Nielsen realization

M closed, oriented, smooth manifold

$$\begin{tikzcd} & \mathrm{Diff}(M) \arrow[d] \\ G \arrow[r, hook] \arrow[ur, dotted, "\exists ?"] & \mathrm{Mod}(M) = \pi_0 \mathrm{Diff}(M) \end{tikzcd}$$

If lift exists, say G is realizable.

Ex $G = \mathbb{Z}/2\mathbb{Z}$ if $f \in \mathrm{Diff}(M)$ and $f^2 \sim id_M$ (isotopic) does there exist $g \sim f$ s.t. $g^2 = id$?

Questions

① Which finite G are realizable?

If G realizable,

② geometric structure?

③ uniqueness?

If G not realizable,

④ bundle monodromy?

$$\begin{tikzcd} & B\mathrm{Diff}(M) \arrow[d] \\ BG \arrow[r] \arrow[ur, dotted, "\exists ?"] & B\mathrm{Mod}(M) \end{tikzcd}$$

(bundles w/ no flat connection)

Surface case

All Kerckhoff 83

Yes hyperbolic metric

Yes (Gabai)

Yes (Earle-Eells) $B\mathrm{Diff}(M) \sim B\mathrm{Mod}(M)$

3-manifolds NR

M^3 (closed oriented)

Previously: sporadic results mostly for M irreducible (every $S^2 \hookrightarrow M$ bounds a 3-ball)

eg • M hyperbolic: every G realizable

$$\mathrm{Isom}(M) \underset{\text{h.e.}}{\overset{\text{(Gabai)}}{\hookrightarrow}} \mathrm{Diff}(M) \longrightarrow \mathrm{Mod}(M) \cong \mathrm{Out}(\pi_1 M)$$

• M Seifert fibered (eg $S^1 \to M \to S_g$)

$$\mathrm{Mod}(M) \cong \mathrm{Out}(\pi_1 M)$$

(Heil–Tollefson) $G \cong \mathbb{Z}/m\mathbb{Z} < \mathrm{Out}(\pi_1 M)$ realizable $\iff$

there is a extension $1 \to \pi_1 M \to \tilde{G} \to G \to 1$

$\iff$ certain obstruction $\theta \in H^3(G;\mathbb{Z})$ vanishes.

Today: For general M,

$$1 \to \mathrm{Ker}(\alpha) \longrightarrow \mathrm{Mod}(M) \xrightarrow{\alpha} \mathrm{Out}(\pi_1 M)$$

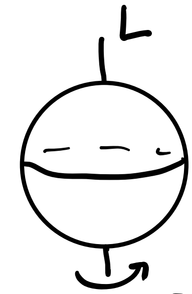
$\mathrm{Ker}(\alpha) \cong \mathrm{Twist}(M)$ (McCullough)

Main Q: What subgroups of $\mathrm{Twist}(M)$ are realizable?

(orthogonal to preceding)

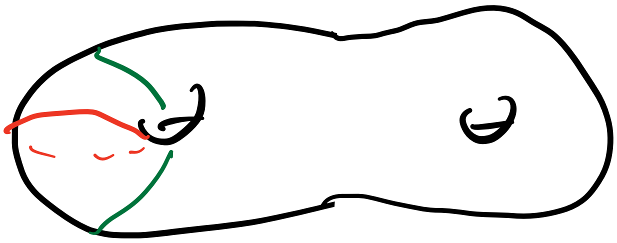
Twist group of M^3

Twist(M) generated by <u>sphere twists</u>



rotate by $2\pi t$

$S^2 \cong S$



M^3

$$T_S : S^2 \times [0,1] \longrightarrow S^2 \times [0,1]$$
$$(x,t) \longmapsto (\mathrm{rot}_L(2\pi t)(x), t)$$

