SYMMETRIES OF EXOTIC ASPHERICAL SPACE FORMS

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Abstract. We study finite group actions on smooth manifolds of the form $M\#\Sigma$, where Σ is an exotic n-sphere and M is a closed aspherical space form. We give a classification result for free actions of finite groups on $M\#\Sigma$ when M is 7-dimensional. We show that if $\mathbb{Z}/p\mathbb{Z}$ acts freely on $T^n\#\Sigma$, then Σ is divisible by p in the group of homotopy spheres. When M is hyperbolic, we give examples $M\#\Sigma$ that admit no nontrivial smooth action of a finite group, even though $\operatorname{Isom}(M)$ is arbitrarily large. Our proofs combine geometric and topological rigidity results with smoothing theory and computations with the Atiyah–Hirzebruch spectral sequence.

1. Introduction

In this paper we study finite group actions and the Nielsen realization problem for smooth manifolds of the form $M\#\Sigma$, where M is an aspherical space form, and Σ is a homotopy sphere. Recall that a space form is a complete Riemannian manifold with constant sectional curvature, and a manifold is aspherical if its universal cover is contractible. Aspherical space forms are precisely flat and hyperbolic manifolds.

1.1. **Nielsen realization for homotopy hyperbolic manifolds.** For N a closed Riemannian manifold with nonpositive curvature, Schoen–Yau [SY79, pg. 378] asked if the natural homomorphism

$$\alpha : \operatorname{Diff}(N) \to \operatorname{Out}\big(\pi_1(N)\big)$$

splits over finite subgroups of $\operatorname{Out}\big(\pi_1(N)\big)$. This is true, for example, if N is a hyperbolic manifold (by Kerckhoff [Ker83] in dimension 2 and by Mostow rigidity in dimensions ≥ 3), but it is false in general. The first counterexamples were provided by Farrell-Jones [FJ90, Thm. 1] who constructed negatively curved manifolds N diffeomorphic to $M\#\Sigma$, where M^n is hyperbolic and Σ is a homotopy n-sphere, and for which the homomorphism $\alpha : \operatorname{Diff}(N) \to \operatorname{Out}\big(\pi_1(N)\big)$ is not surjective, so a fortiori it does not split over $\operatorname{Out}\big(\pi_1(N)\big) \cong \operatorname{Isom}(M)$, which is finite. This result of Farrell–Jones leads to the following questions.

Question 1.1. Let N be a *homotopy aspherical space form*, i.e. a closed smooth manifold that is homotopy equivalent to a closed aspherical space form M. (By known instances of the Borel conjecture, N is homeomorphic to M when $n \neq 4$; see [FH83, FJ89b].)

(a) What is the image of the natural map $\alpha : \operatorname{Diff}(N) \to \operatorname{Out}\big(\pi_1(N)\big)$?
(b) For which subgroups of $\operatorname{Im}(\alpha) < \operatorname{Out}\big(\pi_1(N)\big)$ does α split?
(c) When a splitting exists over G, can one classify the splittings $G \to \operatorname{Diff}(N)$, perhaps in terms of actions on the space form homotopy equivalent to N?

Currently, we do not have a satisfactory answer for Question 1.1(a) in general. The result of Farrell-Jones loc. cit. gives an example where $\operatorname{Im}(\alpha)$ has index 2 in $\operatorname{Out}\big(\pi_1(N)\big)$. In [BT22, pf. of Thm. B] the authors construct examples where the index of $\operatorname{Im}(\alpha)$ in $\operatorname{Out}\big(\pi_1(N)\big)$

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is arbitrarily large. However, if $N \cong M\#\Sigma$, then the restriction to orientation-preserving subgroups $\alpha : \mathrm{Diff}^+(N) \to \mathrm{Out}^+\big(\pi_1(N)\big)$ is surjective (see [BT22, Thm. 1]). We use this fact as a starting point for further analysis of problems (2) and (3).

1.2. **Action classification in dimension 7.** Our first result, which partially answers Questions 1.1(b)–(c), is a classification result for free, odd-order group actions on $M\#\Sigma$ for any 7-dimensional aspherical space form M and homotopy 7-sphere Σ. To state the theorem, we first give a construction.

Regular actions. Let M be an n-dimensional manifold with a (smooth) finite group action $G \curvearrowright M$. Given a homotopy n-sphere Ω, we can construct an action on $N = M\#\Omega^{\#|G|}$ by gluing $|G|$ copies of Ω to M along the G-orbit of a point in M with trivial stabilizer. We say the action $G \curvearrowright N$ is obtained from the action $G \curvearrowright M$ by an equivariant connected sum at a free point. For brevity we call these actions *regular*.

Theorem A (Action classification, dimension 7)**.** *Let M be a 7-dimensional closed aspherical space form and let Σ be a homotopy 7-sphere. Assume that G is a finite group with odd order. Then every free, orientation-preserving action $G \curvearrowright M\#\Sigma$ is regular, i.e. it is obtained from a smooth action $G \curvearrowright M$ by an equivariant connected sum at a free point.*

Next we apply Theorem A to Question 1.1(b). Below we use Θ_n to denote the group of smooth, oriented homotopy n-spheres defined by Kervaire–Milnor [KM63]. If $n \neq 4$, this group is known to be finite and abelian, and every homotopy n-sphere is homeomorphic to S^n. For example, $\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$. If Σ is not diffeomorphic to S^n it's called *exotic*.

Corollary 1.2. *Let M be a hyperbolic 7-manifold. Assume that* $\mathrm{Isom}^+(M)$ *acts freely on M. Fix a homotopy 7-sphere Σ. Then*

$$\alpha : \mathrm{Diff}^+(M\#\Sigma) \to \mathrm{Out}^+\big(\pi_1(M\#\Sigma)\big)$$

splits over $G < \mathrm{Out}^+\big(\pi_1(M\#\Sigma)\big)$ if and only if Σ is divisible by $|G|$ in Θ_7.

There are many examples to which Corollary 1.2 applies. By a result of Belolipetsky-Lubotzky [BL05, Thm. 1.1 and §6.3], for any $n \geq 2$ and any finite group H, there is a hyperbolic n-manifold such that $\mathrm{Isom}(M) \cong \mathrm{Isom}^+(M) \cong H$ and $\mathrm{Isom}(M)$ acts freely on M. (See [BT22, Thm. 6] for an explanation of why one can arrange for the action to be free.)

The proof of Corollary 1.2 using Theorem A is very short, so we explain it now. First, without the assumption that $|G|$ is odd that appears in Theorem A, we can prove the following weaker version of Theorem A.

Addendum 1.3. *Let M and Σ be as in the statement of Theorem A. If G is a finite group with a free, orientation-preserving action on $M\#\Sigma$, then Σ is divisible by $|G|$ in Θ_7.*

The proof of Addendum 1.3 (see §4) makes use of the fact that for hyperbolic 7-manifolds, $M\#\Sigma$ is not diffeomorphic to M when Σ is not diffeomorphic to S^n; see also Remark 3.7.

Proof of Corollary 1.2. Any splitting $G \to \mathrm{Diff}(M\#\Sigma)$ is a free action by Cappell–Lubotzky–Weinberger [CLW18, Thm. 1.5]. By Addendum 1.3, this implies that Σ is divisible by $|G|$ in Θ_7. For the converse, if the divisibility condition holds, then a splitting can be obtained using the regular action construction. □

1.3. **Nielsen realization for homotopy tori.** Next we consider Questions 1.1(a)–(c) for a homotopy n-torus $\mathfrak{T}$. Similar to the hyperbolic case, the homomorphism

$$\alpha : \mathrm{Diff}^+(\mathfrak{T}) \to \mathrm{SL}_n(\mathbb{Z})$$

is frequently not surjective (see [BKKT23, Lem. 3.1]), but is surjective when $\mathfrak{T} \cong T^n \# \Sigma$ for each homotopy n-sphere Σ.

Regarding Question 1.1(b) in this setting, in [BKKT23, Cor. C] Krannich–Kupers and the authors prove that $\alpha : \mathrm{Diff}^+(T^n\#\Sigma) \to \mathrm{SL}_n(\mathbb{Z})$ does not split for certain Σ by showing that there is no splitting of the induced map $\pi_0\big(\mathrm{Diff}(T^n\#\Sigma)\big) \to \mathrm{SL}_n(\mathbb{Z})$ on the mapping class group. This result applies in infinitely many dimensions, but we still do not know whether or not $\mathrm{Diff}(T^7\#\Sigma) \to \mathrm{SL}_7(\mathbb{Z})$ splits when Σ is an exotic 7-sphere.

To make further progress, it is desirable to better understand the structure of finite group actions on the manifolds $T^n\#\Sigma$. Note that $T^n\#\Sigma$ is exotic if Σ is exotic; see §3.3 for more discussion. To state our result in this direction, first we give a definition.

We say an action $G \curvearrowright M^n\#\Sigma^n$ has the *divisibility property* if Σ is divisible by $|G|$ in Θ_n. This is generally weaker than the action being regular; see Example 2.2.

Theorem B (Actions on $T^n\#\Sigma$)**.** *Fix $n \geq 5$ and let $T^n = (S^1)^n$ denote the n-torus. Let Σ be a homotopy n-sphere. Assume that G is a cyclic group of prime order. Then any free, orientation-preserving action of G on $T^n\#\Sigma$ has the divisibility property.*

1.4. **Smoothly asymmetric manifolds.** As a further application of Corollary 1.2, using the result of Belolipetsky–Lubotzky [BL05, loc. cit.], we can show that for any $d > 0$ there exists a 7-dimensional homotopy hyperbolic manifold $N \cong M\#\Sigma$ such that $|\,\mathrm{Out}\big(\pi_1(N)\big)| > d$ and such that $\alpha : \mathrm{Diff}(N) \to \mathrm{Out}\big(\pi_1(N)\big)$ does not split over any nontrivial subgroup of $\mathrm{Out}\big(\pi_1(N)\big)$. The latter condition implies, by a result of Borel [Bor83a], that $\mathrm{Diff}(N)$ does not contain any finite subgroup, i.e. N is *smoothly asymmetric*. This is in contrast with the fact that $\mathrm{Homeo}(N) \cong \mathrm{Homeo}(M)$ contains the subgroup $\mathrm{Out}\big(\pi_1(N)\big) \cong \mathrm{Isom}(M)$ (which is large by construction). In fact we can find examples like this in arbitrarily large dimension.

