Rotation curves

Willie Rush Lim

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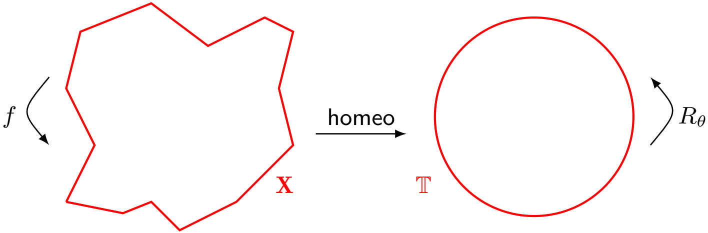
Apr 18, 2026

Rotation curves

An invariant Jordan curve $\mathbf{X} \subset \mathbb{C}$ of a holomorphic map $f : U \to \mathbb{C}$ is a **rotation curve** if $f : \mathbf{X} \to \mathbf{X}$ is conjugate to an irrational rotation $R_\theta(z) = e^{2\pi i\theta} z$.



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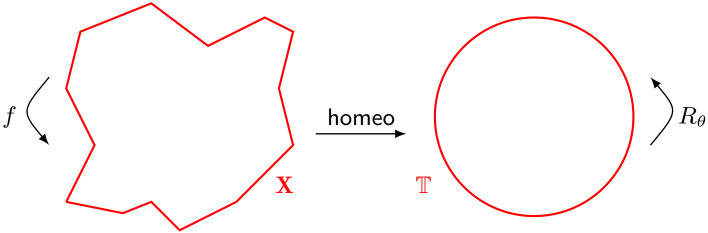


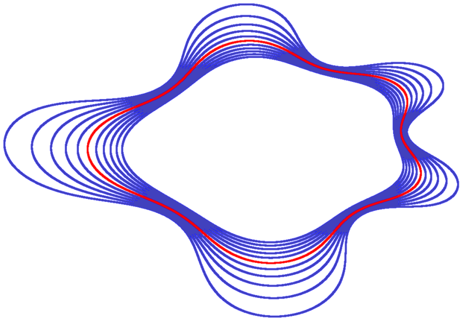
We will fix θ to be an irrational with continued fraction expansion

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \qquad \sup_n a_n < \infty.$$

E.g. quadratic irrationals.

No critical point case

Lemma: If there are no critical points along $\mathbf{X}$, then $\mathbf{X}$ is in a foliation of invariant analytic curves inside a rotation domain.



What if there are critical points?

Everything is possible!

Theorem (wrl '25)

Pick $a + b \geq 1$ *points* $x_1, \ldots, x_a$ *and* $y_1, \ldots, y_b$ *on* $\mathbb{T}$.
There exists a rotation curve $f : \mathbf{X} \to \mathbf{X}$ *with rotation number* θ *admitting*

- *inner critical points with combinatorial location* $(x_1, \ldots, x_a)$,
- *outer critical points with combinatorial location* $(y_1, \ldots, y_b)$.

We can pick f *to be a degree* $a + b + 1$ *rational map and* $\mathbf{X}$ *to be a quasicircle.*

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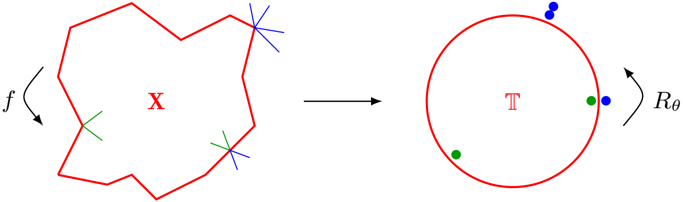

schematic example of $(x_1, x_2) = (0, \frac{3}{5})$, $(y_1, y_2, y_3) = (0, \frac{1}{5}, \frac{1}{5})$

Unicriticality

We call a rotation curve **unicritical** if it only has one critical point.
If so, the combinatorics is encoded by:

- a = inner multiplicity,
- b = outer multiplicity,
- θ = rotation number.

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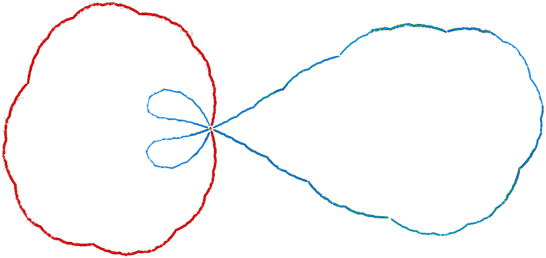

Example: $a = 2$, $b = 1$, $\theta = \frac{\sqrt{5}-1}{2}$

$$f(z) = c_* z^3 \, \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.14421 - 0.96445i$$

Rigidity

Theorem (wrl'26)

Any two unicritical rotation quasicircles $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *with the same combinatorics* (a, b, θ) *are* $C^{1+\alpha}$*-smoothly conjugate.*

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Corollary

For a unicritical rotation quasicircle $\mathbf{X}$ *with combinatorics* (a, b, θ)*,*

- *H-dim*$(\mathbf{X})$ *is universal;*
- *H-dim*$(\mathbf{X}) = 1$ *iff* $\mathbf{X}$ *is* $C^{1+\alpha}$*-smooth iff* $a = b$*.*

References

1. W.R.L. A priori bounds and degeneration of Herman rings with bounded type rotation number. *Invent. Math.* 242 (2025), 827–893.
2. W.R.L. Rigidity of J-rotational rational maps and critical quasicircle maps. *Trans. Amer. Math. Soc.* 379 (2026), 2507–2567.

