Convex cocompact subgroups of Mod(S)

- $S = S_g$ closed, or. surface genus $g \geq 2$
- surface group extension

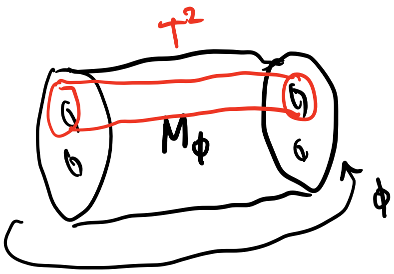
$$\mathrm{Mod}(S) := \pi_0 \mathrm{Homeo}(S)$$

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S) \arrow[r] & \mathrm{Aut}(\pi_1 S) \arrow[r] & \mathrm{Out}(\pi_1 S) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(S) \arrow[r] \arrow[u, equal] & \tilde{G} \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

($\mathrm{Mod}(S) \cong \mathrm{Out}(\pi_1 S)$; G f.g.)

(Main)
Q: Given $G < \mathrm{Mod}(S)$, When is $\tilde{G}$ Gromove hyperbolic?

Ex $G = \langle \phi \rangle$ $\quad \phi \in \mathrm{Mod}(S)$. Here $\tilde{G} \cong \pi_1(M_\phi)$



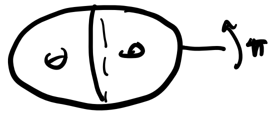
(Thurston) M_ϕ hyp. $\Leftrightarrow$ ϕ pseudo-Anosov (∞ order, irreducible)

$\Updownarrow$

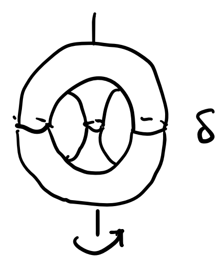
$\pi_1(M_\phi)$ hyperbolic

Ex $G = \langle \beta^2 \delta, \delta \beta^2 \rangle$

$< \mathrm{Mod}(S_2)$



β half twist



G purely pA. Is $\tilde{G}$ hyperbolic?
(already not so obvious ....)

For general f.g. $G < \mathrm{Mod}(S)$:

$\tilde{G}$ hyperbolic $\overset{\text{FM 2002}}{\Longleftrightarrow}$ $G < \mathrm{Mod}(S)$ "convex cocompact"

Hamenstädt 2005

$\Longleftrightarrow$ $G \xrightarrow{\text{orbit}} \mathcal{C}(S)$ ← curve complex

KL 2007 is quasi-isometric embedding.

$\Longleftrightarrow$ $G \to \mathrm{Mod}(S)$ q.i. emb and

BBKL 2020 each ∞-order $g \in G$ is pseudo-Anosov

"purely pA"

Q (FM) $G < \mathrm{Mod}(S)$ f.g. purely pA $\Rightarrow$ $\tilde{G}$ hyperbolic ?

A: Yes if G contained w/in

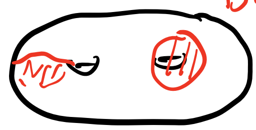
- Veech group $\mathrm{Aff}(X, \omega)$
- certain $\pi_1(M^3) < \mathrm{Mod}(S, *)$ (DKLS)
- certain RAAGs (KMT)
- genus 2 Goeritz group (T-)

Rmk $G < \mathrm{Mod}(S)$ purely pA $\Rightarrow$ $\tilde{G}$ doesn't contain any $BS(p,q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$

Gromov asked if containing $BS(p,q)$ only obstruction to hyperbolicity.

(disproved by Martelli et al)

Goeritz group $\mathbb{G}$

$S =$ 

$\hat{D} \subset \hat{V}$

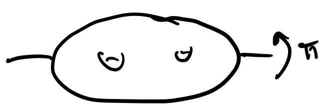
$D \subset V$

$S^3 = V \cup_S \hat{V}$ the genus-2 Heegaard splitting

$\mathbb{G} < \mathrm{Mod}(S)$ mapping classes that extend to both V & $\hat{V}$

$=$ homeos of S^3 preserving HS / isotopy

- Goeritz, Scharlemann, Akbas, Cho — 1933, 2004, 2008 — $\mathbb{G}$ finitely presented, generated by



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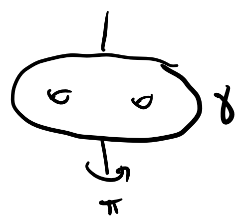
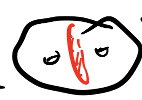
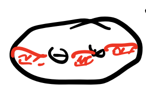
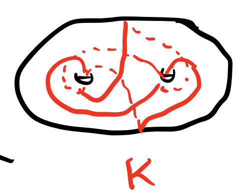


, δ

(open: $\mathbb{G}_g$ f.g for $g \geq 4$?)

