

Geometric cycles and characteristic classes of manifold bundles (with an appendix by M. Krannich)

Bena Tshishiku

Abstract. We produce new cohomology for non-uniform arithmetic lattices $\Gamma < \mathrm{SO}(p, q)$ using a technique of Millson–Raghunathan. From this, we obtain new characteristic classes of manifold bundles with fiber a closed $4k$ -dimensional manifold M with indefinite intersection form of signature (p, q) . These classes are defined on finite covers of $B \mathrm{Diff}(M)$ and are shown to be nontrivial for $M = \#_g(S^{2k} \times S^{2k})$. In this case, the classes produced live in degree g and are independent from the algebra generated by the stable (i.e. MMM) classes. We also give an application to bundles with fiber a K3 surface.

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1. Introduction

The starting point of this paper is the following new result about the cohomology of certain arithmetic groups $\Gamma < \mathrm{SO}(p, q)$.

Theorem 1. *Fix $1 \leq p \leq q$ such that $p + q \geq 3$. Let $\Lambda \subset \mathbb{R}^{p+q}$ be a lattice with an integral bilinear form of signature (p, q) . Consider the group $\mathrm{SO}(\Lambda)$ of automorphisms of Λ with determinant 1. For every $n \geq 1$, there is a finite-index subgroup $\Gamma < \mathrm{SO}(\Lambda)$ so that $\dim H^p(\Gamma; \mathbb{Q}) \geq n$.*

A lattice Λ as in Theorem 1 is determined up to isomorphism by its signature (p, q) and its parity (even or odd) [31, Ch. II, §4]. The group $\mathrm{SO}(\Lambda)$ is a nonuniform lattice in $\mathrm{SO}(p, q) \cong \mathrm{SO}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ and has \mathbb{Q} -rank p . (Note we are using “lattice” in two different ways. This should not cause any confusion.)

Constructing nonzero elements of $H^*(\Gamma; \mathbb{Q})$ is a classical important problem in the theory of arithmetic groups. Our primary interest is to use Theorem 1 to produce new characteristic classes for certain fiber bundles. Recall that a characteristic class for fiber bundles with structure group G is an element of $H^*(BG)$, where BG is the classifying space of G . Our first application is as follows.

Corollary 2. *Fix $g \geq 2$ and fix n even so that $2n \geq g + 4$. Let $W_g^{2n} = \#_g(S^n \times S^n)$, and denote the group of orientation-preserving diffeomorphisms of W_g by $\mathrm{Diff}(W_g)$. For every $m \geq 1$, there is a finite-index subgroup $\mathrm{Diff}^\Gamma(W_g) < \mathrm{Diff}(W_g)$ so that $\dim H^g(B \mathrm{Diff}^\Gamma(W_g); \mathbb{Q}) \geq m$.*

Before discussing further applications, we make several remarks.

The cohomology produced in Theorem 1 is new. Millson–Raghunathan [30] produce *uniform* $\Gamma < \mathrm{SO}(p, q)$ with $1 \leq p \leq q$ and p even such that $H^p(\Gamma; \mathbb{Q}) \neq 0$. Note in particular that [30, p. 103] requires p to be even, whereas in Theorem 1 works for any p . This “improvement” is obtained using some ideas from more recent work of Avramidi–Nguyen-Phan [2].

The particular finite-index subgroup $\Gamma < \mathrm{SO}(\Lambda)$ in Theorem 1 is somewhat difficult to make precise by the nature of the construction. The subgroup

$$\mathrm{Diff}^\Gamma(W_g) < \mathrm{Diff}(W_g)$$

appearing in Corollary 2 is defined as the preimage of a subgroup $\Gamma < \mathrm{SO}(\Lambda)$ under a homomorphism

$$\mathrm{Diff}(W_g) \rightarrow \mathrm{SO}(\Lambda),$$

where Λ is $H_n(W_g^{2n})$ with its intersection form.

When $p = q$, then if $\Gamma < \mathrm{SO}(\Lambda)$ is finite index and $i \leq p - 2$, then

$$H^i(\Gamma; \mathbb{Q}) \cong H^i(\mathrm{SO}(\Lambda); \mathbb{Q}).$$

This follows from Borel’s stability theorem (see [7], and see [42] for the stated range). Moreover, in this range, the cohomology ring $H^*(\mathrm{SO}(\Lambda); \mathbb{Q})$ is a polynomial ring with one generator in each degree $4k > 0$. Theorem 1 shows that the stable range given in Borel’s theorem is nearly sharp in this case. A similar observation for uniform lattices in $\mathrm{SO}(p, q)$ is mentioned in [30].

Another way to state Theorem 1 is that the p -th *virtual Betti number* of $\mathrm{SO}(\Lambda)$ is infinite. If $\mathrm{SO}(\Lambda) < \mathrm{SO}(p, q)$ was uniform, then to prove Theorem 1 it would suffice to prove that there exists $\Gamma < \mathrm{SO}(\Lambda)$ such that $H^p(\Gamma; \mathbb{Q}) \neq 0$. Then since $\mathrm{SO}(\Lambda)$ has a large commensurator one can produce many linearly independent classes in further finite-index subgroups by an argument that appears in [43]. For a non-uniform lattice, the same argument works, but only up to the range in Borel’s stability theorem. In our case, that range is less than p , so that argument cannot be used toward proving Theorem 1. Our approach to showing $\dim H^p(\Gamma; \mathbb{Q})$ can be made large is along the lines of Avramidi–Nguyen-Phan [2, Thm. 1.2], but the argument is different.

For $\Gamma < \mathrm{SO}(\Lambda)$, a class

$$c \in H^*(\Gamma; \mathbb{Q}) \cong H^*(B\Gamma; \mathbb{Q})$$

can be viewed as a characteristic class for vector bundles $W \rightarrow B$ with structure group $\Gamma < \mathrm{GL}_{p+q}(\mathbb{R})$. We explain what the characteristic classes produced in Theorem 1 measure using obstruction theory in §4. This gives a new perspective on the Millson–Raghunathan construction. It also provides an interpretation for the classes in Corollary 2. This becomes relevant in our application to bundles with fiber a K3 surface; see §5.1.

The cohomology produced in Corollary 2 is new. The previously known classes in $H^*(B \operatorname{Diff}^\Gamma(W_g); \mathbb{Q})$ are the *stable classes* (also known as *tautological* or *generalized Miller–Morita–Mumford* classes). For $B \operatorname{Diff}(W_g)$, the stable classes account for all of the cohomology in low degree $* \ll g$ [16–18]. When g is odd, the classes we produce live in odd degree, whereas the stable classes all have even degree, so our classes are not in the algebra generated by the stable classes. A similar statement can be made for g even. The nontriviality of these classes is detected by a nontrivial bundles, so Corollary 2 gives a new way to produce topologically nontrivial bundles $W_g \rightarrow E \rightarrow B^g$.

Corollary 2 illustrates that the *unstable* cohomology of arithmetic groups is a source of cohomology of $B \operatorname{Diff}(W_g)$. This phenomenon is largely unexplored; see also Corollary 30 in the appendix.

About the proof of Theorem 1. The cohomology classes in Theorem 1 are produced using geometric cycles in locally symmetric spaces. Let $X = \operatorname{SO}(p, q)/K$ be the symmetric space associated to $\operatorname{SO}(p, q)$, and let $Y = \Gamma \backslash X$ be the locally symmetric space for $\Gamma < \operatorname{SO}(\Lambda)$. There is an isomorphism

$$H^*(\Gamma; \mathbb{Q}) \cong H^*(Y; \mathbb{Q}).$$

Each class $c \in H^p(\Gamma; \mathbb{Q})$ we produce is Poincaré dual to a cycle $[Z] \in H_{pq-p}^{\text{cl}}(Y; \mathbb{Z})$ in Borel–Moore homology with closed supports, and $[Z]$ is represented by a totally-geodesic, properly-embedded oriented submanifold $Z \subset Y$. To show $[Z] \neq 0$, we find a compact, totally-geodesic oriented submanifold $Z' \subset Y$ of dimension p so that the intersection number $[Z] \cdot [Z'] \in H_0(Y; \mathbb{Z}) \cong \mathbb{Z}$ is nonzero.

The cycles $[Z]$ and $[Z']$ are often called *geometric cycles*. The idea of finding nontrivial homology of a locally symmetric space/arithmetic group by finding a pair of geometric cycles with $[Z] \cdot [Z'] \neq 0$ goes back to Millson [29]; see also [2, 25, 30, 37]. In each of these works, the locally symmetric space $Y = \Gamma \backslash X$ is either compact, or the lattice Γ is commensurable to $\operatorname{SL}_n(\mathbb{Z})$. The spaces $\Gamma \backslash \operatorname{SO}(p, q)/K$ we are interested in do not fall into either of these categories. Theorem 1 extends the known results to this case.

In our argument $Z' \subset Y$ is the quotient of a maximal periodic flat in X . Theorem 1 gives a partial answer to a question of Avramidi–Nguyen-Phan [2, §9].

Characteristic class interpretation. For the application to manifold bundles, the element

$$[Z] \in H_{pq-p}^{\text{cl}}(Y; \mathbb{Q}) \cong H^p(B\Gamma; \mathbb{Q})$$

described above is not of any particular use as an abstract cohomology class. For this reason, one wants a bundle-theoretic construction of $[Z]$ as a characteristic class.

Fix a lattice $\Lambda \subset \mathbb{R}^{p+q}$ as in Theorem 1. For a CW complex B , a map $B \rightarrow B \operatorname{SO}(\Lambda)$ defines a vector bundle $\mathbb{R}^{p+q} \rightarrow W \rightarrow B$ with a fiberwise lattice Λ

and a fiberwise bilinear form β of signature (p, q) . We extend the structure group to $\mathrm{SO}(p, q) > \mathrm{SO}(\Lambda)$ (this amounts to forgetting Λ but remembering β), and then consider the different ways to reduce the structure group from $\mathrm{SO}(p, q)$ to its maximal compact subgroup. Each choice of reduction corresponds to a choice of a rank- p subbundle $U \subset W$ on which β is positive definite. From this setup, we build a characteristic class c that measures the difficulty of choosing $U \subset W$ in a way that is “compatible with Λ .” We make this precise in §4 using classical obstruction theory, and we show that $c \in H^*(B\Gamma; \mathbb{Q})$ is dual to a geometric cycle $[Z]$.

Applications to manifold bundles. Let M^{4k} be a manifold, and let Λ_M denote the lattice $H_{2k}(M; \mathbb{Z})/\text{torsion}$ with its intersection form. Given an M -bundle $\pi: E \rightarrow B$, one can build a vector bundle $W \rightarrow B$ by replacing each fiber $M_b := \pi^{-1}(b)$ with its homology $H_{2k}(M_b; \mathbb{R})$. On the level of classifying spaces, this corresponds to the map

$$\alpha: B \mathrm{Diff}(M) \rightarrow B \mathrm{O}(\Lambda_M)$$

induced by the action $\alpha: \mathrm{Diff}(M) \rightarrow \mathrm{O}(\Lambda_M)$ of the group of orientation-preserving diffeomorphisms $\mathrm{Diff}(M)$ on Λ_M by automorphisms with determinant ± 1 . For $\Gamma < \mathrm{SO}(\Lambda_M)$, we define

$$\mathrm{Diff}^\Gamma(M) = \alpha^{-1}(\Gamma).$$

If $\Gamma < \mathrm{SO}(\Lambda_M)$ is finite index, then $\mathrm{Diff}^\Gamma(M) < \mathrm{Diff}(M)$ is also finite index. In this case, note that any M bundle $E \rightarrow B$ has structure group reducing to $B \mathrm{Diff}^\Gamma(M)$ after passing to a finite cover of B .

To apply Theorem 1 to manifold bundles, we are interested in the homomorphism

$$\alpha^*: H^*(B\Gamma; \mathbb{Q}) \rightarrow H^*(B \mathrm{Diff}^\Gamma(M); \mathbb{Q}). \quad (1.1)$$

Application to $W_g^{4k} = \#_g(S^{2k} \times S^{2k})$. When $M = W_g^{4k}$, information about α^* can be obtained using work of Berglund–Madsen [4]. This is explained in the appendix, written by Manuel Krannich, which studies α^* for the more general class of manifolds $W_g^{2n} = \#_g(S^n \times S^n)$ with $n \geq 3$. It is shown that α^* is injective in a range of degrees growing with n ; see Theorem 29. Corollary 2 follows immediately from Theorems 1 and 29. As a further consequence of Theorem 29, the appendix produces unstable classes in the rational cohomology of $B \mathrm{Diff}_g^{2n}$ for $n \geq 4$; see Corollary 30.

We remark that the homomorphism

$$H^*(B \mathrm{O}(\Lambda_M); \mathbb{Q}) \rightarrow H^*(B \mathrm{Diff}(M); \mathbb{Q})$$

can be completely understood in the stable range using index theory. Morita [32] showed this for M a surface; see also [13]. The techniques used to study α^* outside the stable range rely on surgery theory, Morlet’s lemma of disjunction, and rational homotopy theory; see the appendix.

Application to K3 surfaces. Let M^4 be a manifold diffeomorphic to a K3 surface. In this case $\mathrm{SO}(\Lambda_M)$ is a lattice in $\mathrm{SO}(3, 19)$, and by Theorem 1, we can find finite-index $\Gamma < \mathrm{SO}(\Lambda_M)$ and a nonzero cycle $z \in H_3(B\Gamma; \mathbb{Q})$. Using the global Torelli theorem, we conclude that z is in the image of

$$H_3(B\pi_0 \mathrm{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q}).$$

We are not able to determine if z is in the image of

$$\alpha_*: H_*(B \mathrm{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_*(B\Gamma; \mathbb{Q}),$$

but we relate this problem to another problem of interest. Specifically, we give an example $z \neq 0 \in H_3(B\Gamma; \mathbb{Q})$ so that if z is in the image α_* , then there exists a K3-surface bundle over a 3-manifold that does not admit a fiberwise Einstein metric. This should be contrasted with a theorem of Donaldson [12, Cor. 6.3] that says that every K3 bundle over S^1 admits a fiberwise Einstein metric; when the base has dimension at least 2, the corresponding statement seems to be unknown.

Odd-dimensional manifolds. With the methods of this paper, we can also produce characteristic classes for M bundles when $\dim M$ is odd. A sample application to $M = \#_3(S^d \times S^{d+1})$ is discussed in §5.

Section outline. In §2 we recall the general method of constructing homology of arithmetic groups using geometric cycles. In §3 we apply that method to $\mathrm{SO}(\Lambda)$ and prove Theorem 1. In §4 we explain how to view geometric cycles as characteristic classes. Finally, §5 and the appendix contain the applications to manifold bundles.

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2. Homology of arithmetic groups: geometric cycles

This section provides the setup for proving Theorem 1. We summarize the general strategy to produce geometric cycles in the homology of arithmetic groups/locally symmetric manifolds. We follow [30] and [38] and refer the reader to these sources for further details. In §3 we will apply the material of this section to the specific case of interest $\Gamma < \mathrm{SO}(\Lambda)$.

2.1. Geometric cycles: the general strategy. Fix an algebraic \mathbb{Q} -group G such that $G(\mathbb{R})$ is a semisimple Lie group without compact factors. We are interested in finding some nontrivial homology of a finite-index subgroup $\Gamma < G(\mathbb{Z})$. The exact subgroup Γ will be (unfortunately) out of our control, and at several points we will replace Γ with a further finite-index subgroup (without changing the notation) to ensure that some geometric fact is true.

We begin by describing the locally symmetric model for $B\Gamma$. Choose a maximal compact subgroup $K < G(\mathbb{R})$, and define $X = G(\mathbb{R})/K$. The manifold X is contractible and admits a $G(\mathbb{R})$ -invariant Riemannian metric of nonpositive curvature. Since $\Gamma < G(\mathbb{R})$ is discrete, it acts properly discontinuously on X and each point-stabilizer in Γ is finite. We can replace Γ by a torsion-free, finite-index subgroup [33, Thm. 4.8.2], so then Γ acts freely on X . Then $Y = \Gamma \backslash X$ is a model for $B\Gamma$. The manifold Y may be noncompact, but it has finite volume (because arithmetic subgroups are lattices [33, Thm. 1.3.9]).

The manifold Y has an abundance of totally geodesic immersed submanifolds. Let $G_1 < G$ be a subgroup and take $h \in G(\mathbb{R})$ so that $K_1 := G_1(\mathbb{R}) \cap (hKh^{-1})$ is a maximal compact subgroup of $G_1(\mathbb{R})$. The image of the orbit map

$$G_1(\mathbb{R}) \ni g \mapsto ghK \in X$$

is totally geodesic submanifold

$$G_1(\mathbb{R})/K_1 \cong X_1 \subset X.$$

If G_1 is a \mathbb{Q} -subgroup, then $\Gamma_1 = G_1 \cap \Gamma$ has finite index in $G_1(\mathbb{Z})$, and the natural map $j_1: Y_1 = \Gamma_1 \backslash X_1 \rightarrow Y$ is a proper, totally geodesic immersion [38, §6].

With this setup, we are ready to discuss the general strategy for producing “geometric cycles” in the homology of Y . Let $o = eK$ be the basepoint of X . First choose $G_1, G_2 < G$ so that

(†) X_1 and X_2 have complementary dimension $d_1 + d_2 = \dim X$, the intersection $X_1 \cap X_2 = \{o\}$ is transverse, and Y_1 is compact.

Then choose $\Gamma < G(\mathbb{Z})$ so that

(‡) the quotients Y, Y_1, Y_2 are oriented manifolds, the maps j_1, j_2 are embeddings, and the intersection $Y_1 \cap Y_2$ is transverse and every intersection has positive sign.