Thank you!

Quadratic Topological Matings Exist Along Veins

Eduardo Sodré

Brown University

GATSBY, April 2026

Polynomial Dynamics

$f : \mathbb{C} \to \mathbb{C}$ polynomial: want to understand iterations

$$z \longmapsto f(z) \longmapsto f(f(z)) \longmapsto \cdots \longmapsto f^n(z) \longmapsto \cdots$$

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Filled Julia set K_f: points whose orbit is bounded.

A Filled Julia Set

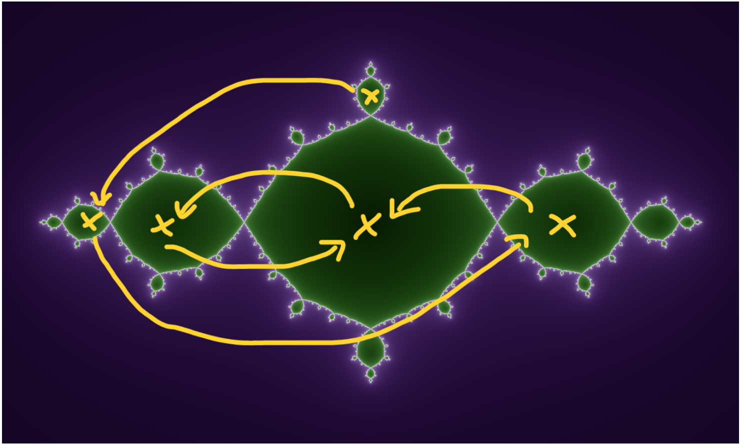

Figure: $f(z) = z^2 - 1$

External Rays

$f : \mathbb{C} \to \mathbb{C}$ polynomial with (locally) connected Julia set K_f.

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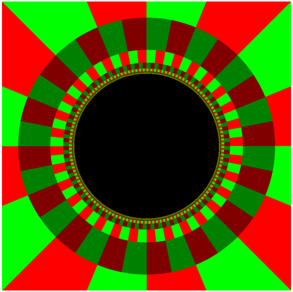
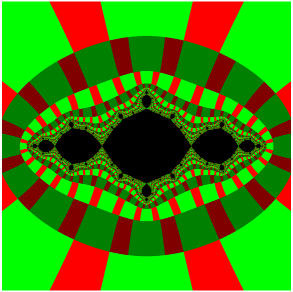
Uniformization of the complement:

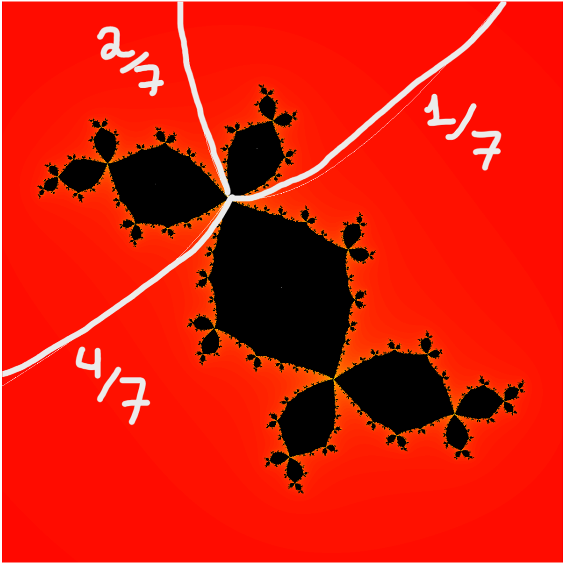
$$\varphi : \mathbb{C} \setminus \overline{\mathbb{D}} \xrightarrow{\cong} \mathbb{C} \setminus K_f$$

Allows us to define the **external rays** of K_f.





External Rays



Matings of Polynomials

Given degree d polynomials f and g, we can construct their **topological mating**.

Glue two copies of $\mathbb{C}$, one for f and another for g, along a circle at infinity to form a 2-sphere.