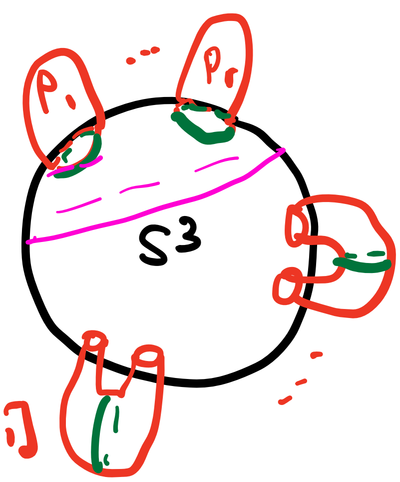
extend to M by id.

$$\mathrm{Twist}(M) = \left\langle [T_S] \;\middle|\; S \subset M \text{ embedded sphere} \right\rangle$$

(McCullough) $\mathrm{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^d$ some d

(Note $[T_S]^2 = 1$ b/c $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$)

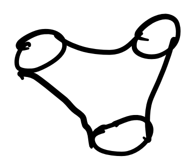
prime decomp $M = \#_d S^1 \times S^2 \# P_1 \# \cdots \# P_r$ P_i irreducible

$M =$ 

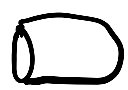
$S^2 \times [0,1]$

$S^2 \times [0,1]$

- Green spheres generate
- relations

(i) pant  $S^3 \setminus \sqcup_3 D^3$

$$[T_{\partial_1} \circ T_{\partial_2} \circ T_{\partial_3}] = 1$$

(ii) Lens space  $L \setminus D^3$ $[T_\partial] = 1$

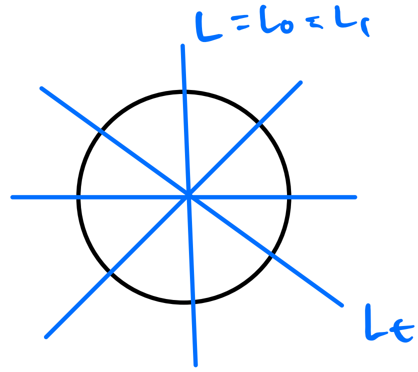
Realizing twists

Initial difficulty: no Thurston normal form. Have to identify key phenomena.

Key example

(this example started whole project)

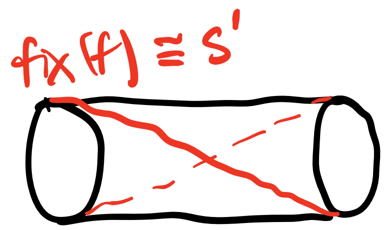
$M = S^2 \times S^1$ $\qquad \mathrm{Twist}(M) \cong \mathbb{Z}/2\mathbb{Z} = \langle [T_S] \rangle$



$T_S = \mathrm{rot}_L(2\pi t)$ on $S^2 \times \{t\}$

$T_S^2 \neq \mathrm{id}$

Q: Is T_S isotopic to f s.t. $f^2 = \mathrm{id}$?



A: yes!

$f = \mathrm{rot}_{L_t}(\pi)$ on $S^2 \times \{t\}$

$f^2 = \mathrm{id}$, $\quad f \sim T_S$

Rmks ① $\nexists$ isometric realization (wrt $S^2 \times \mathbb{R}$ geometry)

counterex to equiv. geometrization...

② Extension of this example:

if $M = \#$ lens spaces, including $S^1 \times S^2$

then every $\mathbb{Z}/2\mathbb{Z} < \mathrm{Twist}(M)$ is realizable

(equiv. glue along fixed pt)

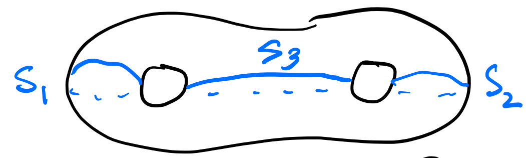
Main Thm

<u>Thm</u> (Chen-T) M any closed or. 3mfld

$1 \neq G < \mathrm{Twist}(M)$ realizable $\iff$

$G \cong \mathbb{Z}/2\mathbb{Z}$ and $M = \#$ lens spaces including $S^1 \times S^2$