Theorem C (Asymmetric smoothings of hyperbolic manifolds)**.** *For every $n_0 \geq 5$ and $d \geq 1$, there exists $n \geq n_0$, a closed hyperbolic n-manifold M, and an exotic sphere $\Sigma \in \Theta_n$, so that $|\,\mathrm{Isom}(M)| \geq d$ and $M\#\Sigma$ is smoothly asymmetric, i.e.* $\mathrm{Diff}(M\#\Sigma)$ *does not contain any nontrivial finite subgroup.*

By construction, the manifolds in Theorem C have many topological symmetries but no smooth symmetries. This is a new phenomenon in the study of asymmetric manifolds; previous results prove asymmetry by controlling $\mathrm{Out}\big(\pi_1(M)\big)$. The first examples of asymmetric aspherical manifolds were constructed by Conner–Raymond–Weinberger [CRW72]. These examples (some of which are solvmanifolds) are *topologically asymmetric* (i.e. $\mathrm{Homeo}(M)$ does not contain any nontrivial finite subgroup), and they are shown to be asymmetric by arranging that $\mathrm{Out}(\pi_1(M))$ is torsionfree. In the Riemannian category, Long–Reid [LR05] gave examples of hyperbolic n-manifolds ($n \geq 2$) that are *isometrically asymmetric* (i.e. $\mathrm{Isom}(M) \cong \mathrm{Out}(\pi_1(M)) = 1$); see also [BL05].

1.5. **About the proofs.** In short, the proofs use geometric rigidity to reduce to smoothing theory problems, assuming we are in dimension $n \geq 5$. A similar approach appears in the argument of Farrell–Jones [FJ90] that shows that $\mathrm{Diff}(M\#\Sigma) \to \mathrm{Out}(\pi_1(M\#\Sigma))$ need not be surjective, but the reduction argument and the specific smoothing theory problems are different. Farrell–Jones essentially reduce to computing the inertia group, which is easy if M is stably parallelizable (and this assumption is fine if one is only looking for examples, rather than a classification). Our results reduce to studying how smooth structures pullback under a covering map $\pi : M \to M/G$, and specifically computing the set of smoothings of M/G that pull back to a smoothing of the form $M\#\Sigma$. This can be formulated in terms of a map $\pi^* : [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}]$. Computing π^* is difficult in general, and it is worth noting that in Theorem A we do not need to assume e.g. that M is stably parallelizable. For Theorems A and C, we do the necessary computations using spectral sequence arguments. For Theorem B, the subtleties in the spectral sequence are obviated by a geometric construction (see Proposition 6.2). Throughout, the tools are classical, but their application in this setting is new.

We summarize the individual arguments in more detail. For the following discussion, recall from smoothing theory that concordance classes of smooth structures on a smooth manifold M are in bijection with homotopy classes of maps $M \to \mathrm{Top/O}$. This is discussed more in §3.

About Theorem A. To show the action $G \curvearrowright M\#\Sigma$ is regular, it suffices to show that the quotient $(M\#\Sigma)/G$ is diffeomorphic to a manifold of the form $(M/G)\#\widehat{\Sigma}$ for some smooth action $G \curvearrowright M$ that is topologically conjugate to the action $G \curvearrowright M\#\Sigma$ (see Lemma 2.1). Using rigidity results for hyperbolic and flat manifolds (Borel conjecture, Mostow and Bieberbach rigidity) we reduce the problem to computing a homomorphism

$$\pi^* : [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}]$$

induced by a covering map $M \to M/G$. Specifically, we need to compute the preimage of $M\#\Sigma$ under π^*. Here we use the Atiyah–Hirzebruch spectral sequence (possible because Top/O is an infinite loop space). The assumptions on $\dim M$ and $|G|$ enable us to analyze the differentials in the spectral sequence.

About Theorem B. To show the action $G \curvearrowright T^n\#\Sigma$ has the divisibility property, we use a similar strategy to Theorem A. Again, we reduce to studying a homomorphism

$$\pi^* : [T^n/G, \mathrm{Top/O}] \to [T^n, \mathrm{Top/O}].$$

From the point-of-view of the Atiyah–Hirzebruch spectral sequence, the divisibility property is related to the filtration

$$F_0 \subset F_1 \subset \cdots \subset F_r = [M, \mathrm{Top/O}]$$

whose associated graded appears on the E_∞ page, and whether or not any of the extensions

$$0 \to F_0 \to F_k \to F_k/F_0 \to 0$$

are split. Here $F_0 \cong \Theta_n/I(M)$, where $I(M) = \{\Sigma \in \Theta_n : M\#\Sigma \cong M\}$ is the inertia group of M. We are unable to resolve the necessary extension problems in general. Instead, we use some structural results about flat manifolds (Propositions 5.1 and 5.3) and a homotopical property of the (suspension of the) quotient map $T^n \to T^n/G$ that we call "compatible splitting of the top cell"; see Proposition 6.2. This is proved by carefully constructing an equivariant Whitney embedding of T^n with its G action.

About Theorem C. To construct smoothly asymmetic examples $M\#\Sigma$ with $|\operatorname{Isom}(M)|$ large, we use a result of Belolipetsky–Lubotzky [BL05] that every finite group is the orientation-preserving isometry group of some hyperbolic n-manifold. With this fact, it would be easy to prove the theorem if we knew that every cyclic action $G \curvearrowright M\#\Sigma$ satisfies the divisibility property, for then we could choose M, Σ with the property that Σ is not divisible by the order of any nontrivial element of $\operatorname{Isom}^+(M)$. Unfortunately, we don't know an analogue of Theorem B for high-dimensional hyperbolic manifolds. Instead, the main idea is to again use the Atiyah–Hirzebruch spectral sequence, but now for $[M, \mathrm{Top/O}_{(p)}]$ where $\mathrm{Top/O}_{(p)}$ is the localization at a prime p.

1.6. **Additional related results.** In contrast to the aspherical setting, there are many results known about Lie group actions on homotopy spheres. For example, the standard smooth structure on S^n is characterized uniquely by its faithful action of $\mathrm{SO}(n+1)$. Furthermore, exotic spheres Σ that bound parallelizable manifolds generally have more symmetry than those that do not; compare [Hsi67, Main Thm.] and [HH69, Thm. 1]. See Schultz [Sch85] for a survey. As is common for positive vs. nonpositive curvature, the known results about group actions on $M\#\Sigma$ when M is spherical vs. aspherical seem to be quite different structurally.

1.7. **Further directions, questions, problems.**

Non-free actions. Understanding non-free actions, e.g. of $\mathbb{Z}/d\mathbb{Z}$ on $T^n\#\Sigma$, is more subtle for a variety of reasons. One difficulty (which is related to our proofs) comes from the fact that topological rigidity (the Borel conjecture) is not generally true for crystallographic groups with torsion [CK91, CDK15]. Thus we are unable to conclude that a smooth finite group action on $T^n\#\Sigma$ is topologically conjugate to a smooth action on T^n. Another complication is that, for non-free actions, any classification of actions should include equivariant connected sum at a point with nontrivial stabilizer. Even so, it may be possible to extend our results to the non-free setting, and this seems potentially most tractable in the case when the action has discrete fixed sets.

Non-regular actions. The restrictions $\dim M = 7$ and $|G|$ odd in Theorem A are due to our inability to analyze differentials in the Atiyah–Hirzebruch spectral sequence.

Question 1.4. Does there exist a free action on $M\#\Sigma$ with M hyperbolic that is not regular (i.e. not an equivariant connected sum)?

It may be possible to construct examples where the smooth structure on $(M\#\Sigma)/G$ differs from $(M/G)\#\Sigma'$ by changing the smooth structure on a neighborhood of a closed geodesic. Smooth structures on hyperbolic manifolds obtained in this way are studied in [FJ93] and [BT22].

Regardless of the answer to Question 1.4, we wonder whether every finite group action on $M\#\Sigma$, with M hyperbolic say, is obtained from a finite group action on M by some concrete family of constructions (that would include equivariant connected sums). See also Remark 2.3.

Divisibility property. We do not know how generally the divisibility property holds.

Question 1.5. Does there exist a smooth, aspherical manifold W, an exotic sphere Σ, and a free, orientation-preserving action of a finite group G on $W\#\Sigma$ that does not satisfy the divisibility property?

If one drops the freeness assumption, then it seems likely that one can obtain examples by an equivariant connected sum at a point with nontrivial stabilizer.

As mentioned above, our argument indicates a connection between the divisibility property and extension problems on the E_∞ page of the Atiyah–Hirzebruch spectral sequence for $[M, \mathrm{Top}/\mathrm{O}]$.

1.8. **Section Outline.** In §2 we give a characterization of regular actions and compare this notion with the divisibility property. In §3 we explain the necessary background from smoothing theory. We prove Theorem A in §4. In §5 we prove two results about flat manifolds, which we use to prove Theorem B in §6. Theorem C is proved in §7.

Conventions/Notations. In the rest of the paper, all manifolds are closed, oriented, and connected; all actions preserve the orientation; G is a finite group; and Σ is an exotic n-sphere. Usually we use M to denote an aspherical space form, and we use W to denote a more general smooth manifold. We use $\mathrm{Homeo}(W)$ and $\mathrm{Diff}(W)$ to denote the homeomorphism and diffeomorphism groups of W with the compact-open and compact-open-C^∞ topologies respectively, and we write $\mathrm{Homeo}^+(W)$ and $\mathrm{Diff}^+(W)$ for the orientation-preserving subgroups.

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2. Regular actions

In this section we carefully define a regular action and we give a characterization of regular actions (Lemma 2.1) that we will use to prove Theorem A.

2.1. G-spaces.

We begin with some terminology. We work in the category of smooth manifolds with a faithful, orientation-preserving G action. By a *smooth G-space* we mean a pair (W, ρ) where W is a smooth manifold and $\rho : G \to \mathrm{Diff}^+(W)$ is an injective homomorphism. Two smooth G-spaces are isomorphic if there is an orientation-preserving diffeomorphism $W \to W'$ that intertwines the G actions. Any smooth G-space has an underlying topological G-space where we forget the smooth structure.

We will always suppress ρ from the notation and refer to W has a G-space; we even write $G < \mathrm{Diff}^+(W)$. This should not cause any confusion because each manifold we consider will only ever have one G action of interest.

2.2. Equivariant connected sum.

Let W be any smooth G-space, and fix an embedded n-disk $D \subset W$ whose translates under G are disjoint. For any manifold U, the equivariant connected sum is obtained by forming the connected sum of W with a copy of U along each of the disks $\{g(D) : g \in G\}$. This makes $W\#U\#\cdots\#U$ ($|G|$ copies of U) into a smooth G-space.

2.3. Regular actions.

Fix a smooth n-manifold W. In what follows it's helpful to think of W as the "preferred" smooth structure on the underlying topological manifold.

Let Σ be an exotic n-sphere, and assume that $W\#\Sigma$ is a smooth G-space. We say that $W\#\Sigma$ is *regular* (as a G-space) if there is a smooth G-space structure on W and an exotic sphere $\widehat{\Sigma}$ so that $W\#\Sigma$ is isomorphic to the equivariant connected sum $W\#\widehat{\Sigma}\#\cdots\#\widehat{\Sigma}$.

The following lemma gives an alternate characterization of regular actions with the additional assumption that the action is free.