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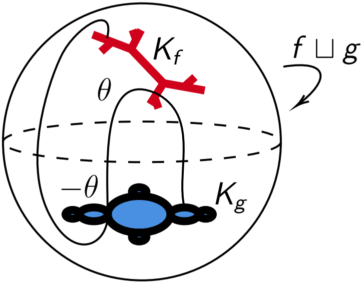
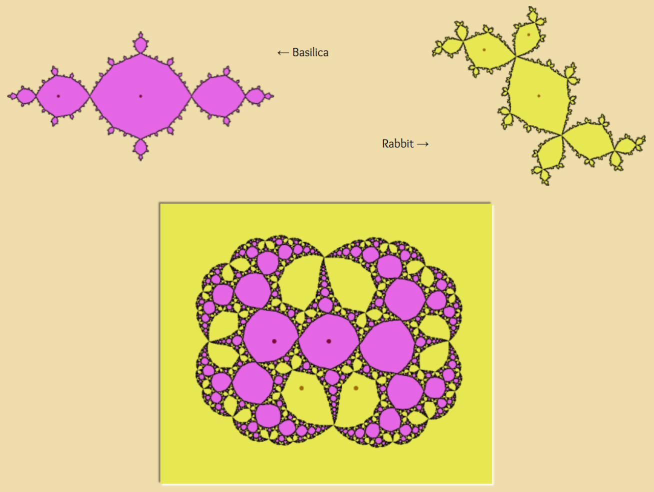


Then, collapse the chains of external rays to points. Identifications correspond to **ray connections**.

Chéritat Movies



Obstructions

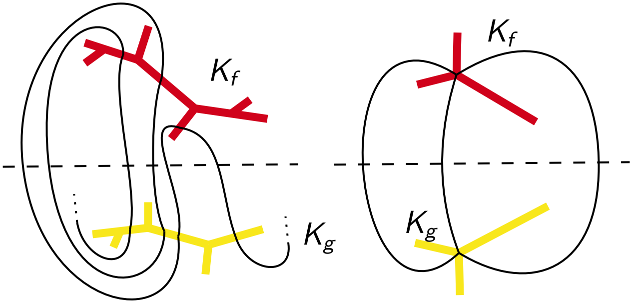
Questions about $S^2/\sim_{\mathrm{ray}}$:

- When is it Hausdorff?
- When is it a 2-sphere?
- When is $f \sqcup g$ on $S^2/\sim_{\mathrm{ray}}$ topologically conjugate to a rational map?

Obstructions

Examples of topological obstructions:

- Infinite ray connection (not Hausdorff);
- Cyclic ray connection (not a 2-sphere).



Postcritically Finite Case

Postcritically finite polynomials: orbit of critical points is finite.

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Theorem (Rees, Shishikura, Tan Lei)

If f and g are **postcritically finite** *quadratic polynomials and* **not** *in conjugate limbs of the Mandelbrot set, then their mating is conjugate to a rational map and it is unique up to conjugacy.*

The Mandelbrot Set

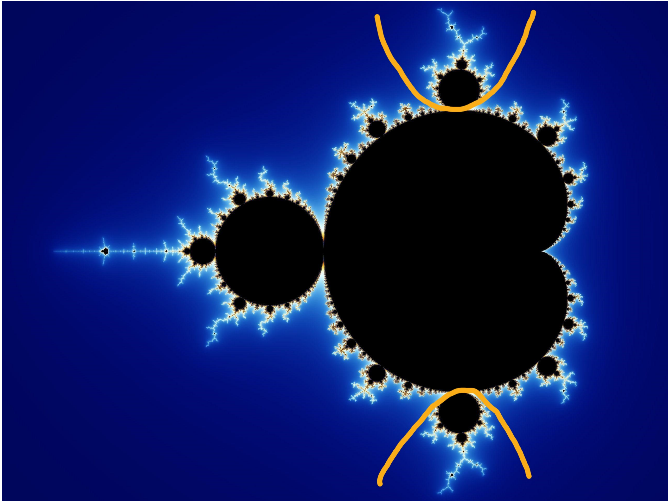
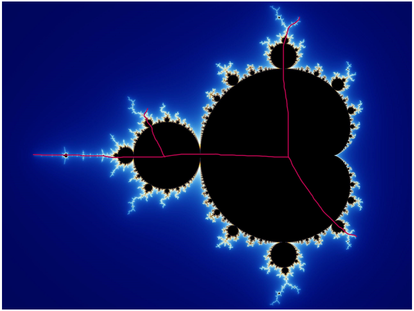
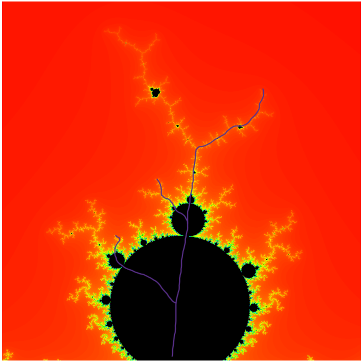


Figure: $M = \{c \in \mathbb{C} \mid K_{z^2+c} \text{ is connected}\}$.

Veins of the Mandelbrot set

Veins of M: Arcs connecting the main cardioid to extremities.





Extremities are postcritically finite parameters.