Given (\dagger) and (\ddagger) , the submanifolds $Y_i \subset Y$ determine classes $[Y_1] \in H_{d_1}(Y; \mathbb{Q})$ and $[Y_2] \in H_{d_2}^{\text{cl}}(Y; \mathbb{Q})$ in homology and homology with closed supports, and the algebraic intersection $[Y_1] \cdot [Y_2]$ is nonzero, so $[Y_1] \neq 0$ in $H_{d_1}(Y; \mathbb{Q}) \cong H_{d_1}(\Gamma; \mathbb{Q})$, and $[Y_2] \neq 0$ in $H_{d_2}^{\text{cl}}(Y; \mathbb{Q}) \cong H^{d_1}(Y; \mathbb{Q}) \cong H^{d_1}(\Gamma; \mathbb{Q})$. This is explained in more detail in [38].

The general strategy does not always work. Indeed, it is not always possible to achieve (\dagger) . One problem is that a totally geodesic subspace $X_1 \subset X$ need not admit a subspace of complementary dimension. (It is shown in [30, Theorem 1.1] that such a complement exists if $X_1 = X^\sigma$ is the fixed set of an involutive isometry.) Another problem is that if Y is non-compact, then there is no reason Y_1 or Y_2 need be compact in general. Nevertheless, in some special cases, one can find G_1, G_2 so that (\dagger) is satisfied. We will see this when G is an indefinite orthogonal group in the next section.

Now we address the difficulty with (\ddagger) . There is a general theorem that ensures the first two clauses of (\ddagger) . (Ensuring that Y is oriented is easy, but ensuring Y_1 and Y_2 are oriented is already nontrivial.)

Theorem 3 ([38, Theorem D]). *Let G be a connected semisimple algebraic \mathbb{Q} -group, let $G_1 < G$ be a connected reductive \mathbb{Q} -subgroup, and let $\Gamma < G(\mathbb{Q})$ be an arithmetic subgroup. Then after replacing Γ by a finite-index subgroup, the map $j_1: Y_1 \rightarrow Y$ is a proper, injective, closed embedding, and each component of the image is an orientable, totally geodesic submanifold of Y .*

Given Theorem 3, the remaining difficulty is showing that, after replacing Γ with a subgroup of large index, $Y_1 \cap Y_2$ is a finite set of points and the intersection number at each point is $+1$. We explain how to approach this problem in the next subsection.

2.2. Intersections and double cosets. Assume that $G_1, G_2 < G$ satisfy (\dagger) and that $\Gamma < G(\mathbb{Z})$ is torsion-free and the associated manifolds Y_1 and Y_2 are oriented, embedded submanifolds of Y . As explained in [30] (to be reviewed below), the components of $Y_1 \cap Y_2$ can be identified with a certain subset Ω of the double coset space $\Gamma_2 \backslash \Gamma / \Gamma_1$. Choosing coset representatives $I(\Gamma) \subset \Gamma$ for Ω , the sign of the intersection corresponding to $\gamma \in I(\Gamma)$ is determined by the double coset of γ in $G_2^+(\mathbb{R}) \backslash G(\mathbb{R}) / G_1^+(\mathbb{R})$, where $G_i^+(\mathbb{R}) < G_i(\mathbb{R})$ is the subgroup that preserves orientation on X_i and on X (these conditions are not automatic if $G_i(\mathbb{R})$ and $G(\mathbb{R})$ are not connected). In particular, if γ can be written $\gamma = g_2 g_1$ with $g_i \in G_i^+(\mathbb{R})$, then the corresponding intersection is positive [30, Cor. to Lem. 2.5]. One wants to show that if $\Gamma < G(\mathbb{Z})$ is a subgroup of large index, then every $\gamma \in I(\Gamma)$ belongs to one of the double cosets in $G_2^+(\mathbb{R}) \backslash G(\mathbb{R}) / G_1^+(\mathbb{R})$ that correspond to a positive intersection number.

Components of $Y_1 \cap Y_2$. First we describe the set $\pi_0(Y_1 \cap Y_2)$ in terms of the double coset space $\Gamma_2 \backslash \Gamma / \Gamma_1$. Denote the projection $\Pi: X \rightarrow Y$. There is a

bijection between $\Pi^{-1}(Y_1 \cap Y_2)$ and

$$T := \{(\gamma, x_1, x_2) : \gamma x_1 = x_2\} \subset \Gamma \times X_1 \times X_2$$

(an intersection downstairs is covered by an intersection upstairs, and we can translate by Γ so that the intersection happens on X_2). The set T has an action of $\Gamma_2 \times \Gamma_1$ given by

$$(\alpha_2, \alpha_1) \cdot (\gamma, x_1, x_2) = (\alpha_2 \gamma \alpha_1^{-1}, \alpha_1 x_1, \alpha_2 x_2).$$

Claim. $Y_1 \cap Y_2 \cong T/(\Gamma_2 \times \Gamma_1)$.

To prove the claim, one shows that if (γ, x_1, x_2) and (γ', x'_1, x'_2) are in T , then $\Pi(x_1) = \Pi(x'_1)$ if and only if there exists $\alpha_i \in \Gamma_i$ so that

$$(\gamma', x'_1, x'_2) = (\alpha_2, \alpha_1) \cdot (\gamma, x_1, x_2).$$

The “if” direction is obvious. For the “only if” direction, one uses the fact that Y_i is embedded in Y , which implies that if $\gamma \in \Gamma$ and $\gamma X_i \cap X_i \neq \emptyset$, then $\gamma \in \Gamma_i$. More details can be found in [30, §2].

A similar argument shows that if (γ, x_1, x_2) and (γ', x'_1, x'_2) are in T , then $\Pi(x_1)$ and $\Pi(x'_1)$ are in the same component of $Y_1 \cap Y_2$ if and only if γ and γ' lie in the same double coset $\Gamma_2 \backslash \Gamma / \Gamma_1$; see [30, Lemma 2.3 and Proposition 2.3]. In other words, $\pi_0(Y_1 \cap Y_2)$ is in bijection with

$$\Omega := \{\Gamma_2 \gamma \Gamma_1 : \gamma X_1 \cap X_2 \neq \emptyset\} \subset \Gamma_2 \backslash \Gamma / \Gamma_1.$$

Note that $\Omega \cong \pi_0(Y_1 \cap Y_2)$ is finite because $Y_1 \cap Y_2 \subset Y_1$ is a submanifold and Y_1 is compact.

Sign of the intersection. Next we explain, for each $y \in \pi_0(Y_1 \cap Y_2)$, whether the intersection is positive, negative, or degenerate. Fix a set of coset representatives $I(\Gamma) \subset \Gamma$ for elements of Ω , and assume that $\gamma_1 = \text{Id}$ represents $\Gamma_2 \Gamma_1$.

Note that for $\gamma \in I(\Gamma)$ we can write $\gamma = a_2 k a_1^{-1}$, where $a_i \in G_i^+(\mathbb{R})$ and $k \in K$. This is because $\gamma X_1 \cap X_2 \neq \emptyset$ implies that there exists $x_i \in X_i$ so that $\gamma x_1 = x_2$. Since $G_i^+(\mathbb{R})$ acts transitively on X_i , we can choose $a_i \in G_i^+(\mathbb{R})$ so that $a_i(o) = x_i$, where $o = eK$ is the basepoint of $X = G(\mathbb{R})/K$ (and is also the intersection of X_1 and X_2). Then $a_2^{-1} \gamma a_1(o) = o$, which means $a_2^{-1} \gamma a_1 = k$ for some $k \in K$.

Since a_i preserves orientation on X_i and X , the sign of the intersection $\gamma X_1 \cap X_2$ is determined by the action of k on $T_o X$:

Lemma 4. *Let $e_1, \dots, e_p \in T_o X_1$ and $e_{p+1}, \dots, e_n \in T_o X_2$ be positively oriented bases. Define $\epsilon(\gamma)$ by*

$$k(e_1) \wedge \dots \wedge k(e_p) \wedge e_{p+1} \wedge \dots \wedge e_n = \epsilon(\gamma) e_1 \wedge \dots \wedge e_n. \quad (2.1)$$

- (a) *The intersection corresponding to $\gamma \in I(\Gamma)$ is positive, negative, or degenerate according to whether $\epsilon(\gamma)$ is positive, negative, or zero.*
- (b) *If $\gamma, \gamma' \in I(\Gamma)$ lie in the same coset $G_2^+(\mathbb{R}) \backslash G(\mathbb{R}) / G_1^+(\mathbb{R})$, then $\epsilon(\gamma) = \epsilon(\gamma')$.*

Proof. We begin with (a). The basis (a_1e_1, \dots, a_1e_p) is positively oriented in $T_{x_1}X_1$, and the basis $(a_2e_{p+1}, \dots, a_2e_n)$ is positively oriented in $T_{x_2}X_2$. We want to know if

$$(\gamma(a_1e_1), \dots, \gamma(a_1e_p), a_2e_{p+1}, \dots, a_2e_n)$$

is positively oriented in $T_{x_2}X$. Since a_2 preserves orientation on X and $\gamma = a_2ka_1^{-1}$, the orientation of this n -tuple is the same as the orientation of

$$(ke_1, \dots, ke_p, e_{p+1}, \dots, e_n).$$

See also [30, Prop. 2.3].

For (b), assume $\gamma, \gamma' \in I(\Gamma)$ and $\gamma' = h_2\gamma h_1$ for some $h_i \in G_2^+(\mathbb{R})$. There are two cases: the intersection $\gamma X_1 \cap X_2$ is either degenerate or not. If $\gamma X_1 \cap X_2$ is degenerate (i.e. has dimension at least 1), then the same is true for $h_2\gamma h_1 = \gamma'$, so

$$\epsilon(\gamma) = 0 = \epsilon(\gamma').$$

If $\gamma X_1 \cap X_2$ and hence also $\gamma' X_1 \cap X_2$ are non-degenerate, then there exists a unique $x_1, x'_1 \in X_1$ and $x_2, x'_2 \in X_2$ so that

$$\gamma x_1 = x_2 \quad \text{and} \quad \gamma' x'_1 = x'_2.$$

Since $\gamma' = h_2\gamma h_1$, it follows that $x_1 = h_1x'_1$ and $x_2 = h_2^{-1}x'_2$. Then if $\gamma = a_2ka_1^{-1}$ where $a_i(0) = x_i$, then we have

$$\gamma' = h_2a_2ka_1^{-1}h_1,$$

and

$$h_1^{-1}a_1(0) = h_1^{-1}(x_1) = x'_1 \quad \text{and} \quad h_2a_2(0) = h_2(x_2) = x'_2,$$

which implies that both $\epsilon(\gamma)$ and $\epsilon(\gamma')$ is computed from the action of k as in (2.1), so

$$\epsilon(\gamma) = \epsilon(\gamma').$$

See also [30, Lem. 2.5]. □

We note that (b) implies that the sign of the intersection doesn't depend on the choice of representative $\gamma \in I(\Gamma)$ as long as $\Gamma_i \subset G_i^+(\mathbb{R})$, which can be arranged by passing to a finite-index subgroup (c.f. Theorem 3). In this case we have the following formula

$$[Y_1] \cdot [Y_2] = \sum_{\gamma \in I(\Gamma)} \epsilon(\gamma).$$

Note in addition that (b) implies that if $\gamma \in G_2^+(\mathbb{R})G_1^+(\mathbb{R})$, then $\epsilon(\gamma) = +1$. We can see this latter fact directly as follows: if $\gamma = g_2g_1 \in G_2^+(\mathbb{R})G_1^+(\mathbb{R})$, then

$$\begin{aligned} \gamma X_1 \cap X_2 &= (g_2g_1X_1) \cap X_2 = (g_2X_1) \cap X_2 \\ &= g_2(X_1 \cap g_2^{-1}X_2) = g_2(X_1 \cap X_2), \end{aligned} \quad (2.2)$$

and since g_2 preserves orientation on X and X_2 , this implies that the sign of the intersection $\gamma X_1 \cap X_2$ is equal to the sign of the intersection $X_1 \cap X_2$, which is positive by assumption.

Writing $\gamma \in I(\Gamma)$ in $G_2^+(\mathbb{R})G_1^+(\mathbb{R})$. As mentioned above, one way to show an intersection $\gamma X_1 \cap X_2$ is positive is to show that $\gamma \in G_2^+(\mathbb{R})G_1^+(\mathbb{R})$. The next two propositions are partial progress to writing γ in this form (which is not always possible in general, but our goal will be to show it can be ensured by passing to a finite-index subgroup a given Γ). The following proposition follows from the argument of [30, Theorem 3.1]; see also [14, Lem. 2.6].

Proposition 5. *Let G be an connected, reductive algebraic \mathbb{Q} -group with $K < G(\mathbb{R})$ a maximal compact subgroup. Fix an arithmetic subgroup $\Gamma < G(\mathbb{Z})$, and let $G_1, G_2 < G$ be connected, reductive \mathbb{Q} -subgroups. There exists a finite-index subgroup $\Gamma' < \Gamma$ so that if $\gamma \in \Gamma'$ and $G_2(\mathbb{R})\gamma \cap KG_1(\mathbb{R}) \neq \emptyset$, then $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$.*

From the above, we see that if $\gamma X_1 \cap X_2 \neq \emptyset$, then $G_2(\mathbb{R})\gamma \cap KG_1(\mathbb{R}) \neq \emptyset$. By Proposition 5, after replacing Γ by a finite-index subgroup, we can ensure that if $\gamma \in I(\Gamma)$, then $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$. We strengthen this with the following proposition.

Proposition 6. *Fix G, G_1, G_2 and Γ as above. Assume that $G_1 \cap G_2 = \{\text{Id}\}$. For $\gamma \in \Gamma$, if $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$, then $\gamma \in G_2(\mathbb{Q})G_1(\mathbb{Q})$.*

Remark 7. Note that if $X_1 \cap X_2 = \emptyset$ and $\gamma = g_2g_1 \in G_2(\mathbb{Q})G_1(\mathbb{Q})$, then also $\gamma X_1 \cap X_2 = \emptyset$ by the computation in (2.2). Thus, as a consequence of Proposition 6 and the above discussion, if X_1 and X_2 are disjoint, then there exists finite index $\Gamma < G(\mathbb{Z})$ so that the quotient submanifolds Y_1, Y_2 are disjoint in Y . This observation will be used in §3.3 to produce linearly independent cycles.

Proof of Proposition 6. We are given $\gamma = h_2h_1$ with $h_i \in G_i(\mathbb{C})$. To show that $h_i \in G_i(\mathbb{Q})$ we show that $h_i\sigma(h_i) = e$ for every $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$. Given $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, since $\sigma(\gamma) = \gamma$, we conclude that

$$h_1\sigma(h_1)^{-1} = h_2^{-1}\sigma(h_2).$$

This equality implies

$$h_1\sigma(h_1)^{-1} \in G_1(\mathbb{C}) \cap G_2(\mathbb{C}),$$

which is trivial assumption. Hence $h_1 = \sigma(h_1)$ and similarly $h_2 = \sigma(h_2)$. \square

In summary, we have shown that $\gamma \in I(\Gamma)$ can be expressed as $\gamma = g_2g_1$ with $g_i \in G_i(\mathbb{Q})$.

Eliminating intersections. Assume Γ satisfies the hypothesis of Proposition 5 and hence the conclusion of Proposition 6. Write $I^+(\Gamma)$ for the subset of $\gamma \in I(\Gamma)$ for which the intersection $\gamma X_1 \cap X_2$ is positive. We finish this section with two easy lemmas that we will use to prove Theorem 1 in §3.

Lemma 8. *Take Γ as in the preceding paragraph. Let $\Gamma' < \Gamma$ be a finite-index subgroup. If $\Gamma' \cap \Gamma_2 \gamma \Gamma_1 = \emptyset$ for every $\gamma \in I(\Gamma) \setminus I^+(\Gamma)$, then for every $\gamma' \in \Gamma'$ for which $\gamma' X_1 \cap X_2 \neq \emptyset$, the sign of the intersection is positive.*

Proof. Denoting $\Gamma'_i = \Gamma' \cap \Gamma_i$, there is a map

$$\Omega' = \Gamma'_2 \backslash \Gamma' / \Gamma'_1 \rightarrow \Gamma_2 \backslash \Gamma / \Gamma_1 = \Omega$$

induced by the inclusion $\Gamma' \hookrightarrow \Gamma$. Fixing γ' as in the statement, by assumption $\Gamma_2 \gamma' \Gamma_1 = \Gamma_2 \gamma \Gamma_1$, where $\gamma \in I^+(\Gamma)$. By Lemma 4, the sign of the intersection $\gamma' X_1 \cap X_2$ is the sign of $\gamma X_1 \cap X_2$, which is positive since $\gamma \in I^+(\Gamma)$. \square

Lemma 9. *Suppose that $\Gamma' \triangleleft \Gamma$ is a normal subgroup. Fix $\gamma \in \Gamma$ and $\Gamma_1, \Gamma_2 < \Gamma$. Then $\Gamma' \cap \Gamma_2 \gamma \Gamma_1 \neq \emptyset$ if and only if $\Gamma' \gamma \cap \Gamma_2 \Gamma_1 \neq \emptyset$.*

Proof. The proof is straightforward. If $\gamma' = \gamma_2 \gamma \gamma_1$ with $\gamma' \in \Gamma'$ and $\gamma_i \in \Gamma_i$, then

$$\gamma_1^{-1} = (\gamma')^{-1} \gamma_2 \gamma = \gamma_2 (\gamma_2^{-1} (\gamma')^{-1} \gamma_2) \gamma.$$

Equivalently, $\gamma_2^{-1} \gamma_1^{-1} = (\gamma_2^{-1} (\gamma')^{-1} \gamma_2) \gamma$, which implies that $\Gamma' \gamma \cap \Gamma_2 \Gamma_1 \neq \emptyset$. The other direction is similar. \square

3. Geometric cycles for $\Gamma < \text{SO}(p, q)$

In this section we prove Theorem 1. Fix $1 \leq p \leq q$ and $\Lambda \subset \mathbb{R}^{p+q}$ as in the statement of the theorem. Let B be the matrix for the bilinear form on Λ with respect to some basis, and consider the algebraic \mathbb{Q} -group

$$G = \text{SO}(B) = \{g \in \text{SL}_{p+q}(\mathbb{C}) : g^t B g = B\}. \tag{3.1}$$

Setting $\Gamma = G(\mathbb{Z}) \cong \text{SO}(\Lambda)$, we split the proof of the theorem into proving two statements:

- (a) Up to replacing Γ by a finite-index subgroup, $H_p(\Gamma; \mathbb{Q})$ is nonzero.
- (b) Given $n \geq 1$, we can replace Γ by a finite-index subgroup so that $\dim H_p(\Gamma; \mathbb{Q}) \geq n$.