Lemma 2.1 (Regular actions). *Let W be a closed smooth n-manifold, and fix $\Sigma \in \Theta_n$. Assume that $W\#\Sigma$ is a smooth G-space with G acting freely. Then $W\#\Sigma$ is regular if and only if there is a smooth G-space structure on W such that (i) $W\#\Sigma$ and W are isomorphic as topological G-spaces, and (ii) the quotient manifold $(W\#\Sigma)/G$ is diffeomorphic to $(W/G)\#\widehat{\Sigma}$ for some $\widehat{\Sigma} \in \Theta_n$.*

Proof of Lemma 2.1. For the forward direction: if $W\#\Sigma$ is regular, then by definition there is a smooth G-space structure on W, an exotic sphere $\widehat{\Sigma}$, and an equivariant diffeomorphism $F : W\#\Sigma \to W\#\widehat{\Sigma}\#\cdots\#\widehat{\Sigma}$ to the equivariant connected sum. This implies (i) because the equivariant connected sum is isomorphic to W as a topological G-space (by equivariant coning/Alexander trick). Furthermore, since F is an equivariant diffeomorphism and the G-actions are free, it descends to a diffeomorphism

$$(W\#\Sigma)/G \to (W\#\widehat{\Sigma}\#\cdots\#\widehat{\Sigma})/G \cong (W/G)\#\widehat{\Sigma}.$$

For the reverse direction, suppose given a G-space structure on W and an exotic sphere $\widehat{\Sigma}$ so that properties (i) and (ii) hold, and write

$$f : (W/G)\#\widehat{\Sigma} \to (W\#\Sigma)/G$$

for the diffeomorphism given by (ii). We want to show there is an equivariant diffeomorphism between $W\#\Sigma$ and the equivariant connect sum $W\#\widehat{\Sigma}\#\cdots\#\widehat{\Sigma}$.

On the one hand, the covering map $W\#\Sigma \to (W\#\Sigma)/G$ pulls back along f to a covering

$$p : X \to (W/G)\#\widehat{\Sigma}.$$

On the other hand, the given action of G on W induces a regular action $G \curvearrowright W\#(\widehat{\Sigma}^{\#|G|})$ which defines a smooth covering space

$$p' : W\#(\widehat{\Sigma}^{\#|G|}) \to (W/G)\#\widehat{\Sigma}.$$

The lemma follows by showing that X and $W\#(\widehat{\Sigma}^{\#|G|})$ are (smoothly) isomorphic covering spaces of $(W/G)\#\widehat{\Sigma}$. Since the smooth structure on a smooth cover of a smooth manifold is uniquely determined, it suffices to observe that p and p' are isomorphic in the topological category. This holds because both are topologically equivalent to the covering $W \to W/G$. This holds for p' by construction, and holds for p by our assumption that $G \curvearrowright W\#\Sigma$ and $G \curvearrowright W$ are topologically conjugate.

Put another way, we have the following diagram.

$$\begin{array}{ccccccc}
X & \xrightarrow{\cong_{\mathrm{Diff}}} & W\#\Sigma & \xrightarrow{\cong_{\mathrm{Top}}} & W & \longleftarrow & W\#(\widehat{\Sigma})^{\#|G|} \\
{\scriptstyle p}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p'} \\
(W/G)\#\widehat{\Sigma} & \xrightarrow{f} & (W\#\Sigma)/G & \longrightarrow & W/G & \longleftarrow & (W/G)\#\widehat{\Sigma}
\end{array}$$

In order for the covering spaces p, p' to be equivalent, we want each square in this diagram to commute. The left-most square commutes by construction. The middle square commutes by the assumption (i) that $W\#\Sigma$ and W are isomorphic as topological G-spaces. In the right-most square the horizontal maps are given by coning on the bottom and equivariant coning on the top, so this square also commutes. $\square$

2.4. **Regular actions vs. divisibility property.** If $G \curvearrowright W\#\Sigma$ is regular, then there exists $\widehat{\Sigma}$ and an orientation-preserving diffeomorphism

$$W\#\Sigma \cong W\#(\widehat{\Sigma}^{\#|G|}).$$

When W is a stably parallelizable aspherical space form, this implies that $[\Sigma] = [\widehat{\Sigma}^{\#|G|}]$ in Θ_n by Proposition 3.5. Hence a regular action on such manifolds has the divisibility property. The following Example 2.2 demonstrates that having the divisibility property is weaker than being regular in general.

Example 2.2. Let $\Sigma \in \Theta_7$ be a nontrivial element of order 7. Let $G = \mathbb{Z}/7\mathbb{Z}$ act on T^7 by rotations and form an equivariant connected sum to get an action of G on $T^7\#\Sigma^{\#7}$. Here $T^7\#\Sigma^{\#7}$ is diffeomorphic to T^7, but they are not isomorphic as G-spaces because the quotients $T^7/G \cong T^7$ and $(T^7\#\Sigma^{\#7})/G \cong T^7\#\Sigma$ are not diffeomorphic (see Proposition 3.5 below). Let now $\Omega \in \Theta_8 \cong \mathbb{Z}/2\mathbb{Z}$ be the nontrivial element, and define a G-space

$$N = \big[(T^7\#\Sigma^{\#7}) \times S^1\big] \# \, \Omega^{\#7},$$

where G acts trivially on the second factor of $(T^7\#\Sigma^{\#7}) \times S^1$, and the connect sum with $\Omega^{\#7}$ is performed equivariantly. By construction N is diffeomorphic to $T^8\#\Omega$, but the action of G on N is not regular by Lemma 2.1 because the quotient

$$N/G \cong \big[(T^7\#\Sigma) \times S^1\big]\#\Omega$$

is not diffeomorphic to $T^8\#\,\Omega$ (if they are, then take connected sum with Ω to deduce that T^8 and $(T^7\#\Sigma) \times S^1$ are diffeomorphic, a contradiction). Note however that the divisibility property is satisfied since $\Omega = \Omega^{\#7}$.

Remark 2.3. It would be interesting to come up with a list of geometric constructions of group actions on exotic smooth structures that includes regular actions and the action in Example 2.2 and to prove a more general classification theorem.

3. Background from smoothing theory

In this section we collect various results from smoothing theory. For an informal treatment of the foundational results, we recommend the notes of Davis [Dav15]; a more formal treatment is given in Siebenmann's [KS77, Essay IV]. In addition to the foundational results, we include a basic result about the concordance classes of smooth structures $W\#\Sigma$ (Lemma 3.2), and a computation of the inertia group for closed aspherical space forms (Proposition 3.5).

3.1. Smooth structures.

A *smooth structure* or *smoothing* of a topological manifold W is a choice of maximal smooth atlas. A *smooth marking* on W is a pair (U, h) where U is a smooth manifold and $h : U \to W$ is a homeomorphism; this in particular determines a smoothing of W by pushing forward a smooth atlas. Two markings (U, h) and (U', h') determine the same smoothing if and only if there is a diffeomorphism $\phi : U' \to U$ so that $h' = h \circ \phi$.

Two smoothings of W are *concordant* if they can be realized as the boundary of a smoothing of $W \times [0, 1]$. A concordance can be represented as a marking $V \to W \times [0, 1]$.

We denote the set of concordance classes of smooth structures on W by $S(W)$.

In the following classification theorem [KS77, Essay IV, Thm. 10.1], Top/O denotes the homotopy fiber of the natural map $B\,\mathrm{O} \to B\,\mathrm{Top}$, where $\mathrm{O} = \operatorname{colim} \mathrm{O}(n)$ is the infinite orthogonal group, and $\mathrm{Top} = \operatorname{colim} \mathrm{Homeo}(\mathbb{R}^n, 0)$, with $\mathrm{Homeo}(\mathbb{R}^n, 0)$ the topological group of homeomorphisms of $\mathbb{R}^n$ that fix the origin. We denote by $[W, \mathrm{Top}/\mathrm{O}]$ the set of homotopy classes of based maps; this is equivalent to the set of homotopy classes of unbased maps since Top/O is connected and is an H-space under Whitney sum and because we assume W is connected.

Theorem 3.1. *Let W be a closed topological manifold of dimension $n \geq 5$. Then a choice of a smooth structure $U \xrightarrow{h} W$ on W determines a bijection $S(W) \xrightarrow{\cong} [W, \mathrm{Top}/\mathrm{O}]$ under which (U, h) is sent to the homotopy class of the constant map.*

To compute $[W, \mathrm{Top}/\mathrm{O}]$, we use that Top/O is in fact an infinite loop space [BV73, p.216]. This affords two different approaches:

(1) For any $k \geq 1$, if we write $\mathrm{Top}/\mathrm{O} = \Omega^k Y$, then

$$[W, \mathrm{Top}/\mathrm{O}] \cong [W, \Omega^k Y] \cong [\Sigma^k W, Y].$$

Thus information about the homotopy type of $\Sigma^k W$ (which can be simpler than that of W) can allow us to make conclusions about $[W, \mathrm{Top}/\mathrm{O}]$.

(2) We can view $[W, \mathrm{Top}/\mathrm{O}]$ as the 0-th group of a cohomology theory. Then the Atiyah–Hirzebruch spectral sequence can be used to gain information about $[W, \mathrm{Top}/\mathrm{O}]$. We elaborate on this more below.

3.2. **Smooth structures of the form $W\#\Sigma$.** View $W\#\Sigma$ as $(W \setminus \mathrm{int}\, D^n) \cup_\phi D^n$, where D^n is glued to $W \setminus \mathrm{int}\, D^n$ along the common boundary $\partial D^n = S^{n-1}$ by $\phi \in \mathrm{Diff}^+(S^{n-1})$ whose isotopy class $[\phi] \in \pi_0(\mathrm{Diff}^+(S^{n-1})) \cong \Theta_n$ corresponds to $[\Sigma]$ via clutching (i.e. $\Sigma \cong D^n \cup_\phi D^n$). From this point-of-view, there is a "standard" homeomorphism $\iota : W\#\Sigma \to W$, which is the identity on $W \setminus \mathrm{int}\, D^n$ and $\iota\big|_{D^n}$ is the cone of ϕ (Alexander trick).

The map $W \to \mathrm{Top}/\mathrm{O}$ that classifies $(W\#\Sigma, \iota)$ can be obtained by a composition

$$W \xrightarrow{c} S^n \xrightarrow{f} \mathrm{Top}/\mathrm{O},$$

where c collapses the complement of $D^n \subset W$ to a point, and f classifies $\Sigma \in \mathcal{S}(S^n) \cong \pi_n(\mathrm{Top}/\mathrm{O})$.

In the following lemma, we examine the action of $\mathrm{Homeo}(W)$ on $\mathcal{S}(W)$ given by post-composition.

Lemma 3.2. *Fix $n \geq 5$. Let W be a smooth n-manifold. Fix $\Sigma \in \Theta_n$ and let $\iota : W\#\Sigma \to W$ be a coning homeomorphism. If every orientation-preserving homeomorphism of W is pseudo-isotopic to a diffeomorphism, then the markings $(W\#\Sigma, \iota)$ and $(W\#\Sigma, h \circ \iota)$ determine concordant smoothings for each $h \in \mathrm{Homeo}^+(W)$.*

Proof of Lemma 3.2. The proof is straightforward. We give the details for completeness. Let g be a diffeomorphism of W that's pseudo-isotopic to h via $F : W \times I \to W \times I$. Then marking

$$W\#\Sigma \times I \xrightarrow{\iota \times \mathrm{id}} W \times I \xrightarrow{F} W \times I$$

is a concordance between the smooth markings $(W\#\Sigma, h \circ \iota)$ and $(W\#\Sigma, g \circ \iota)$.