No Obstructions Along Veins

Theorem

If c_1 and c_2 lie on veins on non-conjugate limbs of M, then the quotient in the topological mating is a 2-sphere.

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Key idea:

Long ray connections are monotonic along veins.

Monotonicity of Ray Connections

Lemma

Suppose that $c_1 \prec c_1'$ *and* $c_2 \prec c_2'$ *on veins.*

(a) *If* $f_{c_1} \sqcup f_{c_2}$ *has a cyclic ray connection, then* $f_{c_1'} \sqcup f_{c_2'}$ *has the same cyclic ray connection.*

(b) *If* $f_{c_1} \sqcup f_{c_2}$ *has a long ray connection*

$$\theta_1 \sim -\theta_1 \sim \theta_2 \sim -\theta_2 \sim \cdots \sim \theta_n,$$

then $f_{c_1'} \sqcup f_{c_2'}$ *has the same ray connection*.*

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Because the extremities are mateable, everything before in the veins will also be.

Lamination models for K_f

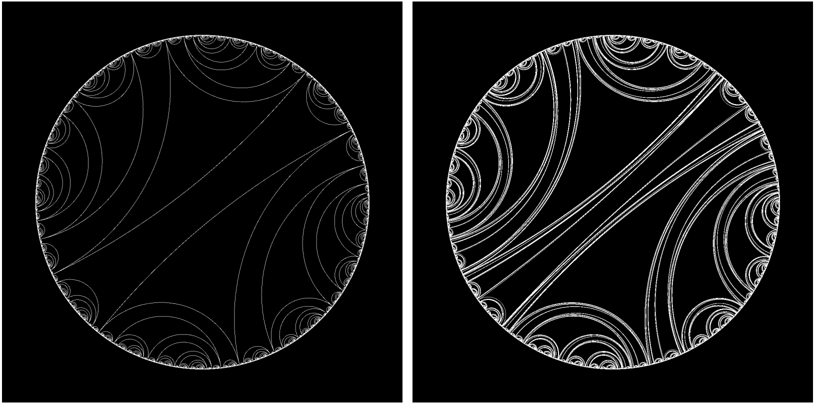


Figure: Laminations for the rabbit and a polynomial succeding it in a vein

End

Thank you!

Ranks of cusped mapping tori

Matthew Zevenbergen (Boston College)

Joint with Ian Biringer (BC) and David Futer (Temple)

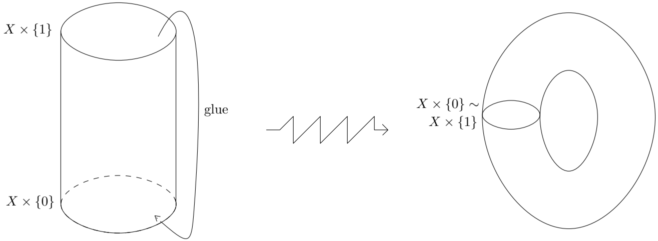
April 2026

Mapping tori

Definition: (Mapping torus) For a self homeomorphism $f : X \longrightarrow X$ of a connected manifold X, define the *mapping torus*

$$M_f := (X \times [0,1])/\sim$$

where $(x,0) \sim (f(x),1)$.



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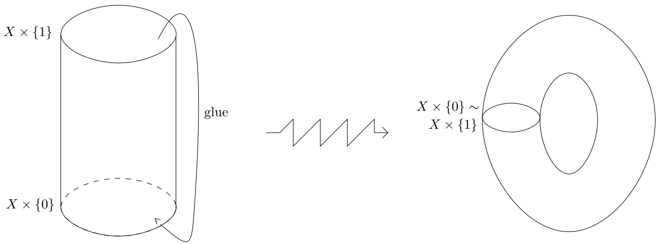

The fundamental group of M_f is an HNN-extension

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(M_f) \longrightarrow \mathbb{Z} \longrightarrow 1.$$

Rank of $\pi_1(M)$

Definition: For a manifold M, $\mathrm{rank}(M)$ is the minimal number of elements needed to generate $\pi_1(M)$.

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Remark: For a mapping torus M_f, we have $\pi_1(M_f) \cong \pi_1(X) \rtimes \mathbb{Z}$, so $\mathrm{rank}(M_f) \leq \mathrm{rank}(X) + 1$.

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Main question: When is $\mathrm{rank}(M_f) = \mathrm{rank}(X) + 1$?