In §3.1, we define algebraic groups G_1, G_2 so that $G_1(\mathbb{R}) \cong \text{SO}(1, 1)^p$ and $G_2(\mathbb{R}) \cong \text{SO}(p, q - 1)$, and we verify that the conditions of (\dagger) from §2 can be satisfied for a good choice of $G_1, G_2 < G$. In §3.2, we show that we can choose $\Gamma < G(\mathbb{Z})$ so that (\ddagger) is also satisfied. Together these prove (a). In §3.3 we prove (b) by showing how to produce many linearly independent flat cycles.

3.1. Choosing $G_1, G_2 < G$.

Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

We choose G_1 as a maximal \mathbb{R} -split torus contained in the centralizer of a *hyper-regular* element $\tau \in G(\mathbb{Z})$, in the sense of Prasad–Raghunathan [35]. The group G_1 is defined and anisotropic over \mathbb{Q} (i.e. G_1 does not contain any nontrivial \mathbb{Q} -split torus, which implies that $G_1(\mathbb{R})/G_1(\mathbb{Z})$ is compact). One can take τ to preserve a decomposition $V = U \oplus U^\perp$ defined over \mathbb{Q} with U^\perp negative definite of dimension $q - p$ and such that the action of τ on U is irreducible and has $2p$ distinct real eigenvalues occurring in $(\lambda, 1/\lambda)$ pairs (irreducible implies in particular that the eigenvectors of τ are not defined over \mathbb{Q}). See below for a concrete example.

For G_2 , we fix $\lambda \in \Lambda$ with $\lambda \cdot \lambda < 0$ and define $G_2 \cong \mathrm{SO}(B')$, where B' is the restriction of B to λ^\perp . This group includes into $G(B)$ in an obvious way, acting trivially on $\langle \lambda \rangle$.

Our groups have real points $G(\mathbb{R}) \cong \mathrm{SO}(p, q)$, $G_1(\mathbb{R}) \cong \mathrm{SO}(1, 1)^p \cong (\mathbb{R}^\times)^p$, and $G_2(\mathbb{R}) \cong \mathrm{SO}(p, q - 1)$. The associated symmetric spaces X, X_1, X_2 have dimensions $pq, p, p(q - 1)$, respectively.

Example. We give an explicit example of the group G_1 for $\mathrm{SO}(2, 2)$ by an ad hoc construction. Consider the number field $F = \mathbb{Q}(\alpha)$, where α is a root of $t^4 - 12t^3 + 23t^2 - 12t + 1$. The elements α and $\frac{1}{7}(16\alpha^3 - 180\alpha^2 + 233\alpha - 12)$ are units in \mathcal{O}_F and act on $\mathcal{O}_F \cong \mathbb{Z}^4$ by the matrices

$$\tau_1 = \begin{pmatrix} -1 & 0 & 0 & -7 \\ 2 & 0 & 0 & 13 \\ -3 & 1 & 0 & -22 \\ 2 & 0 & 1 & 13 \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} -7 & -16 & -12 & -9 \\ 14 & 28 & 20 & 15 \\ -9 & -17 & -12 & -10 \\ 2 & 4 & 3 & 3 \end{pmatrix}.$$

These matrices preserve the bilinear form with matrix

$$B = \begin{pmatrix} 6 & 2 & 2 & 19 \\ 2 & 0 & 1 & 12 \\ 2 & 1 & 0 & 1 \\ 19 & 12 & 1 & 0 \end{pmatrix},$$

which has signature $(2, 2)$ and is unimodular ($\det B = 1$). Let $G_1 < \mathrm{SO}(B)$ be the centralizer of τ_1 . Since τ_1 has 4-distinct real eigenvalues (none with norm 1), $G_1(\mathbb{R}) \cong \mathrm{SO}(1, 1)^2$ is a maximal torus in $\mathrm{SO}(B)(\mathbb{R})$. Furthermore, G_1 is \mathbb{Q} -anisotropic because the eigenvectors of τ_1 are not defined over \mathbb{Q} . Note that $G_1(\mathbb{Z}) < G_1(\mathbb{R})$ is cocompact since it contains $\langle \tau_1, \tau_2 \rangle \cong \mathbb{Z}^2$. This construction can also be used to give examples in $\mathrm{SO}(2, q)$ for $q > 2$ by taking the direct sum with a negative definite form and extending by the identity. In this case, the \mathbb{R} -points of the centralizer of τ_1 is $\mathrm{SO}(1, 1)^p \times \mathrm{SO}(q - p)$.

Generic pairs G_1, G_2 . We return to the general setup $G_1, G_2 < G = \mathrm{SO}(B)$. We want to choose G_1, G_2 so that $X_1 \cap X_2 = \{o\}$ is a single point. This leads us

to define a notion of generic pairs G_1, G_2 , which will be useful at various points. We say that G_1, G_2 are *generic* if $G_1 \cap G_2 = \{\text{Id}\}$. We interpret this condition as an equality of algebraic groups from the “functor of points” point-of-view, so that $G_1(F) \cap G_2(F) = \{\text{Id}\}$ for each field extension F/\mathbb{Q} . In particular, we will use this for $F = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$.

For example, $G_1(\mathbb{R}) \cap G_2(\mathbb{R}) = \{\text{Id}\}$ implies that the intersection $X_1 \cap X_2$ is transverse, i.e. either a point or the empty set (it’s empty if there is no maximal compact subgroup of $G(\mathbb{R})$ that intersects both $G_1(\mathbb{R})$ and $G_2(\mathbb{R})$ in respective maximal compact subgroups). To see this implication, suppose that $\dim X_1 \cap X_2 \geq 1$. Without loss of generality we assume $o \in X_1 \cap X_2$. Then $\dim(T_o X_1 \cap T_o X_2) \geq 1$ implies that $\dim(\mathfrak{g}_1 \cap \mathfrak{g}_2) \geq 1$, where \mathfrak{g}_i is the Lie algebra of $G_i(\mathbb{R})$. Finally, this implies that $\dim(G_1(\mathbb{R}) \cap G_2(\mathbb{R})) \geq 1$.

The following proposition gives a sufficient condition for G_1, G_2 to be generic. To state it, consider the action of $G(\mathbb{Q})$ on $V_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The group $G_1(\mathbb{Q})$ preserves a subspace U of signature (p, p) and acts trivially on U^{\perp} . The group $G_2(\mathbb{Q})$ preserves $\lambda^{\perp} \subset V_{\mathbb{Q}}$ and acts trivially on $\langle \lambda \rangle$.

Proposition 10. *Take $G_1, G_2 < G$ and take $U, \langle \lambda \rangle \subset V_{\mathbb{Q}}$ as in the preceding paragraph. If $\lambda \notin U^{\perp}$, then G_1, G_2 are generic.*

Proof. First we set some notation. Let $\tau \in G(\mathbb{Z})$ be the hyper-regular element used to define G_1 . Let $V_{\mathbb{Q}} = U \oplus U^{\perp}$ the decomposition preserved by τ , where U^{\perp} is negative-definite of dimension $q - p$ and τ acts irreducibly on U . Let $\lambda \in \Lambda$ be the vector used to define G_2 . Observe that G_2 can be described in terms of a centralizer: let $\epsilon \in \text{GL}_{p+q}(\mathbb{Q})$ be the automorphism of $V_{\mathbb{Q}}$ that fixes λ and acts as -1 on its orthogonal complement. Then G_2 is the subgroup of the centralizer of ϵ in $G(\mathbb{Q})$ that acts trivially on $\langle \lambda \rangle$.

For an extension F/\mathbb{Q} , we denote $C_{\tau}(F), C_{\epsilon}(F)$ the centralizers of τ and ϵ in the (vector) space of $(p + q) \times (p + q)$ matrices over F .

Claim. If $a \in C_{\tau}(\mathbb{Q}) \cap C_{\epsilon}(\mathbb{Q})$, then a acts as on U by multiplication by a scalar $x \in \mathbb{Q}$.

Proof of the claim. Since $\epsilon(\lambda) = \lambda$ and a commutes with ϵ , λ is an eigenvector of a , i.e. $a(\lambda) = x\lambda$ for some $x \in \mathbb{Q}$. Write $\lambda = u + v \in U \oplus U^{\perp}$. We know $u \neq 0$ because we’re assuming $\lambda \notin U^{\perp}$. Also a preserves U^{\perp} because $\tau|_{U^{\perp}} = \text{Id}$ and a commutes with τ . Then a also preserves $U = (U^{\perp})^{\perp}$. Then

$$xu + xv = x(u + v) = a(u + v) = a(u) + a(v)$$

implies that u and v are both eigenvectors for a with eigenvalue x . In addition

$$a(\tau^i(u)) = \tau^i(a(u)) = x\tau^i(u)$$

so $\tau^i(u)$ is also an eigenvector for a with eigenvalue x for each $i \in \mathbb{Z}$. Since $u \in U$ and τ acts irreducibly on U , this implies that a acts on U by multiplication by x . This proves the claim. □

Using the claim, it follows that if F/\mathbb{Q} is an extension then any $a \in C_\tau(F) \cap C_\epsilon(F)$ acts on $U \otimes F$ by a scalar $x \in F$. This is because the conditions $a\tau = \tau a$ and $a\epsilon = \epsilon a$ give a system of linear equations defined over \mathbb{Q} , so the set of solutions is described independently of the field.

Now we finish the proof of the proposition by showing that $G_1(F) \cap G_2(F) = \{\text{Id}\}$ for any field extension F/\mathbb{Q} . Since

$$G_1(F) \cap G_2(F) \subset C_\tau(F) \cap C_\epsilon(F),$$

any $a \in G_1(F) \cap G_2(F)$ acts on $U \otimes F$ by multiplication by a scalar $x \in F$. But since a acts by an isometry, $x^2 = 1$ so $x = \pm 1$. To show $a = \text{Id}$ we want to show $x = 1$ (we already know that a acts as Id on U^\perp since $a \in G_1(F)$). Since $a \in G_2(F)$, $a(\lambda) = \lambda$. Writing $\lambda = u + v$ as before, then

$$u + v = \lambda = a(\lambda) = a(u) + a(v) = xu + v.$$

Since $u \neq 0$, this implies $x = 1$. This completes the proof. \square

Note that the hypothesis $\lambda \notin U^\perp$ is automatically satisfied when $p = q$ since then $U^\perp = 0$. In addition, given G_1 and G_2 , we can replace λ with λ' so that the rational lines are $\langle \lambda' \rangle$ and $\langle \lambda \rangle$ are arbitrarily close and G_1, G_2' are generic.

3.2. Eliminating intersections. In this section we start with G_1, G_2 generic with $X_1 \cap X_2 = \{o\}$ and with $\Gamma < G(\mathbb{Z})$ that preserves orientation on X and so that $\Gamma_i = G_i \cap \Gamma$ preserves orientation on X_i and X . We've already explained why this is possible. Here we show that we can find finite-index $\Gamma' < \Gamma$ so that if $\gamma' \in \Gamma'$ and $\gamma'X_1 \cap X_2 \neq \emptyset$, then $\gamma' \in G_2^+(\mathbb{R})G_1^+(\mathbb{R})$. This will prove that Γ' satisfies (\ddagger) and finish part (a) of our proof of Theorem 1.

Orientations and spinor norm. Recall that $G_i^+(\mathbb{R}) < G_i(\mathbb{R})$ denotes the subgroup that preserves orientation on X_i and on X . We explain how to determine these groups in our situation. For this, the spinor norm plays an important role.

Let F/\mathbb{Q} be a field extension (we will only use $F = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$). Then $G(F)$ is a group of orthogonal transformations of the quadratic space $V_F = \Lambda \otimes_{\mathbb{Z}} F$, and the spinor norm $\theta_F: G(F) \rightarrow F^\times / (F^\times)^2$ is a homomorphism, defined as follows. Any $g \in G(F)$ can be expressed as a product of reflections $g = R^{x_1} \cdots R^{x_k}$, where R^x denotes the reflection about the orthogonal complement of $x \in V_F$. Then one defines

$$\theta_F(g) = \prod_{i=1}^k x_i \cdot x_i \text{ mod } (F^\times)^2,$$

which is well-defined independent of the choice of reflections. For more information, see [34, §55] and also [30, §4].

In particular, $G(\mathbb{R}) \cong \text{SO}(p, q)$ has two components, detected by the spinor norm $\theta: G(\mathbb{R}) \rightarrow \mathbb{R}^\times / (\mathbb{R}^\times)^2$.

Lemma 11. *Let $G = \text{SO}(B)$ with $G(\mathbb{R}) \cong \text{SO}(p, q)$. If $p + q$ is even, then $G(\mathbb{R})$ preserves the orientation on $X = G(\mathbb{R})/K$. If $p + q$ is odd, then the orientation preserving subgroup of $G(\mathbb{R})$ is the kernel of the spinor norm homomorphism.*

From the lemma, it follows that $G_2(\mathbb{R}) \cong \text{SO}(p, q-1)$ preserves orientation on X_1 if $p + q$ is odd and preserves orientation on X if $p + q$ is even. Hence $g \in G_2^+(\mathbb{R})$ if and only if $\theta(g) = 1$. Lemma 11 is easy to check; its proof is similar to the proof of the following lemma, whose proof we give.

Lemma 12. *The group $G_1(\mathbb{R})$ preserves orientation on X_1 . If $p + q$ is even, then $G_1(\mathbb{R})$ preserves orientation on X_1 . If $p + q$ is odd, then $g \in G_1(\mathbb{R})$ preserves orientation on X if and only if $\theta(g) = 1$.*

Proof. Whether or not $g \in G_1(\mathbb{R})$ preserves orientation on X_1 or X depends only on the component of $g \in G_1(\mathbb{R}) \cong (\mathbb{R}^\times)^p$. Thus it suffices to consider the action of elements of $G_1(\mathbb{R}) \cap K \cong \{\pm 1\}^p$. This allows one to reduce to the tangent space at the basepoint $o \in X$, where the action of K is the adjoint action.

The Lie algebra of $G(\mathbb{R})$ decomposes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is the Lie algebra of K and $\mathfrak{p} \cong T_oX$. We identify $G(\mathbb{R})$ with the group of isometries of \mathbb{R}^{p+q} with respect to the form whose matrix in the basis $(e_1, \dots, e_p, f_1, \dots, f_q)$ is

$$\begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}.$$

We choose $K \cong S(\text{O}(p) \times \text{O}(q))$ to be the obvious block diagonal subgroup. We can identify \mathfrak{p} with $p \times q$ matrices $M_{p,q}$. The adjoint action of

$$\begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \in K = S(\text{O}(p) \times \text{O}(q)) \quad \text{on } \mathfrak{p}$$

is given by $A \mapsto k_1 A k_2^{-1}$.

Up to conjugation in $G(\mathbb{R})$, we can choose G_1 so that

$$G_1(\mathbb{R}) = \prod_{i=1}^p \text{SO}(\mathbb{R}\{e_i, f_i\}).$$

Then $G_1(\mathbb{R}) \cap K \cong \{\pm 1\}^p$ is generated by maps δ_k , where δ_k acts by $-\text{Id}$ on $\mathbb{R}\{e_k, f_k\}$ and by Id on $\mathbb{R}\{e_k, f_k\}^\perp$. The subspace $T_oX_1 \subset T_oX$ is identified with the diagonal matrices in $M_{p,q} \cong T_oX$ (i.e. matrices with $a_{ij} = 0$ for $i \neq j$).

Now one computes: the adjoint action of δ_k on $M_{p,q}$ is given by

$$(a_{ij}) \mapsto (b_{ij}), \quad \text{where } b_{ij} = \begin{cases} -a_{ij} & i = k \text{ or } j = k \text{ but not both,} \\ a_{ij} & \text{else.} \end{cases}$$

Thus the determinant of δ_k acting on $M_{p,q} \cong T_oX$ is $(-1)^{p+q-2}$, and the action on $T_oX_1 \cong$ (diagonal matrices) is by the identity. This lemma follows directly from this computation. □

As a consequence of Lemmas 11 and 12 we immediately obtain:

Corollary 13. *Fix $\gamma \in \Gamma$ with $\gamma X_1 \cap X_2 \neq \emptyset$. If $\gamma = g_2 g_1 \in G_2(\mathbb{R})G_1(\mathbb{R})$ with $\theta(g_2) = \theta(g_1) = 1$, then the intersection $\gamma X_1 \cap X_2$ is positive.*

Eliminating intersections. The final step in the proof of part (a) of Theorem 1 is the following proposition.

Proposition 14. *Fix Γ as above. There exists a finite-index normal subgroup $\Gamma' \triangleleft \Gamma$ so that $\Gamma' \cap \Gamma_2 \gamma \Gamma_1 = \emptyset$ for every $\gamma \in I(\Gamma) \setminus I^+(\Gamma)$.*

By Proposition 14, after replacing Γ by Γ' , every intersection $\gamma X_1 \cap X_2$ has a positive sign by Lemma 8.

Proof of Proposition 14. For a prime p , consider $G_i(\mathbb{Z}_p)$, and define $G_i^0(\mathbb{Z}_p) \subset G_i(\mathbb{Z}_p)$ the subgroup on which the p -adic spinor norm is trivial. By [30, §4, Cor. 1] after replacing Γ by a finite-index subgroup, we can assume that θ is trivial on Γ (and hence also on $\Gamma_1, \Gamma_2 < \Gamma$). In particular,

$$\Gamma_2 \Gamma_1 \subset G_2^0(\mathbb{Z}_p) G_1^0(\mathbb{Z}_p)$$

for each p .