Thm 1 If $G < \mathbb{G}$ f.g. purely pA then $\tilde{G}$ hyperbolic

Thm 2 $g \in \mathbb{G}$ pA $\iff$ g not conj into

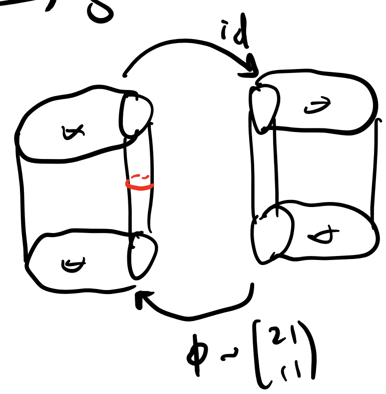
- Stab()
- Stab()
- Stab()
- Stab()

(classify reducible in $\mathbb{G}$, refined NT)

Meaning of K $\quad K \subset S \subset S^3$ fig 8 knot

$T^2 \setminus pt \longrightarrow S^3 \setminus K \longrightarrow S^1$

$S^3 = V \cup \hat{V} =$



$V \cong \hat{V} \cong (T^2 \setminus D^2) \times [0,1]$

monodromy ϕ $\mapsto$ ∞ order reducible elt of $\mathcal{G}$ fixing K

(conj to $\beta \delta \beta^{-1} \delta$)

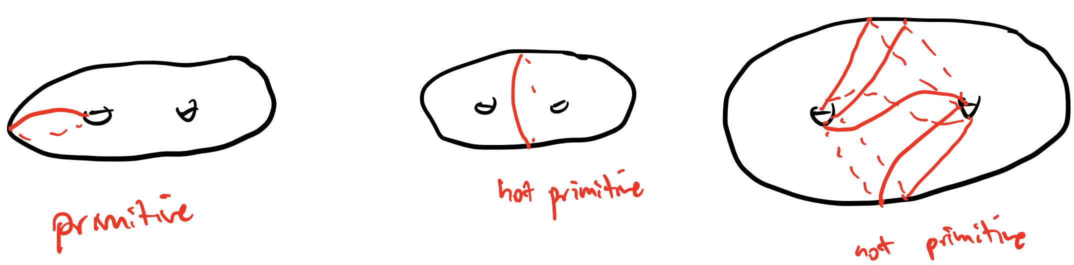
Proof ingredients (for Thm 1)

given $G < \mathcal{G}$ f.g. purely pA WTS $G \xrightarrow{\text{orbit}} \mathcal{C}(S)$ q.i. emb.

basepoint (?)...

- primitive disk complex $\mathcal{P}(V) \subset \mathcal{C}(S)$ spanned by primitive scc's

  a scc is primitive if it bounds disk in V and is part of free basis for $\pi_1(\hat{V}) \cong F_2$



- distance formula

subsurface projection

$$d_{P(V)}(D,E) \underset{k}{\asymp} \sum_{X\subset S} \{ d_X(D,E) \}_c$$

cutoff

certain subsurfaces "holes"

sum includes $X=S$ & $\underbrace{X = \text{comp. of } S\setminus K}$

$\Rightarrow P(V) \hookrightarrow \mathcal{C}(S)$ not qi emb

<u>Proof Sketch Thm 1</u> $G < \mathbb{G}$ purely pA

$$\begin{tikzcd}
\mathbb{G} \arrow[r] & \mathrm{Cone}(\mathbb{G}, \mathbb{G}_D) \arrow[d, phantom, "\wr"] \\
 & P(V) \arrow[d, hook] \\
G \arrow[uu, hook] \arrow[r] & \mathcal{C}(S)
\end{tikzcd}$$

WTS q.i. emb

- dist formula + BBKL $\Rightarrow$ suffices to show $G \to P(V)$ qi emb
- [Cha] $P(V)$ q.i. to cone off of $\mathbb{G}$
- $G \hookrightarrow \mathbb{G}$ q.i. emb b/c $\mathbb{G}$ virt. free
- Abbott–Manning $+\varepsilon$ G purely pA $\Rightarrow$ $G \to \mathrm{Cone}(\mathbb{G}, \mathbb{G}_D)$ q.i. emb. $\square$

<u>Ex</u> $n \geq 2$ $G_n = \langle \beta^n \delta, \delta \beta^n \rangle$ purely pA (Thm 2)

$\Rightarrow \tilde{G}_n$ hyp. (Thm 1)