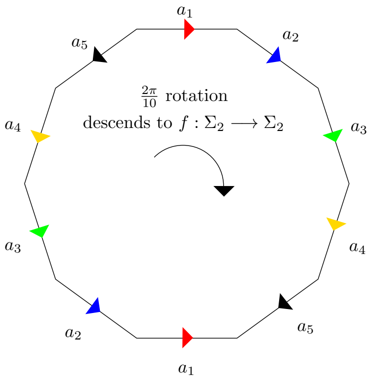
Rank examples

Example 1: For the identity map $\mathrm{id} : X \longrightarrow X$, we have $M_{\mathrm{id}} = X \times S^1$, so

$$\mathrm{rank}(M_{\mathrm{id}}) = \mathrm{rank}(X \times S^1) = \mathrm{rank}(X) + 1.$$

Rank examples

Example 2: Set Σ_g = closed genus g surface. Depict Σ_g by identifying opposite sides of a $4g + 2$-gon, and let $f : \Sigma_g \longrightarrow \Sigma_g$ be the homeomorphism descending from the $2\pi/(4g + 2)$ rotation, as shown for $g = 2$.



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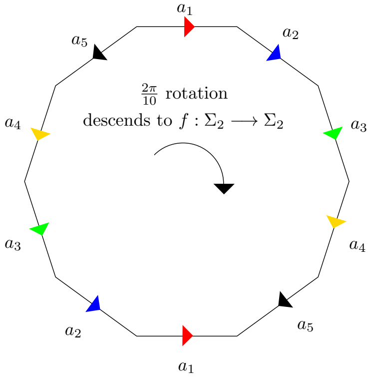

Each $f_*(a_i) = a_{i+1}$, so $\mathrm{rank}(M_f) = 2$.

A rank theorem

Theorem (Biringer-Souto 2016)

For each $g \geq 2$, there is some $L = L(g)$ such that if $f : \Sigma_g \longrightarrow \Sigma_g$ has translation length at least L on the curve complex, then $\operatorname{rank}(M_f) = \operatorname{rank}(\Sigma_g) + 1 = 2g + 1$.

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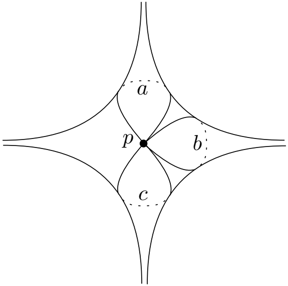
Q: What about mapping tori of punctured surfaces?

Punctured surface counterexample

Example: Let S be a four punctured sphere with $\pi_1(S, p)$ generated by loops a, b, and c.



For $g : S \longrightarrow S$ fixing p such that $g_*(a) = b$ and $f : S \to S$ a "generic" pseudo-Anosov preserving a, the map $g \circ f^n$ has large curve complex translation length for large n, but $\mathrm{rank}(M_{g \circ f^n}) \leq \mathrm{rank}(S)$.

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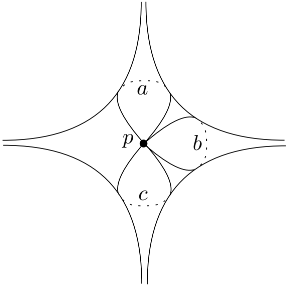

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Hence, the Biringer-Souto theorem doesn't directly extend to punctured surfaces.

Main theorem

Theorem (Biringer-Futer-Z., in progress)

For S a finite type orientable surface with $\chi(S) < 0$, there is some $L = L(S)$ such that if $f : S \longrightarrow S$ has translation length at least L on the curve complex, then $\mathrm{rank}(M_f) \geq \mathrm{rank}(S)$.

Circumference geometry

Question: What is the geometric picture of M_f constructed from $f : S \longrightarrow S$ with large curve complex translation length?

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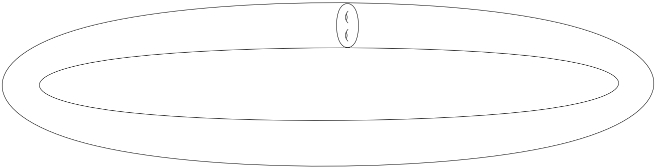
Answer 2: If S is closed, then M_f will have large "circumference": the shortest curve intersecting each fiber will be *long*.



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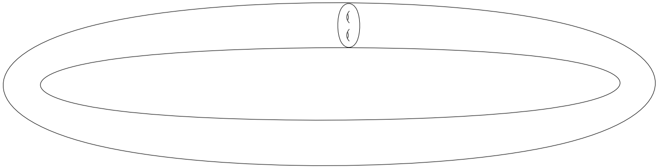

What about if S has punctures?

Circumference geometry

If S has punctures, M_f will have a cusp that intersects each fiber, so the (infimal) circumference is zero.

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If S has punctures, M_f will have a cusp that intersects each fiber, so the (infimal) circumference is zero.

Instead, $M_f \setminus \{\text{cusps}\}$ has large circumference.

Remark: Truncating cusps puts you in a setting with tools similar to those for relatively hyperbolic groups.

Thank you!