Recall (c.f. Lemma 6) that for each $\gamma \in I(\Gamma)$, we can write $\gamma = g_2 g_1$ with $g_i \in G_i(\mathbb{Q})$. This expression is unique because G_1, G_2 are generic. Since $\theta(\gamma) = 1$, we have

$$\theta(g_2) = \theta(g_1) \in \mathbb{R}^\times / (\mathbb{R}^\times)^2.$$

Consider the subset $\{\gamma_1, \dots, \gamma_m\} \subset I(\Gamma)$ of those γ_j for which $\gamma_j = g_{2,j} g_{1,j}$ with $\theta(g_{2,j}) \neq 1$. Note that $\{\gamma_1, \dots, \gamma_m\}$ contains the complement of $I^+(\Gamma)$. To prove the proposition, we will find $\Gamma' \triangleleft \Gamma$ so that $\Gamma' \cap \Gamma_2 \gamma_j \Gamma_1 = \emptyset$ for each j .

For each j , there exists a prime p_j so that $\theta_{p_j}(g_{2,j}) \neq 1$, where

$$\theta_{p_j}: G(\mathbb{Q}_{p_j}) \rightarrow \mathbb{Q}_{p_j}^\times / (\mathbb{Q}_{p_j}^\times)^2$$

is the p -adic spinor norm (i.e. if $x \in \mathbb{Q}$ is not a square, then there exists a prime p so that x is not a square in \mathbb{Q}_p). For each j , there exists n_j so that

$$\Gamma(p_j^{n_j}) \gamma_j \cap G_2^0(\mathbb{Z}_{p_j}) G_1^0(\mathbb{Z}_{p_j}) = \emptyset.$$

To see this, note that the groups $G_i^0(\mathbb{Z}_{p_j})$ are compact in $G(\mathbb{Q}_p)$ and hence so too is their product. Thus since $\theta_{p_j}(g_{2,j}) \neq 1$, the element γ_j is not contained in $G_2^0(\mathbb{Z}_{p_j}) G_1^0(\mathbb{Z}_{p_j})$ (again using that G_1, G_2 is generic so the expression $\gamma_j = g_{2,j} g_{1,j}$ is unique), and so there is a p -adic neighborhood $\Gamma(p_j^{n_j})$ of γ_j that is disjoint from $G_2^0(\mathbb{Z}_{p_j}) G_1^0(\mathbb{Z}_{p_j})$. A similar argument appears in [2, §8].

Consider $\Gamma' = \bigcap \Gamma(p_j^{n_j})$. For each j , by construction

$$\Gamma' \gamma_j \cap G_2^0(\mathbb{Z}_{p_j}) G_1^0(\mathbb{Z}_{p_j}) = \emptyset.$$

This implies that $\Gamma' \gamma_j \cap \Gamma_2 \Gamma_1 = \emptyset$ since $\Gamma_2 \Gamma_1 \subset G_2^0(\mathbb{Z}_{p_j}) G_1^0(\mathbb{Z}_{p_j})$. By Lemma 9 we conclude that $\Gamma' \cap \Gamma_2 \gamma_j \Gamma_1 = \emptyset$. This completes the proof. \square

3.3. Arrangements of flats and proof of Theorem 1(b). So far, we’ve shown that we can find G_1 and $\Gamma < G(\mathbb{Z})$ so that the associated cycle $[Y_1] \in H_p(Y)$ is nontrivial. Here we show that given $n \geq 1$, we can find G_1^1, \dots, G_1^n and $\Gamma < G(\mathbb{Z})$ so that the associated cycles $[Y_1^1], \dots, [Y_1^n] \in H_p(Y)$ are linearly independent. We will assume $2 \leq p \leq q$. The case $p = 1$ (i.e. X is hyperbolic space) is easy.

The argument will mostly take place in the symmetric space X . For this reason we change our notation slightly, denoting maximal flats (previously X_1) by $F \subset X$ and “hyperplanes” (previously X_2) by $H \subset X$. (Calling H an hyperplane is misleading since its codimension is p . However, H is the group preserving a hyperplane $P \subset V$, so in that sense the name is perhaps reasonable.)

Our approach is as follows.

(1) For each $n \geq 1$, we find collections $\{F_i\}_1^n$ and $\{H_i\}_1^n$ of flats and hyperplanes in X so that the intersection matrix $(F_i \cdot H_j)$ is invertible. The groups G_{F_i}, G_{H_i} (the analogues of G_1, G_2 before) will be defined over \mathbb{R} , but not necessarily defined over \mathbb{Q} .

(2) We explain why we can perturb $\{F_i\}$ (resp. $\{H_i\}$) so that they descend to compact (resp. properly embedded) submanifolds \bar{F}_i, \bar{H}_i of $Y = \Gamma \backslash X$ for some Γ . The proof of part (a) of Theorem 1 will then allow us to replace Γ by a finite-index subgroup so that the intersection matrix $(\bar{F}_i \cdot \bar{H}_j)$ is invertible. From this we conclude that the cycles $[\bar{F}_1], \dots, [\bar{F}_n]$ are linearly independent.

Before we carry out this plan, we describe X in terms of a Grassmannian, and explain when a hyperplane and a flat intersect transversely.

For much of this section, the integral structure $\Lambda \subset V$ will not play a role, so we will identify $V \cong \mathbb{R}^{p,q}$ with standard basis of orthogonal vectors

$$\mathbb{R}^{p,q} = \langle e_1, \dots, e_p, f_1, \dots, f_q \rangle$$

with $e_i \cdot e_i = 1$ and $f_j \cdot f_j = -1$.

Flats, hyperplanes, and the Grassmannian of positive p -planes. Define $\text{Gr}_p(V)$ to be the space of p -dimensional subspaces of V on which the form is positive definite, topologized as a subspace of the Grassmannian. The Lie group $\text{SO}(V)$ acts transitively on $\text{Gr}_p(V)$ with stabilizer a maximal compact, so the symmetric space $X = \text{SO}(V)/K$ is isomorphic to $\text{Gr}_p(V)$.

Given a decomposition $V = P \oplus L$, where L is a negative line, we define a “hyperplane”

$$H = \{W \in \text{Gr}_p(V) : W \subset P\}.$$

Given a decomposition $V = U_1 \oplus \dots \oplus U_p \oplus N$, where $U_i \cong \mathbb{R}^{1,1}$ and N is negative-definite, we define a flat

$$F = \{W \in \text{Gr}_p(V) : W = \bigoplus_{i=1}^p W \cap U_i\}$$

As a sanity check, one can see that $F \cong \mathbb{R}^p$ as follows. If $W \in F$, then $W \cap U_i$ is a positive line for each i . The space of positive lines in $\mathbb{R}^{1,1}$ is homeomorphic to \mathbb{R} . As one varies the choice of $W \cap U_i$ for each i , one gets a subspace of $\text{Gr}_p(V)$ homeomorphic to \mathbb{R}^p .

The following lemma characterizes when H and F intersect and when that intersection is transverse. Its proof is easy.

Lemma 15. *Let $V = P \oplus L$ and $V = U_1 \oplus \cdots \oplus U_p \oplus N$ be two decompositions as above, and let H and F be the associated hyperplane and flat.*

- (i) *If $P \cap U_i$ does not contain a positive line for some i , then $H \cap F = \emptyset$.*
- (ii) *If $P \cap U_i$ is equal to a positive line for each i , then H and F intersect transversely in a single point.*
- (iii) *If $P \cap U_i$ contains a positive line for every i , and $P \cap U_j = U_j$ for some j , then $\dim H \cap F \geq 1$.*

In the setup of Proposition 10, the condition $\lambda \notin U^\perp$ and the assumption that τ acts irreducibly on U implies that when one considers the τ -invariant decomposition $U \otimes \mathbb{R} = U_1 \oplus \cdots \oplus U_p$, then the projection of λ to each U_i is nonzero; thus $P \cap U_i$ is a proper subspace of U_i for each i . Consequently, the condition $\lambda \notin U^\perp$ in Proposition 10 corresponds to cases (i) and (ii) in Lemma 15.

A good arrangement. For each $n \geq 1$, we construct a sequence of hyperplanes $\{H_\ell\}_{\ell=1}^n$ and flats $\{F_k\}_{k=1}^n$ defined over \mathbb{R} so that the intersection matrix $(H_\ell \cdot F_k)$ is invertible.

To begin, let F_0 be the flat corresponding to

$$\mathbb{R}^{p,q} = \langle e_1, f_1 \rangle \oplus \cdots \oplus \langle e_p, f_p \rangle \oplus \langle f_{p+1}, \dots, f_q \rangle =: U_1 \oplus \cdots \oplus U_p \oplus N.$$

Next we define a hyperplane H_0 . First let $\phi: \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$ be an automorphism that acts by the identity on $\langle e_1, f_1 \rangle^\perp$, and whose restriction to $\langle e_1, f_1 \rangle$ expands $\langle e_1 + f_1 \rangle$ and contracts $\langle e_1 - f_1 \rangle$. For each $m \geq 0$, define a_m, b_m by

$$\phi^m(e_1) = a_m e_1 + b_m f_1.$$

Then

$$\phi^m(f_1) = b_m e_1 + a_m f_1.$$

We will also use the shorthand $e_1^m := \phi^m(e_1)$ and $f_1^m := \phi^m(f_1)$. By definition, $a_m^2 - b_m^2 = 1$ for each m (hence $a_m > b_m$), and $a_m, b_m \rightarrow \infty$ and $\frac{a_m}{b_m} \rightarrow 1$ as $m \rightarrow \infty$. Fix $m \gg 0$ (to be chosen later, depending on n). Let H_0 be the hyperplane defined by the decomposition $\mathbb{R}^{p,q} = P \oplus L$, where

$$L = \langle f_1^m + f_2 + \cdots + f_p \rangle,$$

and

$$P = L^\perp = \langle e_1^m, e_2, \dots, e_p, f_1^m - f_2, f_2 - f_3, \dots, f_{p-1} - f_p, f_{p+1}, \dots, f_q \rangle.$$

Define flats F_k for $k \geq 1$ by rotating F_0 as follows. Fix $-1 \ll \theta < 0$ (to be chosen later, depending on n). Let $r: \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$ be the rotation that is the identity on $\langle e_1, e_2, f_1, f_2 \rangle^\perp$ and restricts to each of $\langle e_1, e_2 \rangle$ and $\langle f_1, f_2 \rangle$ as a counter-clockwise rotation of angle θ . Note $r \in \text{SO}(p) \times \text{SO}(q)$ (note also that to define r we have used $p, q \geq 2$). For each $k \geq 1$, define $F_k = r^k(F_0)$.

Lemma 16 (Intersection pattern). *The intersection $H_0 \cap F_0$ is nonempty. For $k \geq 1$, if*

$$-(a_m + b_m) \leq \tan(k\theta) \leq -(a_m - b_m), \tag{3.2}$$

then $H_0 \cap F_k = \emptyset$.

Before we prove Lemma 16, we show that it allows us to find a desired arrangement of H_ℓ, F_k .

Observe that for each $n \geq 1$, we can choose $m \gg 0$ and $-\frac{\pi}{4} \ll \theta < 0$ so that (3.2) is true for $k = 1, \dots, n$. Thus for $k = 0, \dots, n$, we have $H_0 \cap F_k \neq \emptyset$ if and only if $k = 0$. Now define $H_\ell = r^\ell(H_0)$. If $k \geq \ell$, then

$$H_\ell \cap F_k \neq \emptyset \Leftrightarrow r^\ell(H_0) \cap r^k(F_0) \neq \emptyset \Leftrightarrow H_0 \cap r^{k-\ell}(F_0) \neq \emptyset \Leftrightarrow k = \ell.$$

Consequently, the intersection matrix $(H_\ell \cdot F_k)$ is lower triangular with 1's on the diagonal. This matrix is invertible, as desired.

Proof of Lemma 16. The first statement is easy: the intersection of H_0 and F_0 is the p -plane $W = \langle e_1^m, \dots, e_p \rangle$.

Now we prove the second statement. The flat F_k corresponds to the decomposition

$$\mathbb{R}^{p,q} = U_1^k \oplus U_2^k \oplus U_3 \oplus \dots \oplus U_p \oplus N,$$

where $U_i^k = r^k(U_i) \subset \mathbb{R}^{p,q}$ for $i = 1, 2$. Note for $i = 1, 2$ that U_i^k is spanned by $r^k(e_i), r^k(f_i)$, and

$$r^k(e_1) = \cos(k\theta)e_1 + \sin(k\theta)e_2 \quad \text{and} \quad r^k(e_2) = -\sin(k\theta)e_1 + \cos(k\theta)e_2,$$

and the same formulas hold when e_1, e_2 are replaced by f_1, f_2 .

We will compute $P \cap U_1^k$ and see under what conditions the intersection is a positive line. If $v \in P \cap U_1^k$, then we can write

$$\begin{aligned} v = & A_1(a_m e_1 + b_m f_1) + A_2 e_2 + B_1(b_m e_1 + a_m f_1 - f_2) \\ & + A_3 e_3 + \dots + A_p e_p + B_2(f_2 - f_3) + \dots + B_{p-1}(f_{p-1} - f_p) \\ & + B_{p+1} f_{p+1} + \dots + B_q f_q \end{aligned} \tag{3.3}$$

and also

$$v = X(\cos(k\theta)e_1 + \sin(k\theta)e_2) + Y(\cos(k\theta)f_1 + \sin(k\theta)f_2). \tag{3.4}$$

Since the coefficients on e_3, \dots, e_p and f_3, \dots, f_q are zero in (3.4), $A_i = 0$ for $i \geq 3$ and $B_j = 0$ for $j \geq 2$. Then setting equations (3.3) and (3.4) equal (and changing notation on the coefficients slightly),

$$\begin{aligned} & (Aa_m + Bb_m)e_1 + (Ab_m + Ba_m)f_1 + Ce_2 - Bf_2 \\ & = X \cos(k\theta)e_1 + Y \cos(k\theta)f_1 + X \sin(k\theta)e_2 + Y \sin(k\theta)f_2. \end{aligned}$$

We can simplify the corresponding system of equations to

$$X \cos(k\theta)b_m + Y \sin(k\theta)b_m^2 = Y \cos(k\theta)a_m + Y \sin(k\theta)a_m^2,$$

so that

$$X = \left[\frac{a_m}{b_m} + \frac{1}{b_m} \tan(k\theta) \right] Y.$$

We want to know if $X^2 - Y^2$ is positive or negative. Since

$$X^2 - Y^2 = \left(\left[\frac{a_m}{b_m} + \frac{\tan(k\theta)}{b_m} \right]^2 - 1 \right) Y^2,$$

this is nonpositive if and only if

$$-1 \leq \frac{a_m}{b_m} + \frac{\tan(k\theta)}{b_m} \leq 1.$$

This inequality is equivalent to (3.2). If it holds, then $H_0 \cap F_k = \emptyset$ by Lemma 15. This completes the proof. \square

Cocompact flats and rational hyperplanes. Now we explain how any flat/hyperplane in X can be perturbed to one that descends to a properly immersed submanifold of $Y = \Gamma \backslash X$. This will allow us to perturb the arrangement constructed above to an arrangement that descends to Y .

Rational hyperplanes We say a hyperplane $H \subset X$ is *rational* or *defined over* \mathbb{Q} if the line L in the corresponding decomposition $V = P \oplus L$ is defined over \mathbb{Q} (equivalently, L is spanned by an integral vector $\lambda \in \Lambda$). In this case, the subgroup of G_H that preserves the decomposition $P \oplus L$ is defined over \mathbb{Q} . Furthermore, since the $G(\mathbb{Q})$ orbit of a negative rational line is dense in the space of all negative lines in V , any hyperplane $H \subset X$ can be approximated by a rational hyperplane (one way to say this: for any neighborhood Ω of ∂H in the visual boundary ∂X , there exists a rational hyperplane H' so that $\partial H' \subset \Omega$).

Rational flats We say a flat $F \subset X$ is *rational* if its stabilizer is defined over \mathbb{Q} . In this case, it descends to a properly embedded submanifold of $Y = X/\Gamma$ by [38, Thm. D], c.f. Theorem 3. The condition that F is rational is not enough for the quotient in Y

to be compact. However, as discussed in §3.1, by [35] there exists $\tau \in G(\mathbb{Z})$ whose centralizer $C_\tau(\mathbb{R})$ is a Cartan subgroup and $C_\tau(\mathbb{R})/(\Gamma \cap C_\tau(\mathbb{R}))$ is compact. The element τ will preserve some decomposition

$$\mathbb{R}^{p,q} = U_{\tau,1} \oplus \cdots \oplus U_{\tau,p} \oplus N_\tau.$$

The $G(\mathbb{Q})$ orbit of $(U_{\tau,1}, \dots, U_{\tau,p})$ in the space of all p -tuples (U_1, \dots, U_p) of orthogonal subspaces $U_i \cong \mathbb{R}^{1,1} \hookrightarrow \mathbb{R}^{p,q}$ is dense (because $G(\mathbb{R})$ acts transitively on such tuples and $G(\mathbb{Q}) \subset G(\mathbb{R})$ is dense). Thus any flat $F \subset X$ can be approximated by a rational flat F' that is compact in the quotient X/Γ .

