical boundary.



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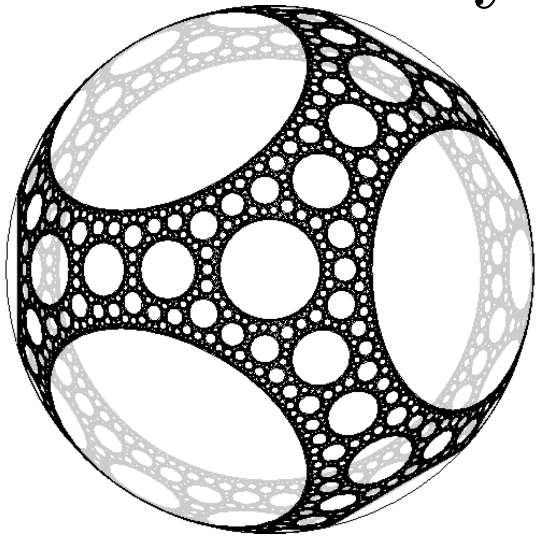

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Theorem (Bestvina-Mess). G a δ-hyperbolic group.

G is PD(n) group $\iff$ $\mathrm{H}^*(\partial G) \simeq \mathrm{H}^*(S^{n\text{-}1})$

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Problem. Extend to relatively hyperbolic groups (G,P).

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Remark: $G_2 = \langle\ a,b,t \mid {}^t a = a^2 b,\ {}^t b = ab\ \rangle$

is not hyperbolic, but is *relatively hyperbolic.*

Duality and the boundary (relatively hyperbolic case)

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Bowditch boundary

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Proof:

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Theorem (Bestvina-Mess). G a δ-hyperbolic group.

$$G \text{ is PD(3) group} \iff \partial G \simeq S^2$$

Theorem (Tshishiku-Walsh). (G,P) relatively hyperbolic.

$$(G,P) \text{ is PD(3) pair} \iff \underbrace{\partial(G,P)}_{\text{Bowditch boundary}} \simeq S^2$$

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Difference in relatively hyperbolic case:

$\partial(G,P)$ not a Z-set compactification of G.

Proof idea

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$$= \pi_1(\ S^3 \setminus \text{figure-8 knot}\ )$$

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cusp subgroup $Z^2 \simeq P \subset G_2 = \langle\ a, b, t \mid {}^t a = a^2 b,\ {}^t b = ab\ \rangle$

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$X = H^3 \setminus (\text{horoballs})$

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$G_\delta = G \underset{P}{*} G$ has $\partial_{\mathrm{D}}(G_\delta,\ P) \simeq$ "tree of Sierpinski carpets" $\simeq S^2$

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2. When $\partial(G,P) \simeq S^3$, does P always have a Z-boundary?

Thank you.