Hyperbolic Origami Surfaces
arxiv:2509.18668

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April 18, 2026

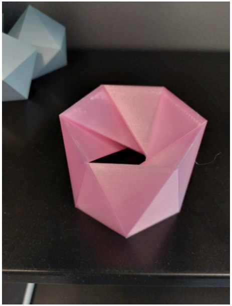
BROWN

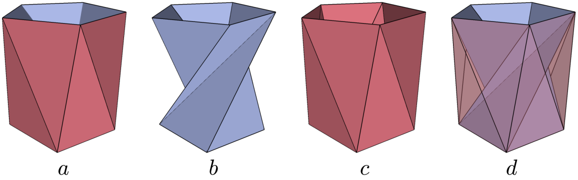
Origami Surfaces

- Does there exists a triangular mesh in $\mathbb{R}^3$ that is an embedded genus-g surface such that the cone angles around all vertices are 2π?
- For the torus (genus $= 1$), **yes**.
- Call a map $\mathbb{R}^2/\Lambda \hookrightarrow \mathbb{R}^3$ a **paper torus** if it's PL with respect to a triangulation of $\mathbb{R}^2/\Lambda$, such that the image of each triangle is a triangle congruent to it.
- First existence result of paper torus: Burago–Zalgaller [BZ60].
- Lazarus–Tallerie [LT24]: a universal triangulation with 1217 vertices (and 2434 faces) that realizes every isometry class of flat tori.

Vertex-Minimal

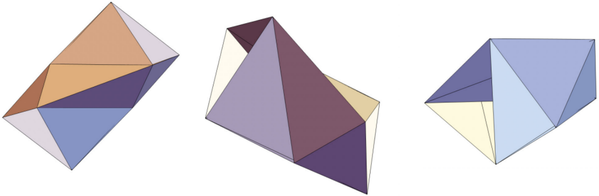
- What is the minimum number of vertices required to obtain a paper torus?
- Diplo-tori: Tsuboi [Tsu20]; Arnoux–Lelievre–Malaga





Vertex-Minimal

- Schwartz [Sch25]: no paper torus with 7 vertices, but there exists paper tori with 8 vertices with 2-fold symmetry that acts by a hyperelliptic involution.
- Doyle–Schwartz [DS25]: a family of 8-vertex paper tori that realizes "almost every class of flat torus."



Embedding a Genus-g Surface

- For $g \geq 2$, no paper genus-g surfaces in $\mathbb{R}^3$ since the Euler characteristic of S_g is negative.
- We should instead consider embeddings of S_g into $\mathbb{H}^3$.

Hyperbolic Origami Genus-g Surface

Definition

A **hyperbolic origami genus-g surface** is a 3-tuple $(X, \mathcal{T}, \phi)$ where:

- X is a hyperbolic genus-g surface.
- $\mathcal{T}$ is a geodesic triangulation of X.
- $\phi : X \hookrightarrow \mathbb{H}^3$ is an embedding sending triangles in $\mathcal{T}$ isometrically to geodesic triangles in $\mathbb{H}^3$.

One can construct hyperbolic origami surfaces from a combinatorial triangulation of S_g by specifying the coordinates of the vertices in $\mathbb{H}^3$ and connect the corresponding geodesic triangles, and then check that no two triangles intersect (except for pairs sharing edges, where they intersect on the edge only), and the cone angles are all 2π.

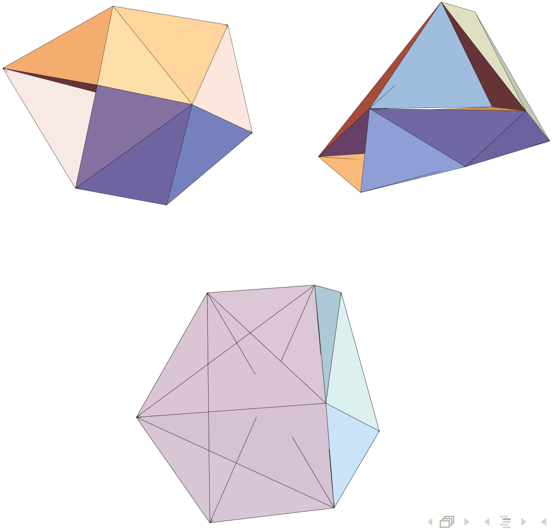
Main Result

Theorem (Z. 2025)

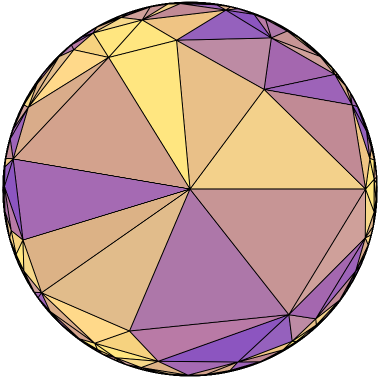
There exists a hyperbolic origami genus-2 surface with 10 vertices.