Now we show directly that $(W\#\Sigma, g \circ \iota)$ is concordant to $(W\#\Sigma, \iota)$. Let $D \subset W$ be the n-disk where the connected sum $W\#\Sigma$ is performed. Since g preserves orientation, there is an isotopy g_t with $g_0 = g$ and such that g_1 is the identity on a neighborhood of D. The homeomorphism $(W\#\Sigma) \times [0,1] \to W \times [0,1]$ defined by $(x,t) \mapsto (g_t \circ \iota(x), t)$ gives a concordance between $(W\#\Sigma, g \circ \iota)$ and $(W\#\Sigma, g_1 \circ \iota)$, but $g_1 \circ \iota = \iota \circ g_1'$ for a diffeomorphism of $W\#\Sigma$ since g_1 is the identity near D. Thus the markings $(W\#\Sigma, g_1 \circ \iota)$ and $(W\#\Sigma, \iota)$ determine the same smoothing, so the smoothings $(W\#\Sigma, g \circ \iota)$ and $(W\#\Sigma, \iota)$ are concordant. □

Remark 3.3. Another way to state the conclusion of Lemma 3.2 is that the concordance class of $(W\#\Sigma, f)$ is independent of the choice of homeomorphism $f : W\#\Sigma \to W$ (since $f = (f \circ \iota^{-1}) \circ \iota$). As such, we can safely omit the marking when referring to the smoothing $W\#\Sigma$.

Remark 3.4 (Applying Lemma 3.2)**.** If M is a closed aspherical space form of dimension ≥ 5, then M satisfies the assumption of Lemma 3.2. By the Borel conjecture [FH83, FJ89b], the quotient of $\mathrm{Homeo}(M)$ by the group of homeomorphism pseudo-isotopic to the identity is $\mathrm{Out}\big(\pi_1(M)\big)$; see [Dav15, proof of Lem. 7]. When M is hyperbolic, this group is $\mathrm{Isom}(M)$ by Mostow rigidity. When M is flat, every automorphism of $\pi_1(M)$ is induced by an affine diffeomorphism by Bieberbach's theorem [Cha86, Ch. I, Thm. 4.1].

3.3. **Inertia group for aspherical space forms.** Recall that for a smooth manifold W, the inertia group $I(W)$ is the subgroup of Θ_n consisting of all homotopy spheres $\Sigma \in \Theta_n$ such that $W\#\Sigma$ is diffeomorphic to W by an orientation-preserving diffeomorphism. In this section we show that if W is a stably parallelizable aspherical space form, then $I(W)$ is trivial. In particular, this shows that there are plenty of exotic aspherical space forms to which our results apply.

A smooth manifold is *stably parallelizable* if it can be embedded with trivial normal bundle in some Euclidean space. We say that a topological manifold with boundary $(X, \partial X)$ is *topologically rigid* if every homotopy equivalence of X that restricts to a homeomorphism on ∂X is homotopic rel ∂X to a homeomorphism.

Proposition 3.5. *Let W be a closed, smooth manifold with dimension $n \geq 5$. Assume*

(i) the natural map $\mathrm{Diff}(W) \to \mathrm{Out}\big(\pi_1(W)\big)$ is surjective.
(ii) $W \times [0,1]$ is topological rigid;
(iii) W is stably parallelizable;

Then the inertia group $I(W)$ is trivial.

A proof of Proposition 3.5 is given in [FJ89a, §2] when W is a closed hyperbolic manifold. The proof we give below in the more general case is similar.

Remark 3.6. Proposition 3.5 can be applied to a closed aspherical space form of dimension ≥ 5 after passing to a finite cover. Condition (i) holds for hyperbolic manifolds by Mostow rigidity and holds for flat manifolds by Bieberbach's (second) theorem [Cha86, Ch. I, Thm. 4.1]. Condition (ii) (topological rigidity of $W^n \times I$, $n \geq 5$) was proved in the flat case by Farrell–Hsiang [FH83, Thm. 5.1-2] and in the hyperbolic case by Farrell–Jones [FJ89b, Cor. 10.6]; see also [Far02, §4]. Regarding condition (iii), every closed hyperbolic manifold has a finite sheeted cover that is stably parallelizable [Sul79, Oku01]. The same statement holds for flat manifolds since every flat manifold is finitely covered by T^n by Bieberbach's (first) theorem [Cha86, Ch. I, Thm. 3.1]. There are also lots of flat manifolds different from T^n that are parallelizable; see [Tho65].

Proof of Proposition 3.5. Fix $\Sigma \in \Theta_n$. Suppose that there is an orientation-preserving diffeomorphism $W \cong W\#\Sigma$. We want to show $[\Sigma] = [S^n]$ in Θ_n.

First we show that if there exists a diffeomorphism $f : W\#\Sigma \to W$, then the markings (W, id) and $(W\#\Sigma, \iota)$ give concordant smooth structures. Here ι is the coning map. By assumption (i), there exists a diffeomorphism $g : W \to W$ so that the homeomorphism $\iota \circ f^{-1} \circ g^{-1} : W \to W$ is homotopic to the identity. Since $g \circ f$ is a diffeomorphism, the markings $(W\#\Sigma, \iota)$ and $(W, \iota \circ f^{-1} \circ g^{-1})$ determine the same smoothing. Therefore, it suffices to show that (W, id) and $(W, \iota \circ f^{-1} \circ g^{-1})$ are concordant. A homotopy from $\iota \circ f^{-1} \circ g^{-1}$ to the identity gives a map $h : W \times [0,1] \to W \times [0,1]$ that restricts to a homeomorphism on the boundary. By assumption (ii), the map h is homotopic rel boundary to a homeomorphism h'. Giving the domain of h' the product smooth structure, the homeomorphism h' gives a smooth marking that is a concordance between (W, id) and $(W, \iota \circ f^{-1} \circ g^{-1})$. Altogether we conclude that $(W, \mathrm{id}) \sim (W, \iota \circ f^{-1} \circ g^{-1}) \sim (W\#\Sigma, \iota)$ are concordant.

It remains to show that if W and $W\#\Sigma$ are concordant, then $[\Sigma] = [S^n]$ in Θ_n. This is where we use assumption (iii). Let $D^n \hookrightarrow W$ be an embedded disk, and let $c : W \to S^n$ be the map that collapses the complement of D^n to a point. This induces a map

$$\Theta_n \cong [S^n, \mathrm{Top/O}] \xrightarrow{c^*} [W, \mathrm{Top/O}] \cong S(W)$$

It suffices to show that c^* is injective. Since Top/O is an infinite loop space, if we write $\mathrm{Top/O} = \Omega^k Y$, then $[-, \mathrm{Top/O}] \cong [\Sigma^k(-), Y]$ (adjunction), so it is equivalent to show that the map

$$[S^{n+k}, Y] \to [\Sigma^k W, Y] \tag{1}$$

is injective for some k. Since W is stably parallelizable, for $k \geq n$, there is a homotopy equivalence $S^{n+k} \vee Z_W \to \Sigma^k W$, where Z_W is a finite CW complex of dimension $< n+k$; see Lemma 6.1. This implies that the map (1) is a (split) injection. □

Remark 3.7. In dimension 7, Proposition 3.5 is true even without assumption (iii). To explain this, let $I_c(W)$ denote the kernel of the map $c^* : \Theta_n \to [W, \mathrm{Top/O}]$ that appears in the proof of Proposition 3.5. This group $I_c(W)$ is known as the concordance inertia group, and in general if $\dim W \geq 5$, then $I_c(W) \subset I(W)$ by the s-cobordism theorem. Therefore to show $I(W)$ is trivial it suffices to show $I_c(W) = I(W)$ and $I_c(W)$ is trivial. Now the proof of Proposition 3.5 shows that assumptions (i) and (ii) imply $I_c(W) = I(W)$, and it follows from Kupers [Kup24] that $I_c(W)$ is trivial for every closed 7-manifold W, so $I(W)$ is trivial. (We remark that Kupers' result is used again in §4 to prove Addendum 1.3.)

Remark 3.8. The proof of Proposition 3.5 shows that (under the assumptions of the proposition) if $W\#\Sigma_1$ is diffeomorphic to $W\#\Sigma_2$, then $[\Sigma_1] = \pm[\Sigma_2]$ in Θ_n. The sign depends on whether the diffeomorphism preserves or reverses orientation. (If $f : W\#\Sigma_1 \to W\#\Sigma_2$ reverses orientation, then by taking connected sum with Σ_2 in the domain, we obtain a diffeomorphism $W\#\Sigma_1\#\Sigma_2 \to W\#\Sigma_2\#\overline{\Sigma}_2 \cong W$.)

3.4. **Localization of** Top/O**.** For the proof of Theorem C, we will use the localization $\mathrm{Top/O}_{(p)}$ of Top/O at an odd prime p. The homotopy groups of the infinite loop space $\mathrm{Top/O}_{(p)}$ are the p-torsion subgroups of Θ_n:

$$\pi_n\big(\,\mathrm{Top/O}_{(p)}\,\big) = \Theta_n \otimes \mathbb{Z}_{(p)}$$

The splitting of the short exact sequence $0 \to bP_{n+1} \to \Theta_n \to \Theta_n/bP_{n+1} \to 0$ was addressed by Brumfiel; c.f. [Bru70]. After localization, there is a splitting on the level of spaces due to Lance [Lan88].

Theorem 3.9 (Localization of Top/O)**.** *Let p be an odd prime. Then there are infinite loop spaces $B = B(p)$ and $C = C(p)$, and infinite loop maps*

$$\beta : B \to \mathrm{Top/O}_{(p)} \quad \textit{and} \quad \alpha : C \to \mathrm{Top/O}_{(p)}$$

such that the map

$$B \times C \xrightarrow{\beta\times\alpha} \mathrm{Top/O}_{(p)} \times \mathrm{Top/O}_{(p)} \xrightarrow{\text{multiplication}} \mathrm{Top/O}_{(p)}$$

is an equivalence of infinite loop spaces. Furthermore, the map β induces an isomorphism from $\pi_n(B)$ onto $bP_{n+1} \otimes \mathbb{Z}_{(p)}$.

This is proved in [Lan88, §5]. To compare the statement of Theorem 3.9 to what appears in [Lan88], one should note that $\mathrm{PL/O}_{(p)} \simeq \mathrm{Top/O}_{(p)}$ when p is an odd prime because $\mathrm{Top/PL}_{(p)} \simeq K(\mathbb{Z}/2\mathbb{Z}, 3)_{(p)}$ is contractible and localization preserves fibrations of simply connected spaces [Sul05, Prop. 2.4].

3.5. **The Atiyah–Hirzebruch spectral sequence.** The infinite loop space $B = B(p)$ from Theorem 3.9 defines a spectrum and also a cohomology theory $\mathbb{E}^*$. In particular, for any space W, $\mathbb{E}^0(W) = [W, B]$. When W is a closed manifold, the groups $\mathbb{E}^*(W)$ can be computed using the Atiyah–Hirzebruch spectral sequence. This is a spectral sequence with E_2 page

$$E_2^{p,q} = H^p(W; \pi_{-q}(B)),$$

that converges to $\mathbb{E}^{p+q}(W)$. See [Ada95, §III.7].