In summary, to prove part (b) of Theorem 1, given $n \geq 1$, we start with the arrangement $\{F_k\}_1^n$ and $\{H_\ell\}_1^n$ of flats and hyperplanes in X with the lower-triangular intersection pattern. Let \bar{F}_k and \bar{H}_ℓ be the images of these submanifolds in $Y = X/\Gamma$. First we perturb to get a new arrangement of rational flats and hyperplanes with the same intersection pattern so that each \bar{F}_k is compact and each \bar{H}_ℓ is properly immersed in Y . By replacing Γ by a finite-index subgroup, we can ensure that \bar{F}_k and \bar{H}_ℓ are oriented, embedded submanifolds (Theorem 3). Next we apply Proposition 6 to each pair (F_k, H_ℓ) (and the corresponding subgroups $G_{F_k}, G_{H_\ell} < G$) to conclude that after replacing Γ by yet another finite-index subgroup, we can ensure that every $\gamma \in I(\Gamma)$ belongs to $G_{F_k}(\mathbb{Q})G_{H_\ell}(\mathbb{Q})$. Then by Remark 7, \bar{H}_ℓ and \bar{F}_k intersect if and only if H_ℓ and F_k intersect, i.e. the intersection matrix $(\bar{H}_\ell \cdot \bar{F}_k)$ is also lower-triangular. Finally, we can pass to a further finite-index subgroup so that the diagonal entries in the intersection matrix are all positive by the argument of §3.2. Therefore, $(\bar{H}_\ell \cdot \bar{F}_k)$ is invertible, which implies that the homology classes $[\bar{F}_1], \dots, [\bar{F}_n]$ are linearly independent in $H_p(Y; \mathbb{Q})$. This proves part (b) of Theorem 1.

4. Vector bundles with arithmetic structure group

By Corollary 2, the classes produced in Theorem 1 give rise to characteristic classes of manifold bundles $W_g \rightarrow E \rightarrow B$ with fiber $W_g = \#_g(S^{2k} \times S^{2k})$. In this section we explain what these characteristic classes measure. This gives a new perspective on the Millson–Ragunathan construction. This will play a role in §5.

Before we begin, we recall the classification of lattices $\Lambda \subset \mathbb{R}^{p+q}$ with integral, unimodular, indefinite bilinear form; see e.g. [31, Ch. II, §4]. This classification is not strictly needed for what follows, but it is helpful to have these examples in mind. If the form is odd, then there exists a basis for Λ , with respect to which the form has matrix $B_{p,q}$, where

$$B_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}. \tag{4.1}$$

If the form on Λ is even, then $q = p + 8\ell$ for some $\ell \geq 0$ and Λ is isomorphic to $H^{\oplus p} \oplus (-E_8)^{\oplus \ell}$, where

$$H = \left(\mathbb{Z}^2, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right) \quad (4.2)$$

and E_8 is the unique positive-definite, even, unimodular lattice of rank 8.

4.1. Vector bundles with structure group $\mathrm{SO}(\Lambda) < \mathrm{SO}(p, q)$. Fix $1 \leq p \leq q$ and set $n = p + q$. Fix a lattice $\Lambda \cong \mathbb{Z}^n$ with an integral, unimodular bilinear form of signature (p, q) . Fix a primitive vector $\lambda \in \Lambda$ such that $\lambda \cdot \lambda < 0$. Set $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. The goal of this section is to construct a characteristic class $c_\lambda \in H^p(B\Gamma; \mathbb{Q})$ for certain $\Gamma < \mathrm{SO}(\Lambda)$ and show that c_λ is dual to a geometric cycle $[Y_2]$ as in §3.

Let $W \rightarrow B$ be a oriented, real vector bundle with rank n . Let W_b denote the fiber over $b \in B$. Assume that the structure group reduces from $\mathrm{GL}^+(V)$ to $\mathrm{SO}(V)$. This is equivalent to the existence of a fiberwise bilinear form $\beta = \{\beta_b\}_{b \in B}$ of signature (p, q) . We can always reduce the structure group from $\mathrm{SO}(V)$ to its maximal compact subgroup

$$K \cong S(O(p) \times O(q))$$

(because they are homotopy equivalent and so are their classifying spaces). Such a reduction defines a decomposition $W \cong U \oplus U^\perp$, where $U = \bigcup_{b \in B} U_b$ is a rank- p subbundle and $\beta_b: U_b \times U_b \rightarrow \mathbb{R}$ is positive definite for each b . Conversely, any positive rank- p subbundle $U \subset W$ defines a reduction of the structure group to $S(O(p) \times O(q))$. The structure group of $W \rightarrow B$ reduces to $\mathrm{SO}(\Lambda)$ if and only if there exists a fiberwise lattice

$$\Lambda = \bigcup_{b \in B} \Lambda_b \subset W,$$

where Λ_b (with its form β_b) is isometric to Λ for each $b \in B$.

Definition. Fix Λ, V and $\lambda \in \Lambda$ and a bundle $W \rightarrow B$ with structure group $\mathrm{SO}(\Lambda)$ as above. We say that a positive rank- p subbundle $U \subset W$ is *orthogonal to λ at $b \in B$* if there exists an isometry $\phi: \Lambda \rightarrow \Lambda_b$ so that $U_b \subset \phi(\lambda)^\perp$. If $U \subset W$ is not orthogonal to λ at any $b \in B$, then we say U is *nowhere orthogonal to λ* .

The characteristic class we define will be an obstruction to finding $U \subset W$ that is nowhere orthogonal to λ . We translate the problem of finding U to a problem about finding a section of an associated bundle.

Set $\pi = \pi_1(B)$. Let $\rho: \pi \rightarrow \mathrm{SO}(\Lambda)$ be the monodromy of $W \rightarrow B$. The symmetric space $X = K \backslash \mathrm{SO}(V)$ is homeomorphic to the Grassmannian

$$\mathrm{Gr}_p(V) = \{V' \subset V : V' \text{ is positive definite and } \dim V' = p\}$$

because $\mathrm{SO}(V)$ acts transitively on $\mathrm{Gr}_p(V)$ and the stabilizer of a point is isomorphic to a maximal compact subgroup $K \subset \mathrm{SO}(V)$.

The group π acts on $X \cong \text{Gr}_p(V)$ via the monodromy ρ . For a space Z with a π -action, we denote the Borel construction

$$Z // \pi := \frac{\tilde{B} \times Z}{\pi},$$

where π acts on the universal cover \tilde{B} by deck transformations and π acts on Z by the given π -action, and the quotient is by the diagonal action. For any such Z , there is a fibration $Z // \pi \rightarrow * // \pi = B$ with fiber Z .

Observe that for $W \rightarrow B$ with monodromy $\rho: \pi \rightarrow \text{SO}(\Lambda)$, a section of the associated bundle $X // \pi \rightarrow B$ is equivalent to a positive rank- p subbundle $U \subset W$.

Let $H_\lambda = \{V' \in \text{Gr}_p(V) : V' \subset \lambda^\perp\} \subset X$. This is the sub-symmetric space corresponding to the subgroup $\text{SO}(\lambda^\perp) < \text{SO}(V)$. The codimension of H_λ in X is p .

By [38, Thm. D], there exists a torsion-free, finite-index subgroup $\Gamma_\lambda < \text{SO}(\Lambda)$ so that the Γ_λ -orbit of H is embedded and admits a Γ_λ -invariant orientation. (The group Γ_λ is not uniquely defined by these properties, e.g. for every prime ℓ , there exists $m > 0$ so that the congruence subgroup $\ker [\text{SO}(\Lambda) \rightarrow \text{SO}(\Lambda/\ell^m \Lambda)]$ satisfies these properties. The construction below works for any choice of Γ_λ .)

Fix a finite index subgroup $\Gamma < \Gamma_\lambda$, and let $H_{\lambda,\Gamma}$ be the Γ -orbit of H_λ in X . By replacing B by a finite cover, we can ensure that $\rho(\pi) < \Gamma$. Set $X_0 = X \setminus H_{\lambda,\Gamma}$ and consider the bundle $X_0 // \pi \rightarrow B$. If $W \rightarrow B$ has a positive rank- p subbundle $U \subset W$ that is nowhere orthogonal to λ , then $X_0 // \pi \rightarrow B$ has a continuous section. Now we can use obstruction theory to extract a characteristic class from this situation. For this, we need to know the first nontrivial homotopy group of X_0 .

Lemma 17. *Fix $k \geq 0$. If $k \leq p - 2$, then $\pi_k(X_0) = 0$. Furthermore,*

$$\pi_{p-1}(X_0) \cong \bigoplus_{\pi_0(H_{\lambda,\Gamma})} \mathbb{Z}.$$

Proof. First assume $k \leq p - 2$. We show any map $S^k \rightarrow X_0$ is homotopically trivial. Since $X \cong \mathbb{R}^{pq}$ is contractible, we obtain a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{i} & X_0 \\ \downarrow & & \downarrow \\ D^{k+1} & \xrightarrow{j} & X \end{array} \tag{4.3}$$

Without loss of generality we may assume that i and j are smooth and j is transverse to $H_{\lambda,\Gamma}$. Since $k + 1 \leq p - 1$ and the codimension of $H_{\lambda,\Gamma}$ is p , if D is transverse to $H_{\lambda,\Gamma}$, then $D \cap H_{\lambda,\Gamma} = \emptyset$, which shows i is homotopically trivial in X_0 .

By the Hurewicz theorem,

$$\pi_{p-1}(X_0) \cong H_{p-1}(X_0).$$

Define a homomorphism

$$\phi: \pi_{p-1}(X_0) \cong H_{p-1}(X_0) \rightarrow \bigoplus_{\pi_0(H_{\lambda,\Gamma})} \mathbb{Z}$$

as follows. Choose an orientation on each component of $H_{\lambda,\Gamma}$. Given $i: S^{p-1} \rightarrow X_0$, extend to $D^p \rightarrow X$ transverse to $H_{\lambda,\Gamma}$, and compute the algebraic intersection of D^p with each component of $H_{\lambda,\Gamma}$.

The map ϕ is obviously surjective: for each component of $H_{\lambda,\Gamma}$, one can choose a $(p-1)$ -sphere in its link, and the image of these generate $\bigoplus_{\pi_0(H_{\lambda,\Gamma})} \mathbb{Z}$. For injectivity, it is well known that if D, H are oriented submanifolds of an oriented manifold X that intersect transversely in a finite collection of points and their algebraic intersection number is 0, then D can be replaced by a homologous submanifold D' with $\partial D = \partial D'$ so that $D' \cap H = \emptyset$. This shows that if $[S^{p-1} \rightarrow X_0]$ is in the kernel of ϕ , then $[S^{p-1} \rightarrow X_0] = 0$ in $H_{p-1}(X_0)$. \square

Applying obstruction theory (see e.g. [11, Ch. 7]), we can try to build a section of $X_0 // \pi \rightarrow B$. Assume that B is a CW complex. We start by choosing a section over the 0-skeleton of B and work our way up inductively defining a section on the k -skeleton for $k \leq p-1$ using the fact that $\pi_{k-1}(X_0) = 0$ for $k \leq p-1$. Once we reach the p -skeleton we meet the first measurable obstruction, which takes the form of a cocycle

$$C_{\lambda,\Gamma}(W) \in H^p(B; \pi_{p-1}(X_0)).$$

If $C_{\lambda,\Gamma}(W) \neq 0$, then $X_0 // \pi \rightarrow B$ has no continuous section, and so $W \rightarrow B$ does not have a positive rank- p subbundle U that is nowhere orthogonal to λ . This is useful, but we are interested in a less-refined, \mathbb{Z} -valued obstruction.

Since $H_{\lambda,\Gamma}$ has a Γ -invariant orientation, there is a preferred generator of each coordinate of $\bigoplus_{\pi_0(H_{\lambda,\Gamma})} \mathbb{Z}$. We use this to define an augmentation map

$$\bigoplus_{\pi_0(H_{\lambda,\Gamma})} \mathbb{Z} \rightarrow \mathbb{Z}.$$

The augmentation map induces a map

$$H^p(B; \pi_{p-1}(X_0)) \rightarrow H^p(B; \mathbb{Z}),$$

which sends $C_{\lambda,\Gamma}(W)$ to a class $c_{\lambda,\Gamma}(W) \in H^p(B; \mathbb{Z})$.

Proposition 18. *Fix $\Lambda, V, \lambda \in \Lambda$, and $\Gamma_\lambda < \text{SO}(\Lambda)$ as above. Let B be a CW complex and let $W \rightarrow B$ be a vector bundle with structure group $\Gamma < \Gamma_\lambda$. If $c_{\lambda,\Gamma}(W) \neq 0$ in $H^p(B; \mathbb{Z})$, then $W \rightarrow B$ has no positive, rank- p subbundle $U \subset W$ that is nowhere orthogonal to λ . Equivalently, for every positive, rank- p subbundle $U \subset W$, there exists $b \in B$ so that U is orthogonal to λ at b .*

Remark. If $C_{\lambda, \Gamma} = 0$, then there exists a section of $X_0 // \pi \rightarrow B$, but this does not ensure that there exists $U \subset W$ that is nowhere orthogonal to λ since $X_0 = X \setminus H_{\lambda, \Gamma}$ is not the complement of the full orbit of H_λ under $SO(\Lambda)$. Note also that if $\Gamma' \subset \Gamma$, then $H_{\lambda, \Gamma'} \subset H_{\lambda, \Gamma}$, so it is possible that $C_{\lambda, \Gamma} \neq 0$ but $C_{\lambda, \Gamma'} = 0$ (and similarly for $c_{\lambda, \Gamma}$ and $c_{\lambda, \Gamma'}$).

If B is a closed, oriented p -manifold, then we can evaluate $c_{\lambda, \Gamma}(W) \in H^p(B; \mathbb{Z})$ on the fundamental class to get an integer $\langle c_{\lambda, \Gamma}(W), [B] \rangle \in \mathbb{Z}$, which is computed as follows. We have a diagram

$$\begin{array}{ccc}
 X & \longrightarrow & X // \pi \xrightarrow{p} X / \Gamma \\
 & & \downarrow \uparrow u \\
 & & B
 \end{array} \tag{4.4}$$

Here u is a section corresponding to a positive, rank- p subbundle $U \subset W$, and the map p is the composition

$$X // \pi = \frac{\tilde{B} \times X}{\pi} \rightarrow X / \rho(\pi) \rightarrow X / \Gamma$$

(the first map collapses \tilde{B} to a point). Let $\bar{H}_{\lambda, \Gamma}$ be the image of H_λ in X / Γ . By our choice of Γ_λ and the assumption $\Gamma < \Gamma_\lambda$, the inclusion $\bar{H}_{\lambda, \Gamma} \hookrightarrow X / \Gamma$ is a proper embedding, c.f. [38, Thm. D]. Now tracing through the definitions, one finds that $\langle c_{\lambda, \Gamma}(W), [B] \rangle$ is equal to the algebraic intersection number of $p \circ u(B)$ with $\bar{H}_{\lambda, \Gamma}$ in X / Γ .

Applying the above construction to the universal bundle over $B\Gamma$, we see that

$$c_{\lambda, \Gamma} \in H^p(B\Gamma) \cong H^p(X / \Gamma)$$

is dual to the cycle $\bar{H}_{\lambda, \Gamma}$, which is a locally symmetric space for a nonuniform lattice in $SO(\lambda^\perp) \cong SO(p, q - 1)$. In §3, we showed that there exists $\Gamma < \Gamma_\lambda$ so that $[\bar{H}_{\lambda, \Gamma}] \in H_{pq-p}^{cl}(X / \Gamma)$ is nonzero. Then $c_{\lambda, \Gamma}$ is also nontrivial.

4.2. Vector bundles with structure group $SL_n(\mathbb{Z})$. The construction of the previous section can be repeated in other situations. Here we remark on a version for vector bundles with structure group $SL_n(\mathbb{Z})$. We will use this in §5 to give an application similar to Corollary 2 to odd-dimensional manifolds.

Fix the standard lattice $\mathbb{Z}^n < \mathbb{R}^n$. Let $\delta = (P, L)$ denote be a pair of subspaces of \mathbb{R}^n defined over \mathbb{Q} such that $\mathbb{R}^n = P \oplus L$ and $\dim L = 1$. For every such δ , we will associate a finite index subgroup $\Gamma_\delta < SL_n(\mathbb{Z})$ and for every $\Gamma < \Gamma_\delta$ we will define a characteristic class $c_{\delta, \Gamma} \in H^{n-1}(B\Gamma; \mathbb{Z})$ for real vector bundles $W \rightarrow B$ with structure group in Γ .

Suppose $W \rightarrow B$ is a real oriented vector bundle of rank n . The structure group reduces from $GL_n^+(\mathbb{R})$ to $SL_n(\mathbb{Z})$ if and only if W admits a fiberwise lattice Λ .

A reduction of the structure group from $\mathrm{GL}_n^+(\mathbb{R})$ to its maximal compact $\mathrm{SO}(n)$ corresponds to a fiberwise inner product β on W .

Definition. Fix $\delta = (P, L)$ and $W \rightarrow B$ and $\mathbf{A} \subset W$ as above. For a fiberwise inner product β , we say that a (P, L) is β -orthogonal at $b \in B$ if there exists an isomorphism $\phi: (\mathbb{R}^n, \mathbb{Z}^n) \rightarrow (W_b, \Lambda_b)$ so that $\phi(P)$ and $\phi(L)$ are orthogonal with respect to β_b . If (P, L) is not β -orthogonal at any $b \in B$, we say (P, L) is *nowhere β -orthogonal*.

We can translate the problem of finding an inner product β so that (P, L) is nowhere β -orthogonal to a problem of finding a section of an associated bundle. Let $X = \mathrm{SO}(n) \backslash \mathrm{SL}_n(\mathbb{R})$. This symmetric space can be identified with the space of unit volume inner products on \mathbb{R}^n . There is a bijective correspondence between fiberwise inner products β on $W \rightarrow B$ and sections of $X // \pi \rightarrow B$, where $\pi = \pi_1(B)$ acts on X via the monodromy $\rho: \pi \rightarrow \mathrm{SL}_n(\mathbb{Z})$.