<u>Ex</u> $M = S^1 \times S^2 \# S^1 \times S^2$

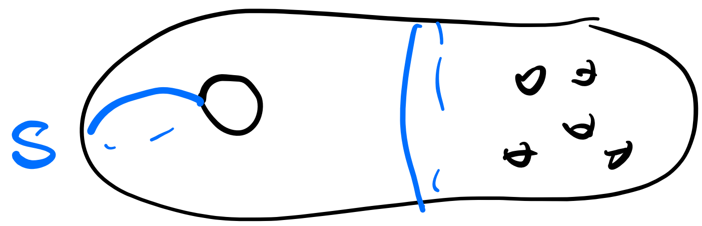


$\mathrm{Twist}(M) = \{id, T_1, T_2, T_3\} \cong (\mathbb{Z}/2\mathbb{Z})^2$

Each $\langle [T_i] \rangle$ realizable, but not simultaneously (!)

<u>Ex</u> $M = S^1 \times S^2 \# (\text{hyperbolic 3mfld})$

$\Rightarrow \langle T_S \rangle \cong \mathbb{Z}/2\mathbb{Z}$ not realizable



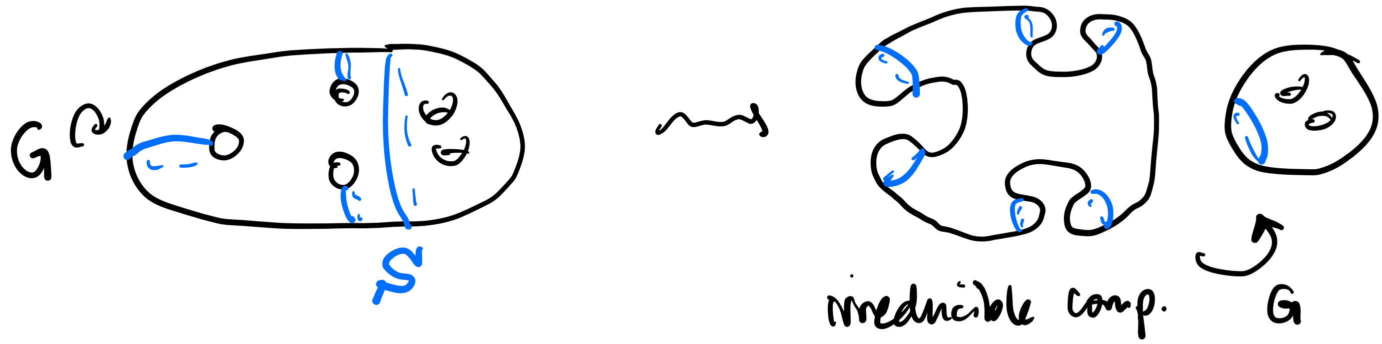
$\Rightarrow \exists$ bundle $M^3 \to E \to \mathbb{R}P^2$ w/ no flat connection

(Open Q: $\exists$? SBS that's not flat?)

<u>Rmk</u> Top = Diff (J. Pardon, Kirby-Edwards) for realizing finite groups of $\mathrm{Mod}(M^3)$

Proof ingredients

• equivariant sphere theorem (Meeks-Yau, Dunwoody)



• sphere twists trivial on π_1, π_2 $\Rightarrow$ $G \curvearrowright S$ trivial.

$(\Rightarrow (\mathbb{Z}/2\mathbb{Z})^k \cong G \hookrightarrow SO(3) \Rightarrow k \leq 2)$

• N^3 closed or. irreducible, $p \in N$

if $\exists$ finite order $f \in \mathrm{Diff}(N,p)$ that acts trivially on $\pi_1(N,p)$ then $N =$ lens space.

<u>Cor</u> M reducible not a connected sum of lens spaces

$\Rightarrow$ $\mathrm{Diff}_0(M)$ is torsion-free

$\Rightarrow$ Every f.g. torsion subgroup of $\mathrm{Diff}(M)$ is finite

ie $\mathrm{Diff}(M)$ doesn't contain Burnside groups...