4. Actions on 7-dimensional aspherical space forms (Theorem A)

In this section we prove Theorem A. We briefly explain the strategy. Assuming that G acts freely on $M\#\Sigma$, we use rigidity results (Mostow, Bieberbach, Farrell–Jones, Farrell–Hsiang) to find a free, isometric action of G on M such that, if $\pi : M \to M/G$ denotes the quotient map, then the smooth structure $M\#\Sigma$ is in the image of the homomorphism

$$\pi^* : [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}].$$

We would like to use this to show that $(M\#\Sigma)/G$ is diffeomorphic to $(M/G)\#\widehat{\Sigma}$ for some $\widehat{\Sigma} \in \Theta_n$. For this we study π^* using the Atiyah–Hirzebruch spectral sequence. Our inability to resolve extension problems in the spectral sequence ultimately forces us to restrict to dimension 7. We argue in two steps, the first of which deals the hyperbolic and flat cases separately.

Step 1: rigidity. In this step we prove Proposition 4.1. The proposition does not use the assumption $\dim M = 7$ and will be used again for the proof of Theorems B and C.

Proposition 4.1. *Fix an exotic n-sphere Σ and fix an aspherical space form M of dimension n. Assume that $M\#\Sigma$ has a faithful, free action of a finite group G.*

(i) There exists an isometric action of G on M and an equivariant homeomorphism $M\#\Sigma \to M$.
(ii) Denoting the quotient map $\pi : M \to M/G$, then the smooth structure $M\#\Sigma$ is in the image of the homomorphism

$$\pi^* : [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}].$$

Proposition 4.1(i) is similar to part of [CLW18, Thm. 1.5], which says that if $\mathrm{Isom}^+(M)$ acts freely on M, then every finite subgroup of $\mathrm{Homeo}^+(M)$ is conjugate into $\mathrm{Isom}^+(M)$. However, in Proposition 4.1, we do not assume that the full group $\mathrm{Isom}^+(M)$ acts freely; nevertheless our proof of Proposition 4.1 is similar to the corresponding part of the proof of [CLW18, Thm. 1.5].

Proof of Proposition 4.1. We treat the hyperbolic and flat cases separately, but the proofs are similar.

Hyperbolic case: Assume first that M is a hyperbolic manifold. The action $G \curvearrowright M\#\Sigma$ induces a homomorphism

$$G \to \mathrm{Out}\left(\pi_1(M\#\Sigma)\right),$$

which is injective by [Bor83b]. By Mostow rigidity $\mathrm{Out}(\pi_1(M)) \cong \mathrm{Isom}(M)$, so we obtain an isometric action of G on M. To prove (i), we construct an equivariant homeomorphism $M\#\Sigma \to M$.

Claim. G acts freely on M.

Proof of Claim. Consider the following pullback diagram induced by the inclusion $G \hookrightarrow \mathrm{Out}(\pi_1(M))$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \mathrm{Aut}(\pi_1(M)) & \longrightarrow & \mathrm{Out}(\pi_1(M)) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array} \tag{2}$$

By Mostow rigidity, the extension in the top row in the Diagram (2) is equivalent to the extension

$$1 \to \pi_1(M) \to N(\pi_1(M)) \to \mathrm{Isom}(M) \to 1,$$

where $N(\pi_1(M))$ denotes the normalizer in $\mathrm{Isom}(\mathbb{H}^n)$. Consequently, Γ can be identified with the group of all lifts of isometries of $G < \mathrm{Isom}(M)$ to the universal cover $\mathbb{H}^n$. Then G acts freely on M if and only if Γ is torsion-free (a finite-order element of Γ acts on $\mathbb{H}^n$ with a fixed point; conversely, an element of G with a fixed point generates a cyclic subgroup that lifts to Γ). We prove the Claim by showing Γ is torsion free.

To show that Γ is torsion free, we give another description of the extension of G in (2). Recall that G acts freely on $M\#\Sigma$, so the quotient $M\#\Sigma \to (M\#\Sigma)/G$ is a covering map, which determines an extension

$$1 \to \pi_1(M\#\Sigma) \to \pi_1\big((M\#\Sigma)/G\big) \to G \to 1.$$

By construction, the homomorphism $G \to \mathrm{Out}\,\big(\pi_1(M\#\Sigma)\big)$ that classifies this extension, induces an isomorphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(M\#\Sigma) & \longrightarrow & \pi_1\big((M\#\Sigma)/G\big) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow\cong & & \downarrow\cong & & \| & & \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

The group $\pi_1\big((M\#\Sigma)/G\big)$ is torsion-free because $(M\#\Sigma)/G$ is a closed aspherical manifold. Thus Γ is torsion free. This completes the proof of the Claim.

It remains to obtain an equivariant homeomorphism $M\#\Sigma \to M$. By the Borel conjecture for hyperbolic manifolds [FJ89b, Cor. 10.5], the isomorphism $\pi_1\big((M\#\Sigma)/G\big) \cong \Gamma \cong \pi_1(M/G)$ is induced by a homeomorphism $f : (M\#\Sigma)/G \cong M/G$. By construction, this homeomorphism lifts to an equivariant homeomorphism $\widetilde{f} : M\#\Sigma \to M$, as desired. This completes the proof of (i).

In order to prove (ii), we note from part (i) that we have a commutative diagram

$$\begin{array}{ccc} M\#\Sigma & \xrightarrow{\ \widetilde{f}\ } & M \\ \downarrow & & \downarrow \pi \\ (M\#\Sigma)/G & \xrightarrow{\ f\ } & M/G \end{array} \tag{3}$$

The vertical maps are covering maps, and the horizontal maps are homeomorphisms. Consider the induced map

$$\pi^* : S(M/G) \cong [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}] \cong S(M)$$

Observe that π^* sends an arbitrary element $(W, g) \in S(M/G)$ to $(\widehat{W}, \widehat{g}) \in S(M)$, where $\widehat{W}$ is the pullback covering space with the smooth structure such that $\widehat{W} \to W$ is smooth;

see the following diagram.

$$\begin{array}{ccc} \widehat{W} & \xrightarrow{\widehat{g}} & M \\ \downarrow & & \downarrow \pi \\ W & \xrightarrow[g]{} & M/G \end{array}$$

From this we deduce that π^* sends $\big((M\#\Sigma)/G, f\big)$ to $(M\#\Sigma, \widetilde{f})$, which shows $M\#\Sigma$ is in the image of π^*.

Flat case: Assume now that M is flat and let Γ denote its fundamental group. The proof is similar to the hyperbolic case, but it uses different rigidity results. Since G acts freely on $M^n\#\Sigma$, the quotient $M^n\#\Sigma \to (M^n\#\Sigma)/G$ is a covering map, so there is an exact sequence

$$1 \to \Gamma \to \pi_1\big((M\#\Sigma)/G\big) \to G \to 1.$$

By [AK57, Thm. 1], the group $\pi_1\big((M\#\Sigma)/G\big)$ is the fundamental group of a flat manifold $\overline{M}$. Consider the corresponding extension of $\pi_1(\overline{M})$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \pi_1\big((M\#\Sigma)/G\big) & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \uparrow \cong & & \| & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi_1(\overline{M}) & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

By the Borel conjecture for flat manifolds [FH83], the isomorphism $\pi_1(M/G) \cong \pi_1(\overline{M})$ above is induced by a homeomorphism $f : (M\#\Sigma)/G \to \overline{M}$. By construction, this homeomorphism lifts to an equivariant homeomorphism $\widetilde{f} : M\#\Sigma \to M$ between the covering spaces with fundamental group Γ (the cover of $\overline{M}$ is diffeomorphic to M by Bieberbach's second theorem [Cha86, Ch. I, Thm. 4.1]). By construction, the codomain M has a flat metric induced from $\overline{M}$ on which G acts isometrically. This proves (i).

To show (ii), we consider the following diagram

$$\begin{array}{ccc} M\#\Sigma & \xrightarrow{\widetilde{f}} & M \\ \downarrow & & \downarrow \pi \\ (M\#\Sigma)/G & \xrightarrow{f} & \overline{M} \end{array}$$

As in the proof of the previous case, from this diagram, we conclude that the homomorphism

$$\pi^* : S(\overline{M}) \cong [\overline{M}, \mathrm{Top/O}] \to [M, \mathrm{Top/O}] \cong S(M)$$

sends $\big((M\#\Sigma)/G, f\big)$ to $(M\#\Sigma, \widetilde{f})$, so $M\#\Sigma$ is in the image of π^*.

This completes the proof of Proposition 4.1. □

Step 2: computation with the Atiyah–Hirzebruch spectral sequence. Consider the Atiyah–Hirzebruch spectral sequence for the cohomology theory $\mathbb{E}^*$ determined by the infinite loop space Top/O, so that $\mathbb{E}^0(M) = [M, \mathrm{Top/O}]$. This spectral sequence has E_2-page

$$E_2^{p,q} = H^p(M; \pi_{-q}(\mathrm{Top/O})).$$

Recall that $\pi_k(\mathrm{Top/O}) = \Theta_k$ if $k \geq 5$ and $\pi_k(\mathrm{Top/O}) = \pi_k\big(K(\mathbb{Z}/2\mathbb{Z}, 3)\big)$ for $k \leq 6$ [KS77, Essay V, Thm. 5.5]. Thus the spectral sequence yields an exact sequence

$$H^7(M; \Theta_7) \xrightarrow{c} [M, \mathrm{Top/O}] \xrightarrow{q} H^3(M; \mathbb{Z}/2\mathbb{Z}) \to 0$$

Identifying $H^7(M;\Theta_7) \cong \Theta_7$ using the fundamental class, the homomorphism c is the map $\Sigma \mapsto (M\#\Sigma, \iota)$. (Aside: We will not need to interpret the homomorphism q, but it can be identified with the natural map $[M, \mathrm{Top/O}] \to [M, \mathrm{Top/PL}]$ that sends a smooth structure to its corresponding PL structure.)

The map $\pi : M \to M/G$ induces a map of spectral sequences and a commutative diagram

$$\begin{array}{ccccc} \Theta_7 & \xrightarrow{c} & [M/G, \mathrm{Top/O}] & \xrightarrow{q} & H^3(M/G;\mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\cdot|G|} & & \downarrow{\pi^*} & & \downarrow{\pi^*} \\ \Theta_7 & \xrightarrow{c} & [M, \mathrm{Top/O}] & \xrightarrow{q} & H^3(M;\mathbb{Z}/2\mathbb{Z}) \end{array}$$

The left vertical map is multiplication by $|G|$, which is the degree of the cover π. As $|G|$ is odd, the right vertical map is injective by a transfer argument [Hat02, Prop. 3G.1].