Consider the submanifold

$$H_\delta = \{\text{inner products such that } \mathbb{R}^n = P \oplus L \text{ is orthogonal}\} \subset X,$$

which is a sub-symmetric space for $\mathrm{SL}_{n-1}(\mathbb{R}) \times \mathbb{R}$. By [38, Thm. D], we can find a torsion-free subgroup $\Gamma_\delta < \mathrm{SL}_n(\mathbb{Z})$ so that the Γ_δ -orbit of H_δ in X is embedded and has a Γ_δ -invariant orientation. Fix a finite-index subgroup $\Gamma < \Gamma_\delta$. Denote the Γ orbit of H_δ in X by $H_{\delta, \Gamma}$, and set $X_0 = X \setminus H_{\delta, \Gamma}$. We replace B with a finite cover so that the monodromy $\rho: \pi \rightarrow \mathrm{SL}_n(\mathbb{Z})$ factors through Γ .

If $W \rightarrow B$ admits an inner product β so that (P, L) is nowhere β -orthogonal, then $X_0 // \pi \rightarrow B$ admits a continuous section. Similar to Lemma 17, we compute

$$\pi_k(X_0) = 0$$

for $k \leq n - 3$ and

$$\pi_{n-2}(X_0) \cong \bigoplus_{\pi_0(H_{\delta, \Gamma})} \mathbb{Z}.$$

Then there is an obstruction class $C_{\delta, \Gamma}(W) \in H^{n-1}(B; \pi_{n-2}(X_0))$, which maps to a class $c_{\delta, \Gamma}(W) \in H^{n-1}(B; \mathbb{Z})$ under the map induced by the augmentation

$$\bigoplus_{\pi_0(H_{\delta, \Gamma})} \mathbb{Z} \rightarrow \mathbb{Z}.$$

We summarize the above discussion with the following proposition.

Proposition 19. Fix $\delta = (P, L)$ and $\Gamma_\delta < \mathrm{SL}_n(\mathbb{Z})$ as above. Let B be a CW complex and let $W \rightarrow B$ be a vector bundle with structure group $\Gamma < \Gamma_\delta$. If $c_{\delta, \Gamma}(W) \neq 0$ in $H^{n-1}(B; \mathbb{Z})$, then $W \rightarrow B$ does not admit an inner product β so that (P, L) is nowhere β -orthogonal. Equivalently, for every inner product β on W there exists $b \in B$ so that (P, L) is β -orthogonal at b .

The class

$$c_{\delta, \Gamma} \in H^{n-1}(B\Gamma) \cong H^{n-1}(X/\Gamma) \cong H_{(n^2-n)/2}^{\text{cl}}(X/\Gamma)$$

is dual to the cycle

$$[\bar{H}_{\delta, \Gamma}] \in H_{(n^2-n)/2}^{\text{cl}}(X/\Gamma)$$

represented by the image of H_{δ} in X/Γ . Compare with the discussion following (4.4). By a theorem of Avramidi–Nguyen-Phan [2] for a subgroup $\Gamma < \Gamma_{\delta}$ of sufficiently large index, the homology class $[\bar{H}_{\delta, \Gamma}] \in H_{(n^2-n)/2}^{\text{cl}}(X/\Gamma; \mathbb{Q})$ is nontrivial.

5. Applications to manifold bundles

In this section and the appendix, we give applications of Theorem 1.

5.1. 4-manifolds, K3 surfaces bundles, and the global Torelli theorem. Let M be a closed oriented 4-manifold. As in the introduction, we use Λ_M to denote $H_2(M; \mathbb{Z})/\text{torsion}$ with its intersection form. Assume that Λ_M is indefinite, and let (p, q) be the signature. Up to switching the orientation, we may assume $p \leq q$. By Theorem 1, when p is odd, there exists a finite-index subgroup $\Gamma < \text{SO}(\Lambda_M)$ so that $H_p(\Gamma; \mathbb{Q}) \neq 0$.

Question 20. Does the image of $H_p(B \text{Diff}^{\Gamma}(M); \mathbb{Q}) \rightarrow H_p(B\Gamma; \mathbb{Q})$ intersect the subspace spanned by flat cycles nontrivially?

One could ask a similar question for homeomorphisms or homotopy automorphisms. There does not seem to be a good reason for the answer to Question 20 to be “Yes”, other than the evidence provided by Corollary 26 and Theorem 29 below.

Example. Fix $1 \leq p \leq q$ and let

$$M = M_{p,q} := (\#_p \mathbb{C}P^2) \# (\#_q \overline{\mathbb{C}P^2}).$$

Then the form on Λ_M has matrix $B_{p,q}$ (defined in (4.1)) and $\alpha: \text{Diff}(M) \rightarrow \text{O}(\Lambda_M)$ is surjective if $p + q \leq 8$ or $p \geq 2$. This follows from [44, Thm. 2]; which shows that α is surjective when $p + q \leq 8$ or $M = N \# (S^2 \times S^2)$ is simply connected and Q_N is indefinite. Since

$$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2} \cong (S^2 \times S^2) \# \overline{\mathbb{C}P^2}$$

(see e.g. [39, pp. 124, 151]), the hypotheses of Wall’s theorem are true for $M_{p,q}$ when $p \geq 2$. This gives many concrete examples to study Question 20.

One does not necessarily need to restrict to flat cycles. In particular, for $M_{1,q}$, the group $\text{O}(\Lambda_M)$ is a nonuniform lattice in $\text{O}(1, q)$. One easy source of homology of finite-index subgroups $\Gamma \subset \text{O}(\Lambda_M)$ are classes

$$[T] \in H_{q-1}(\mathbb{H}^q/\Gamma; \mathbb{Q}) \cong H_{q-1}(\Gamma; \mathbb{Q})$$

represented by a cusp cross-section $T \subset \mathbb{H}^q/\Gamma$. When \mathbb{H}^q/Γ has at least 2 cusps, these classes are always nontrivial. Note that T is finitely covered by a flat torus, but in contrast to flat cycles $T \hookrightarrow \mathbb{H}^q/\Gamma$ is not an isometric embedding. Even so, it would be interesting to understand the analogue of Question 20 for these homology classes. Note that this question is most reasonable for $q < 10$ since the image of α is infinite index in $O(\Lambda_M)$ when $q \geq 10$ [15].

In the remainder of this section we study Question 20 in the case M is a K3 surface (i.e. a smooth 4-manifold diffeomorphic to a K3 surface). Here the form on Λ_M has matrix $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ (the notation is explained in (4.2)). Then $SO(\Lambda_M)$ is a lattice in $SO(3, 19)$. We will be interested in the maps

$$H_*(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \xrightarrow{\alpha_1} H_*(B\pi_0 \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \xrightarrow{\alpha_2} H_*(B\Gamma; \mathbb{Q}), \quad (5.1)$$

for $\Gamma < SO(\Lambda_M)$. By Theorem 1, we can find Γ and $z \neq 0 \in H_3(B\Gamma; \mathbb{Q})$. We will study whether or not z is in the image of α_2 and $\alpha_1 \circ \alpha_2$.

Theorem 21. *Let M be a smooth oriented 4-manifold diffeomorphic to a K3 surface. There exists a finite-index subgroup $\Gamma'_M < SO(\Lambda_M)$ so that for each finite-index subgroup $\Gamma < \Gamma'_M$ and for each $i \geq 0$, the map*

$$\alpha_2: H_i(B\pi_0 \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_i(B\Gamma; \mathbb{Q})$$

is surjective.

Consequently, each flat cycle $z \neq 0 \in H_3(B\Gamma; \mathbb{Q})$ is in the image of α_2 . Theorem 21 is a corollary of the global Torelli theorem ([26] and [5, §12.K]) and can be deduced from the discussion in [19].

Proof of Theorem 21. Let Γ_M be the image of $\operatorname{Diff}(M) \rightarrow O(\Lambda_M)$. It is known [28] that Γ_M is finite index in $O(\Lambda_M)$. To prove the theorem, it suffices to show that the surjection $\operatorname{Diff}(M) \rightarrow \Gamma_M$ splits over a finite-index subgroup Γ'_M .

Let $\operatorname{Ein}(M)$ denote the space of unit-volume Einstein metrics on M , topologized as a subspace of all Riemannian metrics on M . One defines the *homotopy moduli space*

$$\mathcal{M}_{\operatorname{Ein}}(M) := \frac{\operatorname{Ein}(M) \times E \operatorname{Diff}(M)}{\operatorname{Diff}(M)},$$

where $E \operatorname{Diff}(M)$ is the total space of the universal principal $\operatorname{Diff}(M)$ bundle over $B \operatorname{Diff}(M)$. There is a composition of maps

$$\phi: \mathcal{M}_{\operatorname{Ein}}(M) \rightarrow B \operatorname{Diff}(M) \rightarrow B\Gamma_M. \quad (5.2)$$

As explained in [19, §4-5], the group $\pi_1(\mathcal{M}_{\operatorname{Ein}}(M))$ is isomorphic to a finite-index subgroup $\Gamma'_M < \Gamma_M$, and ϕ induces the inclusion on π_1 . \square

The last assertion in the proof will be further explained below (as part of the proof of Proposition 22).

Remark. Since α_2 is surjective, any homology of a lattice $\Gamma < \text{SO}(3, 19)$ in the *stable range* is also in the image of α_2 . Switching to cohomology, the stable cohomology can be described as the cohomology that is pulled back along the map

$$f: B\Gamma \rightarrow B\text{SO}(3, 19) \sim BS(O(3) \times O(19)) \rightarrow B\text{SO}(3).$$

Compare with [19, §3]. Recall

$$H^*(B\text{SO}(3); \mathbb{Q}) \cong \mathbb{Q}[p_1],$$

where $p_1 \in H^4$ is the first Pontryagin class. According to the ranges in [8], f induces an $H_i(-; \mathbb{Q})$ -isomorphism for $i < 1$. Unfortunately, this does not provide nontrivial elements of $H^*(B\Gamma_M; \mathbb{Q})$. (This is incorrectly quoted in [19, Prop. 3.6].)

Theorem 21 reduces Question 20 to studying the image of α_1 . The author does not know of a single nontrivial class in the image of this map (or a single class that is not in the image of this map). In studying α_1 , we will focus on a particular type of flat cycle z .

Set $V = \Lambda_M \otimes \mathbb{R} \cong H_2(M; \mathbb{R})$, and let $X = \text{SO}(V)/K$ be the symmetric space for $G = \text{SO}(V)$. As discussed in §4, there is a homeomorphism $X \cong \text{Gr}_3(V)$. A vector $\delta \in \Lambda$ is called a *root vector* if $\delta \cdot \delta = -2$. As in §4, consider

$$H_\delta = \{V' \in \text{Gr}_3(V) : V' \subset \delta^\perp\} \subset X.$$

Fix a root vector δ , and choose a rational flat $F \subset X$ that intersects H_δ transversely. (This can be done using the arguments of §3.3.) By the construction of Theorem 1, there exists $\Gamma < \Gamma'_M$ so that F and H_δ descend to homology cycles in $Y = \Gamma \backslash X$ that pair nontrivially. In particular, we have a nonzero class $z_0 \in H_3(B\Gamma; \mathbb{Q})$.

Now we discuss whether or not $z_0 \in \text{Im}(\alpha_2 \circ \alpha_1)$. One approach to this question is to consider the map (5.2) from the proof of Theorem 21. For each finite-index subgroup $\Gamma < \Gamma'_M$, define

$$\mathcal{M}_{\text{Ein}}^\Gamma(M) = \frac{T_{\text{Ein}}^0(M) \times E\Gamma}{\Gamma},$$

where $\mathcal{T}_{\text{Ein}}^0(M)$ is one of the two path components of $\mathcal{T}_{\text{Ein}}^0(M)$ (these components are preserved by Γ'_M). If z_0 is in the image of

$$\phi_*: H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M)) \rightarrow H_3(B\Gamma_M),$$

then $z_0 \in \text{Im}(\alpha_2 \circ \alpha_1)$. Unfortunately, the following proposition shows that this approach does not work. Nevertheless, we have an interesting Corollary 23.

Proposition 22. *Let M be a K3 surface. Fix $\Gamma < \Gamma'_M$ and $z_0 \in H_3(B\Gamma; \mathbb{Q})$ as above. The class z_0 is not in the image of $\phi_*: H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$.*

Corollary 23. *If $z_0 \in H_3(B\Gamma; \mathbb{Q})$ is in the image of $H_3(B\text{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$, then there exists a K3 bundle over a 3-manifold that does not admit any fiberwise Einstein metric.*

Of course it may be the case that z_0 is *not* in the image of $H_3(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$, in which case the corollary is vacuously true. In this situation, there is a different interesting corollary.

Corollary 24. *Let M be a K3 surface. If there exists any flat cycle $z \in H_3(B\Gamma; \mathbb{Q})$ is not in the image of $H_3(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$, then the surjection $\operatorname{Diff}(M) \rightarrow \pi_0 \operatorname{Diff}(M)$ is not split.*

Proof of Corollary 24. If a splitting exists, then

$$H_*(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_*(B\pi_0 \operatorname{Diff}^\Gamma(M); \mathbb{Q})$$

would be surjective for every $\Gamma < \Gamma_M$. Combining this with by Theorem 21, then

$$H_*(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_*(B\Gamma; \mathbb{Q})$$

is also surjective for every $\Gamma < \Gamma_M$. This contradicts the assumption that some flat cycle is not in the image of $H_3(B \operatorname{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$. \square

Proof of Corollary 23. Suppose there is a class $w \in H_3(B \operatorname{Diff}^\Gamma(M); \mathbb{Q})$ whose image in $H_3(B\Gamma; \mathbb{Q})$ is z_0 . Up to scaling, we can represent w by a map of a manifold

$$h: B^3 \rightarrow B \operatorname{Diff}^\Gamma(M).$$

The pullback of the universal bundle by h is a K3 bundle $M \rightarrow E \rightarrow B$. We claim it has no fiberwise Einstein metric.

The homotopy moduli space $\mathcal{M}_{\operatorname{Ein}}(M)$ is a classifying space for K3 bundles with a fiberwise Einstein metric, so $E \rightarrow B$ admits a fiberwise Einstein metric if and only if h lifts to a map

$$\tilde{h}: B \rightarrow \mathcal{M}_{\operatorname{Ein}}^\Gamma(M).$$

No such lift can exist by Proposition 22. Thus $E \rightarrow B$ has no fiberwise Einstein metric. \square

As remarked in Theorem 21, Giansiracusa [19, §4-5] proves that

$$\pi_1(\mathcal{M}_{\operatorname{Ein}}(M)) \cong \Gamma'_M.$$

We begin by explaining the proof of this fact, since it will be used to prove Proposition 22. For this, we give a fuller description of the topology of $\mathcal{M}_{\operatorname{Ein}}(M)$, which is illuminated by the global Torelli theorem. For details see [26], [5, §12.K], and [19, §4-5]. In [19, §4.2], Giansiracusa shows that

$$\mathcal{M}_{\operatorname{Ein}}(M) \cong \frac{\mathcal{T}_{\operatorname{Ein}}(M) \times E\Gamma_M}{\Gamma_M}, \quad (5.3)$$

where $\mathcal{T}_{\operatorname{Ein}}(M)$ is the *Teichmüller space*. By definition $\mathcal{T}_{\operatorname{Ein}}(M)$ is the quotient

$$\operatorname{Ein}(M)/\operatorname{Diff}_1(M),$$

where

$$\text{Diff}_1(M) = \ker [\text{Diff}(M) \rightarrow \Gamma_M].$$

Here the action of Γ_M on $\mathcal{T}_{\text{Ein}}(M)$ is induced from the action of $\text{Diff}(M)$ on $\text{Ein}(M)$ (by pulling back metrics). The global Torelli theorem determines $\pi_0(\mathcal{T}_{\text{Ein}}(M))$ and the topology of each component:

- The space $\mathcal{T}_{\text{Ein}}(M)$ has two homeomorphic components

$$\mathcal{T}_{\text{Ein}}(M) \cong \mathcal{T}_{\text{Ein}}^0(M) \sqcup \mathcal{T}_{\text{Ein}}^0(M),$$

and they are permuted by the action of Γ_M .

- Let $\Gamma'_M < \Gamma_M$ be the index-2 subgroup that preserves the components of $\mathcal{T}_{\text{Ein}}(M)$. There is a Γ'_M -equivariant homeomorphism between $\mathcal{T}_{\text{Ein}}^0(M)$ and a dense subspace of $X = \text{SO}(V)/K$:

$$\mathcal{T}_{\text{Ein}}^0(M) \cong X \setminus \bigcup_{\delta \in \Delta} H_\delta, \tag{5.4}$$

where $\Delta \subset \Lambda$ is the set of roots.

Using this description of $\mathcal{T}_{\text{Ein}}(M)$, it follows that $\pi_1(\mathcal{M}_{\text{Ein}}(M)) \cong \Gamma'_M$ by the long exact sequence in homotopy associated to (5.3) together with the fact that the subspaces $H_\delta \subset X$ have codimension-3, so $\mathcal{T}_{\text{Ein}}(M)$ is simply connected.

Proof of Proposition 22. We have fixed a particular torsion-free subgroup $\Gamma < \Gamma'_M$ and a flat cycle $z_0 \in H_3(B\Gamma; \mathbb{Q})$, and we wish to show z_0 is not in the image of $H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M); \mathbb{Q}) \rightarrow H_3(B\Gamma; \mathbb{Q})$. Recall that z_0 has the special property that it pairs nontrivially with the image of a root hyperplane H_δ in $Y = \Gamma \backslash X$.

Suppose for a contradiction that there exists

$$w \in H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M); \mathbb{Q})$$

whose image in $H_3(B\Gamma; \mathbb{Q})$ is z_0 . Since Γ is torsion free,

$$\mathcal{M}_{\text{Ein}}^\Gamma(M) \cong \frac{T_{\text{Ein}}^0(M) \times X}{\Gamma}.$$

There is a diagram that commutes up to homotopy:

$$\begin{array}{ccccc} \mathcal{M}_{\text{Ein}}^\Gamma(M) & \longrightarrow & B \text{Diff}^\Gamma(M) & \longrightarrow & B\Gamma \\ f_1 \downarrow & & & & \parallel \\ \Gamma \backslash \mathcal{T}_{\text{Ein}}^0(M) & \xrightarrow{f_2} & \Gamma \backslash X & \xrightarrow{\sim} & B\Gamma \end{array}$$

The map f_1 is a homotopy equivalence because Γ acts freely on $\mathcal{T}_{\text{Ein}}(M)$ so f_1 is a fibration with contractible fiber $\cong X$. The map f_2 is the inclusion induced by (5.4).