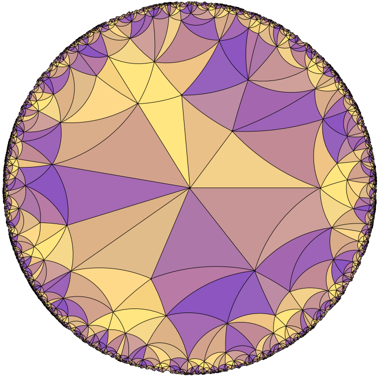
- Push the button: Find an embedded genus-2 surface $\hat{\varphi} : X \hookrightarrow \mathbb{H}^3$ whose cone angles are almost 2π.
- Clean–up: Use the inverse function theorem to show that there exists a hyperbolic origami genus-2 surface around a neighborhood of $\hat{\varphi}$.

Demonstration



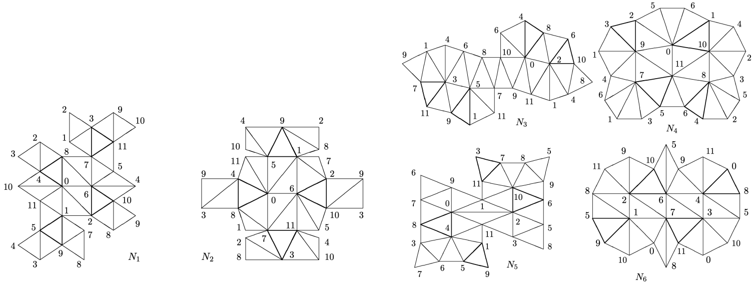
Universal Cover



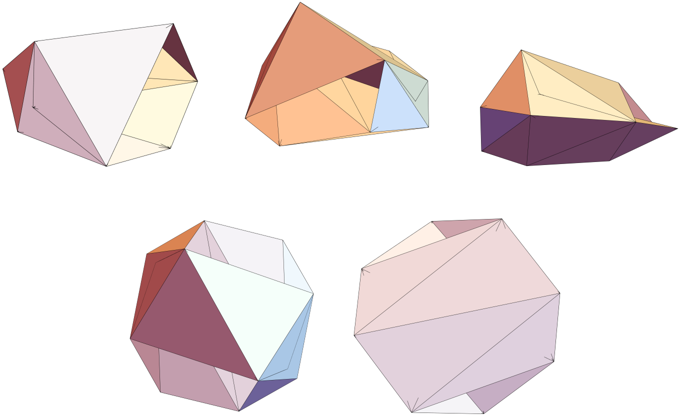


A Model with 2-fold Symmetry

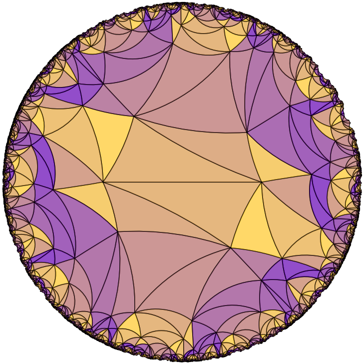
- I found a hyperbolic origami 2-torus with 12 vertices using a triangulation where each vertex has degree 7.
- Datta–Upadhyay [DU06]: six 12-vertex degree-regular triangulations of S_2.

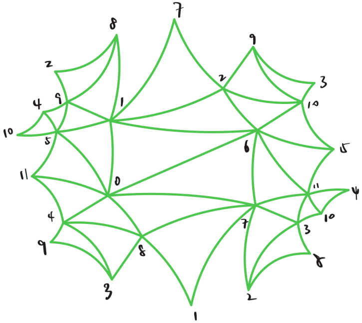


Demonstration



Universal Cover





Approximation

- I found via numerical optimization an embedding of a triangulated S_2 with 10 vertices, where the cone angles differ from 2π by $\leq 10^{-28}$ (computed with floating point arithmetic).

$$
\begin{array}{cccccc}
(0,1,7) & (0,2,1) & (0,3,6) & (0,4,9) & (0,5,3) & (0,6,4) \\
(0,7,8) & (0,8,5) & (0,9,2) & (1,2,4) & (1,3,7) & (1,4,6) \\
(1,5,8) & (1,6,5) & (1,8,3) & (2,3,8) & (2,6,3) & (2,7,4) \\
(2,8,7) & (2,9,6) & (3,4,7) & (3,5,4) & (4,5,9) & (5,6,9)
\end{array}
$$

- Degree sequence of the vertices:

$$(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9) = (9,\ 8,\ 8,\ 8,\ 8,\ 7,\ 7,\ 6,\ 6,\ 5).$$

Approximation

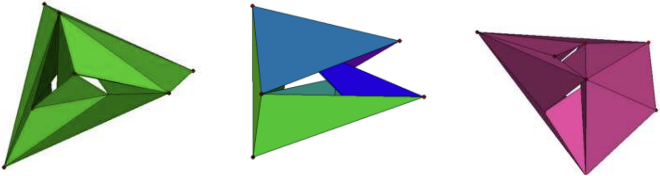
Coordinates of the vertices in $\mathbb{H}^3$:

0.7315	0.0202	0.28688022781563440615364787558404
−0.316	0.5792	−0.2252919753182150895621576631383
0.3426	−0.592	−0.22851917827575874828458055465199
−0.4323	−0.592	−0.23272863894943839798113773793091
−0.7303	0.04	−0.22959077803009316431117678104662
0.1464	0.6149	0.13588682780065943976868637469759
−0.5154	0.0395	0.46102777383206591809202059407883
0.6649	−0.1156	−0.22651115997910956793325851442687
0.152	0.2539	−0.23985732806740791506457015735734
−0.03	0.0606	0.64396456614136038886542316026992

Record the z-coordinates of this approximation as a vector $\hat{Z} \in \mathbb{R}^{10}$.