To complete the proof of Theorem A, we note that by Step 1, $M\#\Sigma = \pi^*(x)$, where $x = ((M\#\Sigma)/G, f)$; c.f. Diagram (3). Since $M\#\Sigma = c(\Sigma)$, we have $0 = q(\pi^*(x)) = \pi^*(q(x))$, which implies that $q(x) = 0$ since the right vertical map is injective. Thus $x = c(\widehat{\Sigma})$ for some $\Sigma \in \Theta_7$, i.e. $(M\#\Sigma)/G$ is concordant, hence diffeomorphic (by [KS77, Essay I, Theorem 4.1]), to $(M/G)\#\widehat{\Sigma}$. By Proposition 4.1(i), the action $G \curvearrowright M\#\Sigma$ is topologically conjugate to an isometric action on M. Therefore, we can apply Lemma 2.1 to conclude that the action of G on $M\#\Sigma$ is regular.

To prove Addendum 1.3 we use the fact, shown in [Kup24], that for all closed 7-manifolds W there is a splitting

$$[W, \mathrm{Top/O}] \cong H^7(W;\Theta_7) \oplus H^3(W;\mathbb{Z}/2\mathbb{Z}).$$

With respect to this splitting the map π^* is diagonal. Hence writing

$$M\#\Sigma = (a, 0) \in H^7(M;\Theta_7) \oplus H^3(M;\mathbb{Z}/2\mathbb{Z})$$

with $a \neq 0$ and

$$x = (u, v) \in H^7(M;\Theta_7) \oplus H^3(M;\mathbb{Z}/2\mathbb{Z}),$$

we have $\pi^*(u, v) = (a, 0)$ and $u \neq 0$. Since π is a degree p map, we have $\pi^*(u, 0) = (p \cdot u, 0)$ and the divisibility property follows. □

5. Two facts about flat manifolds

In this section we prove two results that are needed in preparation for the proof of Theorem B. These results (Propositions 5.1 and 5.3) concern the structure of flat manifolds with prime cyclic holonomy.

Proposition 5.1 (Prime cyclic holonomy implies mapping torus)**.** *Fix a prime p. Let N be a compact flat manifold with holonomy group $G = \mathbb{Z}/p\mathbb{Z}$. Then N has the structure of a mapping torus*

$$T^{n-1} \to N \to S^1$$

whose monodromy has order p.

Proof of Proposition 5.1. By assumption $\pi_1(N)$ is isomorphic to a discrete torsion-free subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{SO}(n)$, and Γ is torsion-free because N is an aspherical manifold (not an orbifold).

Step 1. First we construct a surjection $\Gamma \to \mathbb{Z}$ with kernel $\mathbb{Z}^{n-1}$. This step uses an argument of [CW89, Lem. 1].

Let $A < \Gamma$ be the maximal abelian subgroup, $A \cong \mathbb{Z}^n$. There is a short exact sequence

$$1 \to A \to \Gamma \to G \to 1. \tag{4}$$

Let $\xi \in H^2(G; A)$ denote the Euler class of the extension (c.f. [Bro82, Ch. IV, Thm. 3.12]), where the coefficient group A has the $\mathbb{Z}[G]$-module structure coming from the extension (4). We know that $\xi \neq 0$ because otherwise $\Gamma \cong \mathbb{Z}^n \rtimes G$ is not torsion-free.

Fix a generator g of G, and set

$$\delta = 1 + g + \cdots + g^{p-1} \in \mathbb{Z}[G].$$

For a fixed $\mathbb{Z}[G]$-module M, write $m_\delta : M \to M$ for the homomorphism defined by $m_\delta(x) = \delta \cdot x$. Recall that

$$H^2(G; M) \cong M^G / \operatorname{Im}\big[m_\delta : M \to M\big] \tag{5}$$

by the standard resolution for computing cohomology of cyclic groups (c.f. [Bro82, I.6]).

Set

$$A' = \ker\big[m_\delta : A \to A\big] \quad \text{and} \quad A'' = \operatorname{Im}\big[m_\delta : A \to A\big],$$

so there is a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to A' \to A \to A'' \to 0.$$

Consider the following portion of the associated long exact sequence in cohomology with coefficients

$$H^2(G; A') \to H^2(G; A) \to H^2(G; A'')$$

Observe that $(A')^G = 0$ (if $a \in A'$ and $g \cdot a = a$, then $0 = m_\delta(a) = p \cdot a$). Then

$$H^2(G; A') = 0$$

by (5), which implies that $H^2(G; A) \to H^2(G; A'')$ is injective. Furthermore, A'' is a trivial module (because $g \cdot \delta = \delta$), so

$$H^2(G; A'') \cong H^2(G; \mathbb{Z}) \oplus \cdots \oplus H^2(G; \mathbb{Z}).$$

Since $\xi \neq 0$, there is surjective composition $q : A \twoheadrightarrow A'' \twoheadrightarrow \mathbb{Z}$ so that the image of ξ under the induced homomorphism

$$q_* : H^2(G; A) \hookrightarrow H^2(G; A'') \to H^2(G; \mathbb{Z})$$

is nonzero.

Set $B = \ker(q)$. This is a $\mathbb{Z}[G]$-submodule of A, which implies that B is a normal subgroup of Γ. By construction, the extension

$$1 \to A/B \to \Gamma/B \to \Gamma/A \to 1 \tag{6}$$

does not split (its Euler class is $q_*(\xi) \neq 0$). Here $\Gamma/A \cong G \cong \mathbb{Z}/p\mathbb{Z}$ and $A/B \cong \mathbb{Z}$ so the preceding central extension has the form

$$1 \to \mathbb{Z} \to \Gamma/B \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

Since this sequence does not split and p is prime, this implies that $\Gamma/B \cong \mathbb{Z}$.

Thus we have a surjection $\Gamma \to \Gamma/B \cong \mathbb{Z}$ with kernel $B \cong \mathbb{Z}^{n-1}$, as desired.

Step 2. We explain why the short exact sequence $1 \to B \to \Gamma \to \mathbb{Z} \to 1$ from Step 1 is realized topologically as a fiber bundle $T^{n-1} \to N \to S^1$. For this, it suffices to show that N is obtained from a quotient of $T^{n-1} \times \mathbb{R}$ by a homeomorphism of the form

$$h(x, t) = (h_1(x), t + t_0) \tag{7}$$

for some $h_1 \in \operatorname{Homeo}(T^{n-1})$ and $t_0 \neq 0 \in \mathbb{R}$.

Consider the action of Γ on $\mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be the subspace spanned by the orbit of $0 \in \mathbb{R}^n$ under B. Decompose $\mathbb{R}^n = V \oplus V^\perp$. Since B is normal in Γ, the action of Γ on $\mathbb{R}^n$ descends to an action of $\Gamma/B \cong \mathbb{Z}$ on $\mathbb{R}^n/B \cong V/B \times V^\perp \cong T^{n-1} \times \mathbb{R}$; furthermore, N is the quotient of this action.

Fix $\gamma \in \Gamma$ that projects to a generator under $\Gamma \to \mathbb{Z}$. Write $\gamma = (u, f) \in \mathbb{R}^n \rtimes \mathrm{SO}(n)$. The element $f \in \mathrm{SO}(n)$ generates the image of the holonomy $\Gamma \to \mathrm{SO}(n)$, so f has order p. Also, γ normalizes B, so f preserves V and the decomposition $\mathbb{R}^n = V \oplus V^\perp$. Write $f = f_1 \oplus f_2$ for the action on $V \oplus V^\perp$, and write $u = (u_1, u_2) \in V \oplus V^\perp$. Since f_2 is an isometry of $V^\perp \cong \mathbb{R}$, $f_2 = \pm 1$. It suffices to show $f_2 = 1$ because then γ acts on $V \oplus V^\perp$ by

$$\gamma(x,t) = (f_1(x) + u_1, t + u_2) \tag{8}$$

and this descends to an action on $\mathbb{R}^n/B \cong T^{n-1} \times \mathbb{R}$ of the form (7).

Suppose for a contradiction that $f_2 = -1$. Since f has order p, this is only possible when $p = 2$. Then the (induced) action of γ on $T^{n-1} \times \mathbb{R}$ acts by a reflection in the $\mathbb{R}$ direction (γ preserves the foliation $T^{n-1} \times \{t\}$, so this action on $\mathbb{R}$ is well-defined). But then the quotient $N = \big(T^{n-1} \times \mathbb{R}\big)/\langle\gamma\rangle$ is not compact, which is a contradiction.

Step 3. Finally we explain why we can assume the monodromy of $T^{n-1} \to N \to S^1$ has order p. From the equation (8), the monodromy is the map $x \mapsto f_1(x) + u_1$ on V/B. This map is isotopic to $x \mapsto f_1(x)$ since $x \mapsto x + u_1$ is a rotation on $T^{n-1} \cong V/B$ and rotations are isotopic to the identity. Since f_1 has order p, this completes the proof. □

Remark 5.2. Our proof does not work if $G = \mathbb{Z}/d\mathbb{Z}$ when d is not prime. One reason is that when d is not prime there are non-split extensions $1 \to \mathbb{Z} \to \Lambda \to \mathbb{Z}/d\mathbb{Z} \to 1$ with $\Lambda \neq \mathbb{Z}$. In this case it seems the only possible conclusion would be that N is a mapping torus $N_1 \to N \to S^1$ where N_1 is a $(n-1)$-dimensional flat manifold.

Proposition 5.3 (Flat mapping torus is parallelizable)**.** *Let M_f be a flat manifold that has the structure of a mapping torus $T^{n-1} \to M_f \to S^1$ whose monodromy f is orientation-preserving and has finite order d. Then M_f is parallelizable.*

A version of this is proved by [Tho65] with a different assumption. We give an alternate argument.

Proof of Proposition 5.3. To show M_f is parallelizable, it suffices to construct an n-frame field on $\widetilde{M_f} \cong \mathbb{R}^n$ that is Γ-invariant, where $\Gamma = \pi_1(M_f)$. Write $\Gamma = \mathbb{Z}^{n-1} \rtimes_{f_*} \mathbb{Z}$, and decompose $\mathbb{R}^n$ accordingly as $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$.

The group Γ has a generating set $\gamma_1, \ldots, \gamma_{n-1}, \eta$, where $\langle\gamma_1, \ldots, \gamma_{n-1}\rangle \cong \mathbb{Z}^{n-1}$ acts by translations of $\mathbb{R}^{n-1} \times \mathbb{R}$ that are trivial in the second factor, and η acts on $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ by $\eta(x,t) = (f_*(x) + \beta, t + \frac{1}{d})$, where $(\beta, f_*) \in \mathbb{R}^{n-1} \rtimes \mathrm{SO}(n-1)$.

Define an orthonormal n-frame field on $\mathbb{R}^n$ as follows. First define a frame field along $\mathbb{R}^{n-1} \times 0$ by choosing an orthonormal frame at the origin, and moving it along $\mathbb{R}^{n-1}$ by parallel transport with respect to the standard Riemannian metric. Choose this frame compatible with the decomposition $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ (i.e. for the frame at the origin, $n-1$ vectors belong to the subspace $\mathbb{R}^{n-1}$ and the last vector belongs to the subspace $\mathbb{R}$).