The diagram commutes up to homotopy because the two compositions induce the same map on π_1 and $B\Gamma$ is Eilenberg–Maclane space.

Let \bar{H}_δ be the image of H_δ in $\Gamma \backslash X$, and let \bar{F} be a totally geodesic submanifold representing the flat cycle z_0 . By our choice of H_δ and F , the algebraic intersection $\bar{F} \cdot \bar{H}_\delta$ is nonzero. On the other hand, the existence of w implies, by the diagram above, that there is a cycle

$$Z \rightarrow \Gamma \backslash \mathcal{T}_{\text{Ein}}^0(M) \subset \Gamma \backslash X$$

that is homologous to \bar{F} . Since the $Z \rightarrow \Gamma \backslash X$ factors through $\Gamma \backslash \mathcal{T}_{\text{Ein}}^0(M)$, the image of Z is disjoint from \bar{H}_δ , which implies that

$$\bar{F} \cdot \bar{H}_\delta = Z \cdot \bar{H}_\delta = 0.$$

This is a contradiction, so the class w does not exist. \square

Remark 25. In the Teichmüller space $\mathcal{T}_{\text{Ein}}^0(M) \subset X$, as one approaches one of the subsets $H_\delta \subset X$, topologically there is an embedded sphere $f: S^2 \hookrightarrow M$ with $f_*[S^2] = \delta$ that is being collapsed to a point [1]. The flat cycle $\bar{F} \subset \Gamma \backslash X$ in the proof of Proposition 22 only lifts to $\Gamma \backslash \mathcal{T}_{\text{Ein}}^0(M)$ after finitely many points are removed. From this, one obtains a K3-surface bundle over a 3-torus with finitely many punctures (note \bar{F} is finitely covered by \mathbb{T}^3). One cannot extend the bundle over the punctures without introducing singularities. The natural object that exists over the torus with the punctures filled is a “singular” K3 bundle, i.e. it is a fiber bundle away from finitely many points in the base, and at each point in this finite collection, the fiber is the space obtained from a K3 surface by collapsing some embedded 2-sphere (with self-intersection -2) a point.

5.2. 2-dimensional cycles and the Mather–Thurston theorem. To end this section, we mention another example/application of our ideas.

Corollary 26. *Let M be smooth manifold. Suppose that $\pi_0 \text{Diff}(M)$ is commensurable with $\text{SL}_3(\mathbb{Z})$. Then for each $n \geq 1$ there exists a finite-index subgroup $\Gamma < \text{SL}_3(\mathbb{Z})$ so that*

$$\dim H_2(B \text{Diff}^\Gamma(M); \mathbb{Q}) \geq n.$$

In fact, as we will see in the proof, Corollary 26 remains true if $B \text{Diff}^\Gamma(M)$ is replaced by the classifying space $B \text{Diff}^\Gamma(M)^\delta$ of $\text{Diff}^\Gamma(M)$ with the discrete topology. This is a stronger conclusion than in Corollary 2.

For manifolds satisfying the hypothesis of Corollary 26 one could consider

$$M = \#_3(S^k \times S^{k+1}) \quad \text{for } k \geq 4,$$

c.f. [40, Thm. 13.3].

Corollary 26 follows by applying the following two theorems. The first theorem, due to Avramidi–Nguyen-Phan [2], is analogous to Theorem 1. For a prime p and $\ell \geq 1$, denote the congruence subgroup

$$\ker [\mathrm{SL}_d(\mathbb{Z}) \rightarrow \mathrm{SL}_d(\mathbb{Z}/p^\ell\mathbb{Z})]$$

by $\Gamma_d(p^\ell)$.

Theorem 27 (Avramidi–Nguyen-Phan). *Given a prime p and an integer $n \geq 1$, there exists $\ell_0 > 0$ so that if $\ell > \ell_0$, then $\dim H_{d-1}(\Gamma_d(p^\ell); \mathbb{Q}) \geq n$.*

Like in Theorem 1, their homology comes from maximal flats in the associated symmetric space. We will focus on the case $d = 3$, which is special because we can use the following theorem.

Theorem 28 (Mather, Thurston [41]). *Let M be a smooth closed manifold. The group $\mathrm{Diff}_0(M)$ of diffeomorphisms isotopic to the identity is a simple group.*

Proof of Corollary 26. Denote $\mathrm{Mod}(M) := \pi_0 \mathrm{Diff}(M)$. The exact sequence

$$1 \rightarrow \mathrm{Diff}_0(M) \rightarrow \mathrm{Diff}(M) \rightarrow \mathrm{Mod}(M) \rightarrow 1$$

gives a 5-term exact sequence in group homology

$$\begin{aligned} H_2(\mathrm{Diff}(M)) \xrightarrow{\phi} H_2(\mathrm{Mod}(M)) &\rightarrow H_1(\mathrm{Diff}_0(M))_{\mathrm{Mod}(M)} \\ &\rightarrow H_1(\mathrm{Diff}(M)) \rightarrow H_1(\mathrm{Mod}(M)) \rightarrow 0. \end{aligned}$$

By the Mather–Thurston theorem,

$$H_1(\mathrm{Diff}_0(M)) = \mathrm{Diff}_0(M)^{\mathrm{ab}} = 0$$

so ϕ is surjective. Since the obvious composition

$$B \mathrm{Diff}(M)^\delta \rightarrow B \mathrm{Diff}(M) \rightarrow B \mathrm{Mod}(M)$$

is induced by the quotient $\mathrm{Diff}(M) \rightarrow \mathrm{Mod}(M)$, the map

$$H_2(B \mathrm{Diff}(M)) \rightarrow H_2(B \mathrm{Mod}(M))$$

is also surjective. The same argument applies to $\mathrm{Diff}^\Gamma(M)$. \square

One could go further and try to extend the computations of [3] to the manifolds $M = \#_g(S^k \times S^{k+1})$ to show that the classes in Theorem 27 are in the image of

$$H_*(B \mathrm{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_*(B\Gamma; \mathbb{Q})$$

as in Theorem 29. For another example, one could look at the diffeomorphism groups of handlebodies, i.e. boundary-connected-sums of $D^{k+1} \times S^k$ for $k \geq 4$, c.f. [10].

A. Lifting cycles from arithmetic groups to diffeomorphism groups by Manuel Krannich

We denote by $\text{Diff}(W_g)$ the topological group of orientation-preserving diffeomorphisms of the iterated connected sum

$$W_g = \sharp^g (S^n \times S^n)$$

in the smooth topology. Fixing an embedded disc $D^{2n} \subset W_g$, we also consider the manifold

$$W_{g,1} = W_g \setminus \text{int}(D^{2n})$$

and its group of diffeomorphisms $\text{Diff}_\partial(W_{g,1})$ fixing a neighborhood of the boundary pointwise, which is related to $\text{Diff}(W_g)$ by a map

$$\text{Diff}_\partial(W_{g,1}) \rightarrow \text{Diff}(W_g)$$

given by extending diffeomorphisms of $W_{g,1} \subset W_g$ via the identity.

A.1. The action on homology. The action of the group of diffeomorphisms $\text{Diff}(W_g)$ on the middle homology $H_n(W_g) \cong \mathbb{Z}^{2g}$ preserves the unimodular hyperbolic $(-1)^n$ -symmetric intersection form

$$\lambda: H_n(W_g) \otimes H_n(W_g) \rightarrow \mathbb{Z}$$

and thus gives rise to a map

$$\pi_0 \text{Diff}(W_g) \longrightarrow \begin{cases} \text{Sp}_{2g}(\mathbb{Z}) & n \text{ odd,} \\ \text{O}_{g,g}(\mathbb{Z}) & n \text{ even,} \end{cases}$$

whose image we denote by $G_g \leq \text{GL}_{2g}(\mathbb{Z})$. If n is even or $n = 1, 3, 7$, this map is surjective and G_g coincides with $\text{Sp}_{2g}(\mathbb{Z})$ or $\text{O}_{g,g}(\mathbb{Z})$ depending on the parity of n , whereas the image $G_g \leq \text{Sp}_{2g}(\mathbb{Z})$ for $n \neq 1, 3, 7$ odd is the finite index subgroup $\text{Sp}_{2g}^q(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z})$ of matrices preserving the standard theta-characteristic (see e.g. [4, Ex. 5.5]). As the space of orientation-preserving embeddings $\text{Emb}(D^{2n}, W_g)$ is connected, the map

$$\pi_0 \text{Diff}_\partial(W_{g,1}) \rightarrow \pi_0 \text{Diff}(W_g)$$

is surjective, so the images of these two groups in $\text{GL}_{2g}(\mathbb{Z})$ agree. Given a subgroup $\Gamma \leq G_g$, we denote by

$$\text{Diff}^\Gamma(W_g) \leq \text{Diff}(W_g) \quad \text{and} \quad \text{Diff}_\partial^\Gamma(W_{g,1}) \leq \text{Diff}_\partial(W_{g,1})$$

the preimages of Γ with respect to the canonical maps to G_g .

The primary goal of this appendix is to prove the following.

Theorem 29. *Let $2n \geq 6$ and $\Gamma \leq G_g$ a finite index subgroup. The natural map*

$$H^*(B\Gamma; \mathbb{Q}) \longrightarrow H^*(\text{BDiff}_\partial^\Gamma(W_{g,1}); \mathbb{Q})$$

is injective in degrees

$$* \leq \begin{cases} 2 & \text{for } 2n = 6, \\ n & \text{for } 2n > 6 \text{ and } g < 4 - c, \\ n + \min(n - 4, g - 1 + c) & \text{otherwise,} \end{cases}$$

where $c = 0$ if n is even and $c = 1$ if n is odd.

Remark. Since the action of $\text{Diff}_\partial^\Gamma(W_{g,1})$ on $H_n(W_g)$ factors through $\text{Diff}^\Gamma(W_g)$, the same conclusion holds for $\text{BDiff}^\Gamma(W_g)$ instead of $\text{BDiff}_\partial^\Gamma(W_{g,1})$.

A.2. Stable and unstable cohomology. The block-inclusion

$$\text{GL}_{2g}(\mathbb{Z}) \subset \text{GL}_{2g+2}(\mathbb{Z})$$

is covered by a map

$$\text{Diff}_\partial(W_{g,1}) \rightarrow \text{Diff}_\partial(W_{g+1,1})$$

given by extending diffeomorphisms via the identity, so $\text{GL}_{2g}(\mathbb{Z}) \subset \text{GL}_{2g+2}(\mathbb{Z})$ restricts to an inclusion $G_g \subset G_{g+1}$ and we obtain a map

$$\text{BDiff}_\partial(W_{\infty,1}) \rightarrow \text{BG}_\infty$$

by taking (homotopy) colimits of $\text{BDiff}_\partial(W_{g,1}) \rightarrow \text{BG}_g$ in g . By the work of Borel, Galatius–Randal-Williams, and Madsen–Weiss [7, 8, 16, 27], the cohomology rings of BG_∞ and $\text{BDiff}_\partial(W_{\infty,1})$ is a polynomial algebra concentrated in even degrees. Moreover, there are natural choices of polynomial generators for these rings with respect to which the induced map

$$H^*(\text{BG}_\infty; \mathbb{Q}) \rightarrow H^*(\text{BDiff}_\partial(W_{\infty,1}); \mathbb{Q})$$

corresponds to an inclusion of generators; this can for instance be seen by an index-theoretic argument (see e.g. [13, Sect. 2.4]). Given a subgroup $\Gamma \leq G_g$ of finite index, we have a commutative square

$$\begin{array}{ccc} H^*(\text{BG}_\infty; \mathbb{Q}) & \longrightarrow & H^*(B\Gamma; \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*(\text{BDiff}_\partial(W_{\infty,1}); \mathbb{Q}) & \longrightarrow & H^*(\text{BDiff}_\partial^\Gamma(W_{g,1}); \mathbb{Q}), \end{array}$$

whose upper horizontal arrow is an isomorphism in a range of degrees growing with g by a result of Borel [7, 8]. In light of work of Harer and Galatius–Randal-Williams [18, 21], the same holds for the lower horizontal morphism if $\Gamma = G_g$ and $2n \neq 4$.¹ Moreover, the proof of Theorem 29 will make apparent that the vertical arrows in the diagram are (compatibly) split injective in a range increasing with n and any finite index subgroup $\Gamma \leq G_g$ if $2n \geq 6$. As a result, the cokernel of the upper horizontal map — the so-called *unstable cohomology* of $B\Gamma$ — injects in this range of degrees into the cokernel of the lower horizontal map and thus provides a source for unstable cohomology of $\text{BDiff}_\partial^\Gamma(W_{g,1})$.

When varying $\Gamma \leq G_g$ over finite index subgroups, the rational cohomology of $B\Gamma$ in degree g is arbitrarily large for n even and g odd; this is the main result of the body of this work (see Theorem 1). For the full group $\Gamma = G_g$ on the other hand, little is known about the unstable cohomology, aside from some scattered classes: for instance, computations of Hain [20] and Hulek–Tomassi [22] show that for n odd, there is a nontrivial unstable class in $H^6(BG_3; \mathbb{Q})$ and one in $H^{12}(BG_4; \mathbb{Q})$. By the above discussion, these classes remain nontrivial (and unstable) when pulled back to $\text{BDiff}_\partial(W_{g,1})$ as long as n is sufficiently large, so we obtain the following corollary.

Corollary 30. *For n odd, the cokernel of the natural morphism*

$$H^i(\text{BDiff}_\partial(W_{\infty,1}); \mathbb{Q}) \longrightarrow H^i(\text{BDiff}_\partial(W_{g,1}); \mathbb{Q})$$

is nontrivial for $(i, g) = (6, 3)$ as long as $n > 5$, and for $(i, g) = (12, 4)$ if $n > 8$.

Remark. To the knowledge of the author, these are the first known unstable rational cohomology classes of $\text{BDiff}_\partial(W_{g,1})$, aside from the case $2n = 2$ of surfaces.

A.3. The work of Berglund–Madsen. The proof of Theorem 29 crucially relies on work of Berglund and Madsen [4], who used a combination of classical surgery theory and rational homotopy theory to construct rational models for the classifying spaces $\text{BhAut}_\partial^{\text{id}}(W_{g,1})$ and $\widetilde{\text{BDiff}}_{\partial,o}(W_{g,1})$ of the spaces of homotopy automorphisms and block diffeomorphisms homotopic to the identity. Using these models, they proved that the rational cohomology rings of the classifying spaces of the full automorphism spaces $\text{hAut}_\partial(W_{g,1})$ and $\widetilde{\text{Diff}}_\partial(W_{g,1})$ are independent of g in a range of degrees and studied the rational cohomology in this *stable* range. As explained above, Theorem 29 yields some information on $H^*(\text{BDiff}_\partial(W_{g,1}); \mathbb{Q})$ in the *unstable* range. Its proof involves relating $H^*(\text{BDiff}_\partial(W_{g,1}); \mathbb{Q})$ to $H^*(\widetilde{\text{BDiff}}_\partial(W_{g,1}); \mathbb{Q})$ by combining [4] with Morlet’s lemma of disjunction as in [36], extending some arguments in [4] for spaces of automorphisms homotopic to the identity to the full automorphism spaces, in particular to show that the cohomology ring $H^*(\text{BhAut}_\partial(W_{g,1}); \mathbb{Q})$

¹Recent work of Kupers–Randal-Williams [24] shows that, this holds for any finite index subgroup $\Gamma \leq G_g$ as long as $2n \geq 6$ and Γ is not completely contained in the subgroup $\text{SO}_{g,g}(\mathbb{Z}) \subset \text{O}_{g,g}(\mathbb{Z})$ if n is even.

injects into $H^*(\text{BDiff}_\partial(W_{g,1}); \mathbb{Q})$ also in the unstable range (it is even a retract), and carrying out a spectral sequence argument involving low-degree computations of $\text{BhAut}_\partial(W_{g,1})$ and some facts from the theory of arithmetic groups.

A.4. The proof of Theorem 29. We divide the proof of Theorem 29 into three steps corresponding to three maps in a factorisation

$$\text{BDiff}_\partial^\Gamma(W_{g,1}) \xrightarrow{\textcircled{1}} \widetilde{\text{BDiff}}_\partial^\Gamma(W_{g,1}) \xrightarrow{\textcircled{2}} \text{BhAut}_\partial^{\cong, \Gamma}(W_{g,1}) \xrightarrow{\textcircled{3}} \text{B}\Gamma, \quad (\text{A.1})$$

which we explain in the following. Up to canonical equivalences, the topological group of block diffeomorphisms fixing a neighborhood of the boundary $\widetilde{\text{Diff}}_\partial(W_{g,1})$ fits between $\text{Diff}_\partial(W_{g,1})$ and the topological monoid of homotopy automorphisms fixing the boundary, so there are natural maps

$$\text{Diff}_\partial(W_{g,1}) \rightarrow \widetilde{\text{Diff}}_\partial(W_{g,1}) \rightarrow \text{hAut}_\partial(W_{g,1})$$

(see e.g. [4, Sect. 4]), which explain the maps $\textcircled{1}$ and $\textcircled{2}$ for $\Gamma = G_g$, denoting by

$$\text{hAut}_\partial^{\cong}(W_{g,1}) \subset \text{hAut}_\partial(W_{g,1})$$

the union of components in the image of the map

$$\widetilde{\text{Diff}}_\partial(W_{g,1}) \rightarrow \text{hAut}_\partial(W_{g,1}).$$

By definition of the block-diffeomorphism group, the map

$$\text{Diff}_\partial(W_{g,1}) \rightarrow \widetilde{\text{Diff}}_\partial(W_{g,1})$$

is surjective on path components (in fact, it is an isomorphism by Cerf’s “concordance implies isotopy”), so in particular the image of $\widetilde{\text{Diff}}_\partial(W_{g,1})$ in $\text{GL}_{2g}(\mathbb{Z})$ coincides with the image G_g of $\text{Diff}_\partial(W_{g,1})$, which explains the map $\textcircled{3}$ for $\Gamma = G_g$. For a general subgroup $\Gamma \leq G_g$, this composition is defined analogously by restricting the components of the automorphism spaces involved to the preimage of Γ of the canonical maps to G_g .