Approximation

- Lutz [Lut08] found PL embeddings of every 10-vertex triagulation of a 2-torus into $\mathbb{R}^3$.
- Starting with these embeddings, normalize so that they are in the unit ball. Then, implement a hillclimbing algorithm to minimize the difference between cone angles and 2π.
- One of the 865 triangulations worked!



Future directions

- Can we find explicit coordinates for these hyperbolic origami surfaces?
- Can we find higher genus hyperbolic origami surfaces?
- Can we find an immersion of a non-orientable triangulated surface into $\mathbb{R}^3$ or $\mathbb{H}^3$ whose cone angles are all 2π, and the link around each vertex is embedded?
- Which points in the moduli space of the 2-torus are obtained from hyperbolic origami 2-tori? In other words, is the triangulation we used for the 2-torus a "universal triangulation?"

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Finiteness of closed arithmetic surface bundles over the circle

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GATSBY Spring ‘26

introduction...

- *(Thurston, BAMS, 1982) "Find topological and geometric properties of quotient spaces of* **arithmetic** *subgroups of* $PSL(2, \mathbb{C})$. *These manifolds often seem to have* **special beauty**"

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- *(Thurston, BAMS, 1982) "Find topological and geometric properties of quotient spaces of* **arithmetic** *subgroups of* $PSL(2,\mathbb{C})$*. These manifolds often seem to have* **special beauty** *"*
- Arithmetic hyperbolic 3-manifolds are constructed using number theory. How? Their fundamental groups in $PSL(2,\mathbb{C})$ mimic how $SL(2,\mathbb{Z})$ sits in $SL(2,\mathbb{R})$ and how $SL(2,\mathbb{Z}[i])$ sits in $SL(2,\mathbb{C})$.

special beauty could be .... cool properties/examples?

Arithmetic hyperbolic 3-manifolds include:

- examples of closed hyperbolic 3-manifold with no hyperbolic elements, like the **Weeks** manifold (Maclachlan-Reid)

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- examples of hyperbolic 3-manifolds with **all** closed geodesics simple (Chinburg-Reid)

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Arithmetic hyperbolic 3-manifolds include:

- examples of closed hyperbolic 3-manifold with no hyperbolic elements, like the **Weeks** manifold (Maclachlan-Reid)
- examples of hyperbolic 3-manifolds with **all** closed geodesics simple (Chinburg-Reid)
- examples of towers of hyperbolic rational homology 3-spheres (Calegari-Dunfield, Boston-Ellenberg)

special beauty could be .... scarcity?

- Finitely many arithmetic hyperbolic 3-manifolds with bounded volume (Borel)

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special beauty could be .... scarcity?

- Finitely many arithmetic hyperbolic 3-manifolds with bounded volume (Borel)
- Finitely many arithmetic hyperbolic 3-manifolds (up to taking finite-sheeted covers) with bounded injectivity radius *below* and rank of fundamental group *above* (Biringer-Souto)
- Finitely many arithmetic surface bundles (up to taking finite-sheeted cyclic covers) over S^1 of fixed topological type and bounded (invariant) trace field degree *above* (Bowditch-Machlaclan-Reid)

surface bundles over S^1

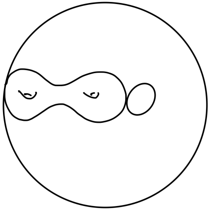

Figure 1: a fibered 3-manifold

- a fibered 3-manifold is the mapping torus M_f of a surface diffeomorphism $f : \Sigma \to \Sigma$.

surface bundles over S^1

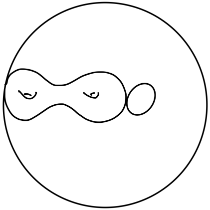

Figure 1: a fibered 3-manifold

- a fibered 3-manifold is the mapping torus M_f of a surface diffeomorphism $f : \Sigma \to \Sigma$.
- When the diffeomorphism f is isotopic to a **pseudo-Anosov** diffeomorphism of Σ, M_f is (finite-volume) hyperbolic (Thurston, Sullivan)

main theorem

Theorem (C-Lee-Miller)

Fix a natural number $g \geq 2$. There are at most finitely arithmetic hyperbolic surface bundles over S^1 with fiber a genus g surface up to taking finite-sheeted cyclic covers.

and...

Thank you for listening.