Now let α_t be a path from the identity I to f_* in $\mathrm{SO}(n-1)$, defined for $t \in [0, 1/d]$. Define a frame on $\mathbb{R}^{n-1} \times \{t\}$ by acting by α_t on the framing of $\mathbb{R}^{n-1} \times \{0\}$.

This extends in an obvious way to a framing on $\mathbb{R}^{n-1} \times \mathbb{R}$ that is η-invariant. The resulting framing is γ_i-invariant for each i, since by construction it is constant on $\mathbb{R}^{n-1} \times \{t\}$ for each $t \in \mathbb{R}$. Since Γ is generated by η and the γ_i, the framing is Γ invariant. □

6. Actions on exotic tori (Theorem B)

In this section we prove Theorem B. We begin with a brief sketch of the argument, and then give the details in the following subsections.

Fix $\Sigma \in \Theta_n$ and suppose that $G \cong \mathbb{Z}/p\mathbb{Z}$ acts freely on $T^n \# \Sigma$ by orientation-preserving diffeomorphisms. As T^n is a flat manifold, Proposition 4.1 gives an action $G \curvearrowright T^n$ such that $T^n \# \Sigma$ is in the image of a homomorphism

$$\pi^* : [Q, \mathrm{Top/O}] \to [T^n, \mathrm{Top/O}],$$

induced by the quotient $\pi : T^n \to T^n/G =: Q$.

We show that there are isomorphisms

$$\begin{array}{ccccc} [T^n, \mathrm{Top/O}] & \cong & [S^n, \mathrm{Top/O}] & \oplus & A \\ \pi^* \uparrow & & & & \\ [Q, \mathrm{Top/O}] & \cong & [S^n, \mathrm{Top/O}] & \oplus & \bar{A} \end{array} \tag{9}$$

where A and $\bar{A}$ are finite abelian groups. With respect to this splitting, we show that the homomorphism π^* is diagonal; see Corollary 6.3.

Given this, the proof is completed as follows. With respect the isomorphisms in (9), we write $T^n \# \Sigma$ as $(x, 0) \in [S^n, \mathrm{Top/O}] \oplus A$. From Proposition 4.1, we have $(x, 0) = \pi^*(y, a)$. Using that π^* is diagonal, we deduce that $\pi^*(0, a) = 0$ and $y \neq 0$; furthermore, since π is a degree-p covering map, $\pi^*(y, 0) = (py, 0)$. This implies the divisibility property.

6.1. Compatible splitting of the top cell.

We now provide details for the existence of a "diagonal" homomorphism as in (9).

For an open embedding $e : X \hookrightarrow Y$ of manifolds, we denote $e' : Y' \to X'$ the induced map of 1-point compactifications. Recall that a smooth n-manifold X is *stably parallelizable* if it can be smoothly embedded in $\mathbb{R}^{n+k}$, for some $k \geq 1$, with trivial normal bundle.

The following lemma is well known. We give a proof, which will be helpful in preparation for Proposition 6.2.

Lemma 6.1 (Splitting the top cell)**.** *Fix $n \geq 1$, and let W be a stably parallelizable closed n-manifold. Then for $k \geq n$, there is a homotopy equivalence $S^{n+k} \vee Z_W \to \Sigma^k W$, where Z_W is a finite CW-complex of dimension $< n + k$.*

Proof. Give W a cell structure with a single n-cell, and give $\Sigma^k W$ the induced cell structure, which has a single $(n + k)$-cell. Let $Z = (\Sigma^k W)^{(n+k-1)}$ denote the $(n + k - 1)$-skeleton. There is a cofiber sequence

$$Z \xrightarrow{i} \Sigma^k W \xrightarrow{q} S^{n+k}.$$

Since W is stably parallelizable, there is a framed Whitney embedding $j : W \times D^k \hookrightarrow \mathbb{R}^{n+k}$. Here we use the assumption $k \geq n$. The map

$$p : S^{n+k} \xrightarrow{j'} \Sigma^k(W_+) \to \Sigma^k W$$

is a right inverse to the map q (up to homotopy) since the composition $q \circ p : S^{n+k} \to S^{n+k}$ has degree 1.

Claim. The map $p \vee i : S^{n+k} \vee Z \to \Sigma^k W$ is a homotopy equivalence.

Proof of Claim. Since the domain and codomain are simply connected, it suffices to show that $p \vee i$ is a homology equivalence by Whitehead's theorem [Hat02, Cor. 4.33]. It is easy to see that $p \vee i$ induces an isomorphism on H_ℓ for $\ell \leq n+k-2$ since Z is the $(n+k-1)$ skeleton; see e.g. [Hat02, Lem. 2.34(c)]. It remains to treat the cases $\ell = n+k-1$ and $\ell = n+k$.

For $\ell = n+k$, the composition $S^{n+k} \vee Z \xrightarrow{p\vee i} \Sigma^k W \xrightarrow{q} S^{n+k}$ induces an isomorphism on H_{n+k} (since $q \circ p$ has degree 1), and since each of these spaces has $H_{n+k} = \mathbb{Z}$, it follows that $(p \vee i)_* : H_{n+k}(S^{n+k} \vee Z) \to H_{n+k}(\Sigma^k W)$ is an isomorphism (and also that $q_* : H_{n+k}(\Sigma^k W) \to H_{n+k}(S^{n+k})$ is an isomorphism – we use this in the next paragraph).

For $\ell = n+k-1$, it suffices to show that $i_* : H_{n+k-1}(Z) \to H_{n+k-1}(\Sigma^k W)$ is injective. Considering the long-exact sequence of the pair $(\Sigma^k W, Z)$, it is equivalent to show that the homomorphism $H_{n+k}(\Sigma^k W) \to H_{n+k}(\Sigma^k W, Z)$ is surjective. This homomorphism can be identified with $q_* : H_{n+k}(\Sigma^k W) \to H_{n+k}(\Sigma^k W/Z)$, which we observed is an isomorphism in the preceding paragraph.

This proves the claim, and finishes the proof of the lemma. □

Proposition 6.2 (Compatible splitting)**.** *Let $\pi : T^n \to Q$ be the quotient by a free action of $G = \mathbb{Z}/p\mathbb{Z} \curvearrowright T^n$. For $k \geq n$ there exists splittings $\Sigma^k T^n \simeq S^{n+k} \vee Z_{T^n}$ and $\Sigma^k Q \simeq S^{n+k} \vee Z_Q$ as in Lemma 6.1 so that the map*

$$S^{n+k} \hookrightarrow S^{n+k} \vee Z_{T^n} \xrightarrow{\simeq} \Sigma^k T^n \xrightarrow{\Sigma^k(\pi)} \Sigma^k Q \xrightarrow{\simeq} S^{n+k} \vee Z_Q \twoheadrightarrow Z_Q, \tag{10}$$

where the fourth map is a homotopy inverse to the splitting of Lemma 6.1, is homotopically trivial.

Proof of Proposition 6.2. By Proposition 5.1, Q has the structure of a mapping torus $T^{n-1} \to Q \to S^1$ with order-p monodromy f. Set $H = \langle f \rangle$.

Fix an equivariant embedding $T^{n-1} \hookrightarrow V$ into a $\mathbb{R}[H]$-module V; this is possible by an equivariant version of the Whitney embedding theorem [Pal57, Mos57]. Without loss of generality, we can assume that the action $f \curvearrowright V$ is orientation-preserving (if not, we can replace V by $V \oplus V$).

Let f_t be a homotopy in $\mathrm{SO}(V)$ from f to the identity. We assume this homotopy takes place for $t \in (0, \infty)$, with $f_t = \mathrm{id}$ for $t \leq 1$ and $f_t = f$ for $t \geq 2$.

The map $F(v,t) = (f_t(v), t)$ is a homeomorphism of $V \times (0,\infty)$. Let M_F be the mapping torus of F. We can view M_F as a bundle

$$V \to M_F \to \mathbb{R}^2 \setminus \{0\}.$$

Since the bundle is trivial near 0, we can extend M_F to a bundle $V \to \widehat{M_F} \to \mathbb{R}^2$. This latter bundle is trivial $\widehat{M_F} \cong V \times \mathbb{R}^2$ since $\mathbb{R}^2$ is contractible. By construction, $Q = M_f$ embeds in $M_F \subset \widehat{M_F}$. (Explicitly, choose the copy of T^{n-1} in $V \times \{2\}$. Then the image of $T^{n-1} \times \{2\} \times [0,1]$ in $M_F = \big(V \times (0,\infty) \times [0,1]\big)/\sim$ is an embedding of M_f in M_F.)

Now we lift to an equivariant embedding of T^n. Identify $\mathbb{R}^2 \cong \mathbb{C}$, and consider the p-fold cover

$$\phi : V \times \mathbb{C} \to V \times \mathbb{C} \quad \text{given by} \quad (v,z) \mapsto (v, z^p).$$

This is a regular cover, branched over $V \times \{0\}$ with deck group $G = \mathbb{Z}/p\mathbb{Z}$. The pre-image of M_f under this cover is $M_{f^p} = M_{\mathrm{id}} = T^n$. Thus we have an embedding $T^n \hookrightarrow V \times \mathbb{R}^2$ that is equivariant with respect to the deck group actions of the coverings $\pi : T^n \to M_f$ and $\phi : V \times \mathbb{R}^2 \to V \times \mathbb{R}^2$, and the quotient by these actions yields an embedding $Q = M_f \hookrightarrow V \times \mathbb{R}^2$. Since T^n and Q are stably parallelizable by Proposition 5.3, the normal bundles of $T^n \subset V \times \mathbb{R}^2$ and $Q \subset V \times \mathbb{R}^2$ are trivial. (For this it's possible we need to first take product with $\mathbb{R}^m$ (with trivial G action); c.f. [KM63, Lem. 3.3].) Thus we have a commutative diagram:

$$\begin{array}{ccc} T^n \times D^k & \xhookrightarrow{\;j_{T^n}\;} & V \times \mathbb{R}^2 \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ Q \times D^k & \xhookrightarrow{\;j_Q\;} & V \times \mathbb{R}^2 \end{array}$$

We use collapse maps of the embeddings j_{T^n}, j_Q to get a commutative diagram

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{\;p_{T^n}\;} & \Sigma^k T^n \\ {\scriptstyle \phi'}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma^k(\pi)} \\ S^{n+k} & \xrightarrow{\;p_Q\;} & \Sigma^k Q \end{array} \tag{11}$$

By the proof of Lemma 6.1, for $W = T^n$ or Q, the map $p_W : S^{n+k} \to \Sigma^k W$ splits the top cell, i.e. there is a homotopy equivalence so $\Sigma^k W \simeq S^{n+k} \vee Z_W$ so that the composition

$$S^{n+k} \xrightarrow{p_W} \Sigma^k W \simeq S^{n+k} \vee Z_W \twoheadrightarrow Z_W \tag{12}$$

is homotopically trivial. Then the composition (10) is homotopically trivial because it factors through (12) by virtue of diagram (11). $\square$