In what follows, we examine the quality of the maps $\textcircled{1}$ – $\textcircled{3}$ in rational cohomology.

$\textcircled{1}$ Extending (block) diffeomorphisms of an embedded disc $D^{2n} \subset W_{g,1}$ to all of $W_{g,1}$ by the identity induces a commutative square

$$\begin{array}{ccc} \text{BDiff}_\partial(D^{2n}) & \longrightarrow & \text{BDiff}_\partial(W_{g,1}) \\ \downarrow & & \downarrow \\ \widetilde{\text{BDiff}}_\partial(D^{2n}) & \longrightarrow & \widetilde{\text{BDiff}}_\partial(W_{g,1}) \end{array}$$

whose induced map on vertical homotopy fibres

$$\widetilde{\text{Diff}}_{\partial}(D^{2n})/\text{Diff}_{\partial}(D^{2n}) \longrightarrow \widetilde{\text{Diff}}_{\partial}(W_{g,1})/\text{Diff}_{\partial}(W_{g,1})$$

is $(2n-4)$ -connected by an application of Morlet's lemma of disjunction. For $g \gg 0$, this can be combined with Berglund–Madsen's work [4] to conclude that

$$\widetilde{\text{Diff}}_{\partial}(D^{2n})/\text{Diff}_{\partial}(D^{2n})$$

is $(2n-5)$ -connected, as observed by Randal-Williams [36, Sect. 4]. This in turn implies that

$$\widetilde{\text{Diff}}_{\partial}(W_{g,1})/\text{Diff}_{\partial}(W_{g,1})$$

has no rational cohomology in degrees $* < 2n-4$ for all $g \geq 0$, which we combine with the homotopy pullback

$$\begin{array}{ccc} \text{BDiff}_{\partial}^{\Gamma}(W_{g,1}) & \longrightarrow & \text{BDiff}_{\partial}(W_{g,1}) \\ \downarrow & & \downarrow \\ \text{B}\widetilde{\text{Diff}}_{\partial}^{\Gamma}(W_{g,1}) & \longrightarrow & \text{B}\widetilde{\text{Diff}}_{\partial}(W_{g,1}) \end{array}$$

for a fixed subgroup $\Gamma \leq G_g$ to conclude the following.

Proposition 31. *For $2n \geq 6$ and a subgroup $\Gamma \leq G_g$, the induced map*

$$H^*(\text{B}\widetilde{\text{Diff}}_{\partial}^{\Gamma}(W_{g,1}), \mathbb{Q}) \longrightarrow H^*(\text{BDiff}_{\partial}^{\Gamma}(W_{g,1}); \mathbb{Q})$$

is an isomorphism for $* < 2n-4$ and a monomorphism for $* = 2n-4$.

② Our discussion of the second map in the composition (A.1) is not specific to the manifold $W_{g,1}$, so we phrase it more general.

Proposition 32. *For a compact, simply-connected, stably parallelisable manifold M of dimension $d \geq 5$ with sphere boundary $\partial M \cong S^{d-1}$, the natural map*

$$\text{B}\widetilde{\text{Diff}}_{\partial}(M) \longrightarrow \text{BhAut}_{\partial}^{\cong}(M)$$

is split injective on rational cohomology rings.

Proof. Following [4, Sect. 4.4], we pick a base point $*$ $\in \partial M$ in the boundary and denote by $\text{hAut}_{\partial}^*(\tau_M^S)$ the topological monoid of homotopy automorphisms $f: M \rightarrow M$ which are the identity on the boundary together with a bundle automorphism $\tilde{f}: \tau_M^S \rightarrow \tau_M^S$ of the stable tangent bundle which restricts to the identity over the fixed basepoint and whose underlying self-map of M agrees with f . We denote by

$$\text{hAut}_{\partial, \circ}^*(\tau_M^S) \subset \text{hAut}_{\partial}^*(\tau_M^S)$$

the kernel of the map

$$\mathrm{hAut}_\partial^*(\tau_M^s) \rightarrow \pi_0 \mathrm{hAut}_\partial(M)$$

given by taking homotopy classes and forgetting the bundle map and by

$$\mathrm{hAut}_\partial^{*,\cong}(\tau_M^s) \subset \mathrm{hAut}_\partial^*(\tau_M^s)$$

the preimage of the subgroup

$$\pi_0 \mathrm{hAut}_\partial^{\cong}(M) \subset \pi_0 \mathrm{hAut}_\partial(M)$$

given by the image of $\widetilde{\mathrm{Diff}}_\partial(M)$ in $\mathrm{hAut}_\partial(M)$. Up weak equivalence, there is a canonical map of topological monoids

$$\widetilde{\mathrm{Diff}}_\partial(M) \rightarrow \mathrm{hAut}_\partial^*(\tau_M^s)$$

given by taking derivatives [4, p. 116]. This map induces a map of fibre sequences:

$$\begin{array}{ccccc} \mathrm{B}\widetilde{\mathrm{Diff}}_{\partial,\circ}(M) & \longrightarrow & \mathrm{B}\widetilde{\mathrm{Diff}}_\partial(M) & \longrightarrow & \mathrm{B}\pi_0 \mathrm{hAut}_\partial^{\cong}(M) \\ \downarrow & & \downarrow & & \parallel \\ \mathrm{BhAut}_{\partial,\circ}^*(\tau_M^s) & \longrightarrow & \mathrm{BhAut}_\partial^{*,\cong}(\tau_M^s) & \longrightarrow & \mathrm{B}\pi_0 \mathrm{hAut}_\partial^{\cong}(M), \end{array}$$

where $\widetilde{\mathrm{Diff}}_{\partial,\circ}(M) \subset \widetilde{\mathrm{Diff}}_\partial(M)$ denotes kernel of the map

$$\widetilde{\mathrm{Diff}}_\partial(M) \rightarrow \pi_0 \mathrm{hAut}_\partial^{\cong}(M).$$

Berglund and Madsen showed that the two fibres in this diagram are nilpotent spaces [4, Prop. 4.8, Cor. 4.14] and that the map between them is a rational equivalence [4, Cor. 4.21]. By an application of the Serre spectral sequence, this implies that the middle vertical map is a rational cohomology isomorphism, so to finish the proof, it suffices to show that the map

$$\mathrm{BhAut}_\partial^{*,\cong}(\tau_M^s) \rightarrow \mathrm{BhAut}_\partial^{\cong}(M)$$

admits a section. Fixing a trivialisaton of τ_M^s , every homotopy automorphism of M is covered by a unique bundle automorphism that agrees with the identity on each fibre with respect to the chosen trivialisaton, which induces a section of the map

$$\mathrm{hAut}_\partial^*(\tau_M^s) \rightarrow \mathrm{hAut}_\partial(M).$$

Restricting components and taking classifying spaces, this induces a section as required. \square

In the case $M = W_{g,1}$, the argument for Proposition 32 shows more generally that

$$\mathrm{B}\widetilde{\mathrm{Diff}}_\partial^\Gamma(M) \rightarrow \mathrm{BhAut}_\partial^{\cong,\Gamma}(M)$$

is split injective on rational cohomology for any subgroup $\Gamma \leq G_g$.

③ Combining the previous two propositions, we conclude that the map

$$H^*(\mathrm{BhAut}_\partial^{\cong, \Gamma}(W_{g,1}); \mathbb{Q}) \rightarrow H^*(\mathrm{BDiff}_\partial^\Gamma(W_{g,1}); \mathbb{Q})$$

is injective in degrees $* \leq 2n - 4$, which finishes the proof of Theorem 29 when combined with the following proposition.

Proposition 33. *For $2n \geq 6$ and a finite index subgroup $\Gamma \leq G_g$, the induced map*

$$H^*(\mathrm{B}\Gamma; \mathbb{Q}) \longrightarrow H^*(\mathrm{BhAut}_\partial^{\cong, \Gamma}(W_{g,1}); \mathbb{Q})$$

is an isomorphism for $ < n$ and a monomorphism for $* = n$. Moreover, if $g \geq 4 - c$, then this map is an isomorphism for*

$$* < n + \min(n - 1, g - 1 + c)$$

and a monomorphism for

$$* = n + \min(n - 1, g - 1 + c),$$

where $c = 0$ if n is even and $c = 1$ if n is odd.

Proof. We consider the rational Serre spectral sequence

$$\begin{aligned} E_2^{p,q} &= H^p(\mathrm{B}\pi_0 \mathrm{hAut}_\partial^{\cong, \Gamma}(W_{g,1}); H^q(\mathrm{BhAut}_\partial^{\mathrm{id}}(W_{g,1}); \mathbb{Q})) \\ &\implies H^{p+q}(\mathrm{BhAut}_\partial^{\cong, \Gamma}(W_{g,1}); \mathbb{Q}). \end{aligned} \quad (\text{A.2})$$

of the fibration sequence

$$\mathrm{BhAut}_\partial^{\mathrm{id}}(W_{g,1}) \longrightarrow \mathrm{BhAut}_\partial^{\cong, \Gamma}(W_{g,1}) \longrightarrow \mathrm{B}\pi_0 \mathrm{hAut}_\partial^{\cong, \Gamma}(W_{g,1}).$$

By [4, Prop. 5.6], the rational homotopy Lie algebra $\pi_{*+1}(\mathrm{BhAut}_\partial^{\mathrm{id}}(W_{g,1})) \otimes \mathbb{Q}$ is isomorphic, as a module over $\mathbb{Q}[\pi_0 \mathrm{hAut}_\partial^{\cong, \Gamma}(W_{g,1})]$, to a graded sub Lie algebra of the Lie algebra $\mathrm{Der}^+ \mathbf{L}(H)$ of positive degree derivations of the free graded Lie algebra $\mathbf{L}(H)$ on the graded vector space

$$H := H_n(W_{g,1}; \mathbb{Q}) \cong \mathbb{Q}^{2g}$$

concentrated in degree $(n - 1)$. In particular, the action of $\pi_0 \mathrm{hAut}_\partial^{\cong, \Gamma}(W_{g,1})$ on

$$\pi_{*+1}(\mathrm{BhAut}_\partial^{\mathrm{id}}(W_{g,1})) \otimes \mathbb{Q}$$

factors through the homology action

$$\pi_0 \mathrm{hAut}_\partial^{\cong, \Gamma}(W_{g,1}) \longrightarrow \Gamma \subset \mathrm{GL}_{2g}(\mathbb{Z}).$$

Moreover, as a result of [4, Prop. 7.9], there is an isomorphism of $\mathbb{Q}[\pi_0 \text{hAut}_\partial^{\cong, \Gamma}(W_{g,1})]$ -modules

$$H^q(\text{BhAut}_\partial^{\text{id}}(W_{g,1}); \mathbb{Q}) \cong H_{\text{CE}}^q(\pi_{*+1}(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}),$$

where the right hand side is the Chevalley–Eilenberg cohomology of the graded homotopy Lie algebra. Consequently, the action on these groups factors through Γ as well. Since the homology action $\pi_0 \text{hAut}_\partial^{\cong, \Gamma}(W_{g,1}) \rightarrow \Gamma$ is surjective by definition and has finite kernel as a result of [4, Prop. 5.3], it induces an identification of the E^2 -page of (A.2) of the form

$$\begin{aligned} E_2^{p,q} &= H^p(\text{B}\pi_0 \text{hAut}_\partial^{\cong, \Gamma}(W_{g,1}); H^q(\text{BhAut}_\partial^{\text{id}}(W_{g,1}); \mathbb{Q})) \\ &\cong H^p(\text{B}\Gamma; H^q(\text{BhAut}_\partial^{\text{id}}(W_{g,1}); \mathbb{Q})). \end{aligned} \quad (\text{A.3})$$

As H is concentrated in degree $(n - 1)$, the Lie algebra

$$\text{Der}^+ \mathbf{L}(H) \cong \pi_{*+1}(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q}$$

is concentrated in degrees $k(n - 1)$ for $k \geq 1$. In particular the space $\text{BhAut}_\partial^{\text{id}}(W_{g,1})$ is rationally $(n - 1)$ -connected, so the E_2 -page (A.3) vanishes for $q \leq n - 1$ except for the bottom row $p = 0$, which implies the first claim. To prove the second part of the statement, note that in degrees $* < 2n - 2$, the Lie algebra $\text{Der}^+ \mathbf{L}(H)$ is only nontrivial in degree $n - 1$, so the reduced rational cohomology

$$\tilde{H}^q(\text{BhAut}_\partial^{\text{id}}(W_{g,1}); \mathbb{Q}) \cong \tilde{H}_{\text{CE}}^q(\pi_{*+1}(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q})$$

is for $* < 2n - 1$ only nontrivial in degree n where it is isomorphic to the dual

$$(\pi_n(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q})^\vee.$$

To prove the claim, it thus suffices to show that

$$H^p(\text{B}\Gamma; (\pi_n(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q})^\vee)$$

vanishes for $p < g - 1 + c$ and $g \geq 4 - c$, since this would imply that the E^2 -page of (A.2) is for $p + q < n + \min(n - 1, g - 1 + c)$ concentrated in the bottom row. By the discussion above, we have

$$\begin{aligned} (\pi_n(\text{BhAut}_\partial^{\text{id}}(W_{g,1})) \otimes \mathbb{Q})^\vee &\subset \text{Der}^+ \mathbf{L}(H)_{n-1}^\vee \\ &\cong \text{Hom}(H, \mathbf{L}^2(H))^\vee \\ &\cong H \otimes \mathbf{L}^2(H)^\vee \subset H \otimes (H^{\otimes 2})^\vee \cong H^{\otimes 3} \end{aligned}$$

as a $\mathbb{Q}[\Gamma]$ -module, where $\mathbf{L}^2(H) \subset \mathbf{L}(H)$ is the subspace spanned by brackets of length two. Here we used that $\mathbf{L}(H)$ is a free Lie algebra for the first isomorphism and

the standard (symplectic or orthogonal) form for the last isomorphism. Since short exact sequences of finite dimensional $\mathbb{Q}[\Gamma]$ -modules split (this uses $g \geq 2$ and that \mathbb{Q} -algebraic groups $\mathrm{Sp}_{2g}(\mathbb{Q})$ and $\mathrm{SO}_{2g}(\mathbb{Q})$ are simple, see for instance [4, Thm. A.1, Prop. B.4]), we conclude that

$$(\pi_n(\mathrm{BhAut}_\partial^{\mathrm{id}}(W_{g,1})) \otimes \mathbb{Q})^\vee$$

is a direct summand of $H^{\otimes 3}$, so it suffices to show that $H^p(\mathrm{B}\Gamma; H^{\otimes 3})$ vanishes for $p < g - 1 + c$ and $g \geq 4 - c$.

By enhancements due to Tshishiku [42, Thm. 1.1, Thm. 1.2] of Borel's vanishing ranges for the cohomology of arithmetic groups [7, 8], the cohomology of Γ with coefficients in an irreducible finite-dimensional algebraic representation vanishes in this range, so we have

$$H^p(\mathrm{B}\Gamma; H^{\otimes 3}) \cong H^p(\mathrm{B}\Gamma; \mathbb{Q}) \otimes (H^{\otimes 3})^\Gamma,$$

where $(-)^{\Gamma}$ denotes taking invariants. The proof will conclude by arguing that these invariants vanish. This can be simplified in two ways: firstly, if n is even, after possibly restricting to the subgroup $\Gamma \cap \mathrm{SO}_{g,g}(\mathbb{Z}) \leq \Gamma$, we may assume that Γ is contained in the connected algebraic subgroup

$$\mathrm{SO}_{g,g}(\mathbb{Q}) \subset \mathrm{O}_{g,g}(\mathbb{Q})$$

and secondly, it suffices to show that the invariants vanish after tensoring with \mathbb{R} . Since $\mathrm{Sp}_{2g}(\mathbb{Q})$ and $\mathrm{SO}_{g,g}(\mathbb{Q})$ are connected, semi-simple and have no compact factors, the subgroup $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{R})$ or $\Gamma \subset \mathrm{SO}_{g,g}(\mathbb{R})$ (depending on whether n is even or odd) is a lattice [9, Thm. 7.8] and moreover Zariski dense by Borel's density theorem [6]. This implies that the Γ -invariants of $H^{\otimes 3} \otimes \mathbb{R}$ agree with those of $\mathrm{Sp}_{2g}(\mathbb{R})$ or $\mathrm{SO}_{g,g}(\mathbb{R})$, which in turn vanish by classical invariant theory. \square

Remark. (i) Note that the first part of the proof of Proposition 33 did not require $\Gamma \leq G_g$ to be of finite index. Since Propositions 31 and 32 are valid in this generality as well, Theorem 29 holds for arbitrary subgroups $\Gamma \leq G_g$ in the range $* \leq n$ for $2n \geq 6$ and $* \leq 2$ for $2n = 2$.

(ii) Similar arguments to those of Proposition 33 were used in [23] to improve the ranges for rational homological stability of $\mathrm{BhAut}_\partial(W_{g,1})$, $\widehat{\mathrm{BDiff}}_\partial(W_{g,1})$, and a truncation of $\mathrm{BDiff}_\partial(W_{g,1})$.

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B. Tshishiku, Department of Mathematics, Harvard University,
Cambridge, MA 02138, USA
E-mail: tshishikub@gmail.com

M. Krannich, Centre for Mathematical Sciences, University of Cambridge,
Cambridge CB3 0WB, UK
E-mail: krannich@dpms.cam.ac.uk