We now use Proposition 6.2 to prove that there is a "compatible splitting" of

$$\pi^* : [Q, \mathrm{Top/O}] \to [T^n, \mathrm{Top/O}].$$

Let $\pi : T^n \to Q$ be the quotient by a free action of $G = \mathbb{Z}/p\mathbb{Z}$. Let $u_Q : D^n \hookrightarrow Q$ be an embedded disk, chosen sufficiently small so that it lifts to an embedding $\tilde{u} : \sqcup_p D^n \hookrightarrow T^n$. Let $u_{T^n} : D^n \hookrightarrow T^n$ be an embedded disk that contains the image of $\tilde{u}$, so $\tilde{u}$ factors as $\sqcup_p D^n \xrightarrow{j} D^n \xrightarrow{u_{T^n}} T^n$. This leads to the following commutative diagram of spaces.

$$\begin{array}{ccc} T^n & \xrightarrow{\;u'_{T^n}\;} & S^n \\ & \searrow^{\tilde{u}'} & \big\downarrow{\scriptstyle j'} \\ {\scriptstyle \pi}\big\downarrow & & \bigvee_p S^n \\ & & \big\downarrow{\scriptstyle \Delta} \\ Q & \xrightarrow[\;u'_Q\;]{} & S^n \end{array}$$

Here the map Δ is the identity map on each S^n. The composition $\Delta \circ j' : S^n \to S^n$ has degree p. Suspending this diagram and combining with Diagram (11), we obtain

$$\begin{array}{ccccc} S^{n+k} & \xrightarrow{\;p_{T^n}\;} & \Sigma^k T^n_+ & \xrightarrow{\;u'_{T^n}\;} & S^{n+k} \\ {\scriptstyle \deg = p}\big\downarrow & & \big\downarrow & & \big\downarrow \\ S^{n+k} & \xrightarrow{\;p_Q\;} & \Sigma^k Q_+ & \xrightarrow{\;u'_Q\;} & S^{n+k} \end{array} \tag{13}$$

The composition $S^{n+k} \to S^{n+k}$ in each row is a degree-1 map. Recalling that Top/O is an infinite loop space, let Y be a space such that $\Omega^k Y \simeq \mathrm{Top/O}$. Apply $[-, Y]$ to Diagram

(13), and use the adjunction $[A, \Omega B] \cong [\Sigma A, B]$ to arrive at the following diagram.

$$\begin{array}{ccccc} [S^n, \mathrm{Top/O}] & \xrightarrow{(u'_Q)^*} & [Q, \mathrm{Top/O}] & \xrightarrow{p_Q^*} & [S^n, \mathrm{Top/O}] \\ \downarrow & & \downarrow \pi^* & & \downarrow \mu_p \\ [S^n, \mathrm{Top/O}] & \xrightarrow{(u'_{T^n})^*} & [T^n, \mathrm{Top/O}] & \xrightarrow{p_{T^n}^*} & [S^n, \mathrm{Top/O}] \end{array}$$

Here μ_p is multiplication by p. We have proved the following corollary.

Corollary 6.3. *The maps $(u'_Q)^*$ and $(u'_{T^n})^*$ are split injections, and there exist splittings $p_Q^*, p_{T^n}^*$ with the property that*

$$p_{T^n}^* \circ \pi^* = \mu_p \circ p_Q^*. \tag{14}$$

6.2. **Finishing the proof of Theorem B.** In Proposition 4.1(ii), we showed that $T^n \# \Sigma = \pi^*(z)$ for some $z \in [Q, \mathrm{Top/O}]$. Using Corollary 6.3, we have

$$\Sigma = p_{T^n}^*(T^n \# \Sigma) = p_{T^n}^*(\pi^*(z)) = \mu_p(p_Q^*(z))$$

which shows that Σ is divisible by p in $\Theta_n = [S^n, \mathrm{Top/O}]$, as desired.

7. Asymmetric smoothings (Theorem C)

Recall from Proposition 4.1 that if M is a closed hyperbolic manifold and G acts freely on $M \# \Sigma$, then the smoothing $M \# \Sigma$ is in the image of a certain homomorphism

$$\pi^* : [M/G, \mathrm{Top/O}] \to [M, \mathrm{Top/O}].$$

We would like to use this to conclude that M satisfies the divisibility property, similar to Step 2 in the proof of Theorem B. If we could show this, then it would suffice to find M and Σ with the property that Σ is not divisible by $|G|$ for any nontrivial group $G < \mathrm{Isom}^+(M)$.

Unfortunately, we don't know how to prove that hyperbolic manifolds satisfy the divisibility property in general (it seems difficult to produce a geometric construction that would yield a version of Proposition 6.2 in the hyperbolic case). Instead, as in the proof of Theorem A, we study π^* using the Atiyah–Hirzebruch spectral sequence. Here the difficulty is (as usual) potentially nontrivial differentials and extension problems, but we show these issues can be avoided for a proper choice of M, Σ and by localizing Top/O at an odd prime.

Proof of Theorem C.

Step 1: the construction. Fix $n_0 \geq 5$, $d \geq 1$, and choose $n = 4k - 1 \geq n_0$ and an odd prime p such that the p-torsion subgroup of bP_{n+1} is nontrivial and the p-torsion subgroup of bP_{m+1} is trivial for $m < n$. This is possible because the set of primes that divide $|bP_{4k}|$ for some k is infinite. For example, $|bP_{4k}|$ is divisible by $2^{2k-1} - 1$ (see [KM63, §7]), and it is not difficult to show that if s, t are relatively prime, then $2^s - 1$ and $2^t - 1$ are relatively prime.

Let $\mathbb{Z}_{(p)}$ denote the set of rational numbers with denominator relatively prime to p. The group $(bP_{n+1})_{(p)} := bP_{n+1} \otimes \mathbb{Z}_{(p)}$ is the p-torsion of bP_{n+1}. Since bP_{n+1} is cyclic, $(bP_{n+1})_{(p)} \cong \mathbb{Z}/p^a\mathbb{Z}$ for some $a \geq 1$. Choose a generator $\Sigma \in (bP_{n+1})_{(p)}$.

Next choose a closed oriented hyperbolic n-manifold M such that (i) $\mathrm{Isom}^+(M) = \mathrm{Isom}(M)$, (ii) $\mathrm{Isom}(M)$ is a p-group where every element has order divisible by p^a, and (iii) $\mathrm{Isom}(M)$ acts freely on M. Such examples exist by the construction of Belolipetsky–Lubotzky [BL05, Thm. 1.1]; see also [BT22, Thm. 6].

Step 2: the computation. Take M and Σ as in Step 1. We claim that $N = M\#\Sigma$ is asymmetric. Fix a finite order element $g \in \mathrm{Diff}^+(M\#\Sigma)$ and denote $G = \langle g\rangle$. Suppose for a contradiction that $g \neq \mathrm{id}_N$. By a result of Borel [Bor83b], the induced map $G \to \mathrm{Out}(\pi_1(N)) \cong \mathrm{Isom}(M)$ is injective, so the order of g is p^b for some $b \geq a$.

By Proposition 4.1 , there is a degree $|G|$ covering map $\pi : M \to \overline{M}$ and $x \in [\overline{M}, \mathrm{Top/O}]$ such that $\pi^*(x) = M\#\Sigma$, where $\pi^* : [\overline{M}, \mathrm{Top/O}] \to [M, \mathrm{Top/O}]$.

To explain the remainder of the argument, we consider the following commutative diagram.

$$\begin{array}{ccccc} [\overline{M}, \mathrm{Top/O}] & \xrightarrow{\pi^*} & [M, \mathrm{Top/O}] & \xleftarrow{(i')^*} & [S^n, \mathrm{Top/O}] \cong \Theta_n \\ \downarrow & & \downarrow & & \downarrow \\ [\overline{M}, \mathrm{Top/O}_{(p)}] & \longrightarrow & [M, \mathrm{Top/O}_{(p)}] & \longleftarrow & [S^n, \mathrm{Top/O}_{(p)}] \cong \Theta_n \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ [\overline{M}, B] & \xrightarrow{0} & [M, B] & \xleftarrow{\cong} & [S^n, B] \cong bP_{n+1} \otimes \mathbb{Z}_{(p)} \end{array} \tag{15}$$

In the top row, the map $i' : M \to S^n$ is the collapse map induced by the inclusion of a disk $i : D^n \hookrightarrow M$. The vertical maps are induced by maps

$$\mathrm{Top/O} \to \mathrm{Top/O}_{(p)} \to B \times C \twoheadrightarrow B.$$

where the second arrow is a homotopy inverse to the equivalence of Theorem 3.9.

Claim. $[M, B] \cong H^n(M; \pi_n(B)) \cong bP_{n+1} \otimes \mathbb{Z}_{(p)}$ and similarly for $\overline{M}$.

Proof of Claim. We prove this using the Atiyah–Hirzebruch spectral sequence. As discussed in §3, this spectral sequence has E_2-page

$$E_2^{i,-j} = H^i(M; \pi_j(B)),$$

and converges to $\mathbb{E}^{i-j}(M)$, where $\mathbb{E}^*$ denotes the cohomology theory associated to the infinite loop space B. In particular, to determine, $[M, B] = \mathbb{E}^0(M)$, we focus on the terms $E_2^{i,-i} = H^i(M; \pi_i(B))$. By our choice of p, and from the fact that M is a closed, oriented n-manifold, we have

$$H^i(M; \pi_i(B)) = \begin{cases} \mathbb{Z}/p^a\mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Thus to prove the claim, it suffices to show that $E_2^{n,-n}$ survives to the E_∞-page. This term $E_2^{n,-n}$ receives no nontrivial differentials because $\pi_k(B) = bP_{k+1} \otimes \mathbb{Z}_{(p)}$ is 0 for $k < n$ (by construction), and the differentials out of $E_2^{n,-n}$ also vanish because $H^i(M; -)$ vanishes for $i > n$.

By the claim and functoriality in the construction of the spectral sequence [Ada95, §III.7], the map $[\overline{M}, B] \to [M, B]$ induced by $\pi : M \to \overline{M}$ agrees with the induced map

$$H^n(\overline{M}; \pi_n(B)) \cong \mathbb{Z}/p^a\mathbb{Z} \to \mathbb{Z}/p^a\mathbb{Z} \cong H^n(M; \pi_n(B)),$$

which is the zero map because it is multiplication by $\deg(\pi) = |G|$ and p^a divides $|G|$ by construction. This explains the arrow labeled "0" in Diagram (15). Similarly, we conclude that the map $[S^n, B] \to [M, B]$ is an isomorphism since i' is a degree-1 map, hence induces an isomorphism $H^n(S^n; \pi_n(B)) \cong H^n(M; \pi_n(B))$.

Now we conclude. On the one hand, the image of $M\#\Sigma$ under $[M, \mathrm{Top/O}] \to [M, B]$ is nonzero because $M\#\Sigma = (i')^*(\Sigma)$, and the "right side" of Diagram (15) commutes. On the other hand, $M\#\Sigma$ is in the kernel of $[M, \mathrm{Top/O}] \to [M, B]$ because $M\#\Sigma = \pi^*(x)$, and the "left side" of Diagram (15) commutes. This contradiction implies that our finite order element $g \in \mathrm{Diff}(N)$ must have been trivial, and this completes the proof of Theorem C. □

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