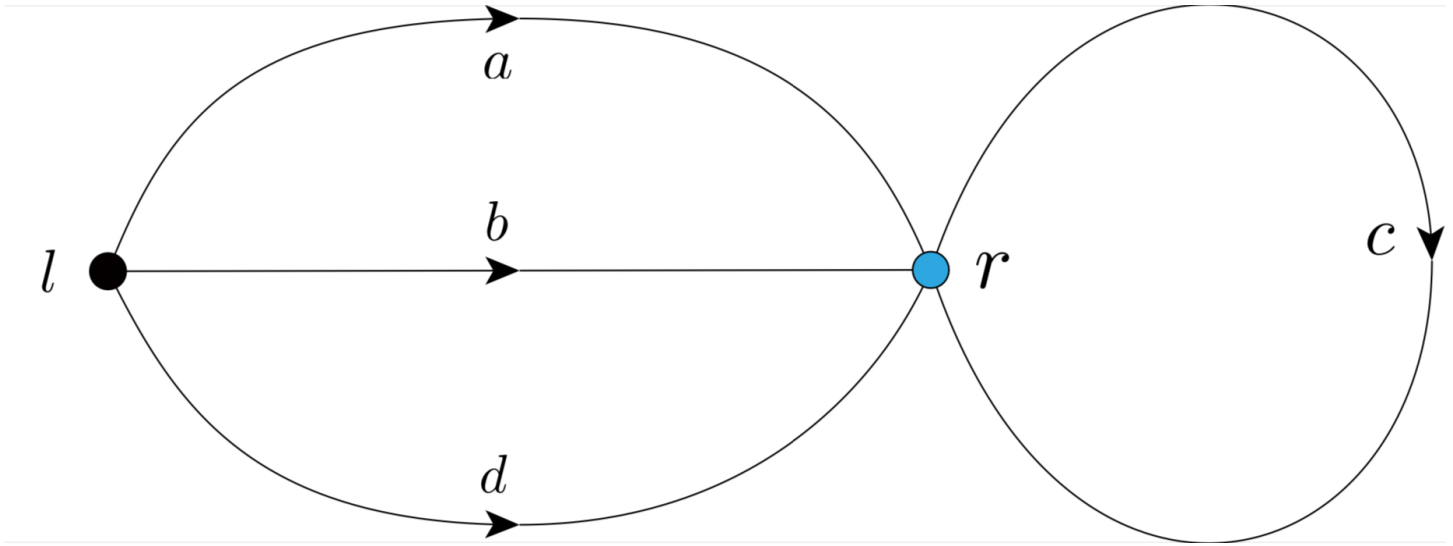
# TRANSVERSE GRAPHS AND SPIRALING LAMINATIONS IN GRAPH MAPPING TORI — AN EXAMPLE

Joint work with Khanh Le
Junzhi Huang

The example is from work by Dowdall-Kapovich-Leininger [DKL]

Let $\Gamma$ be this graph:

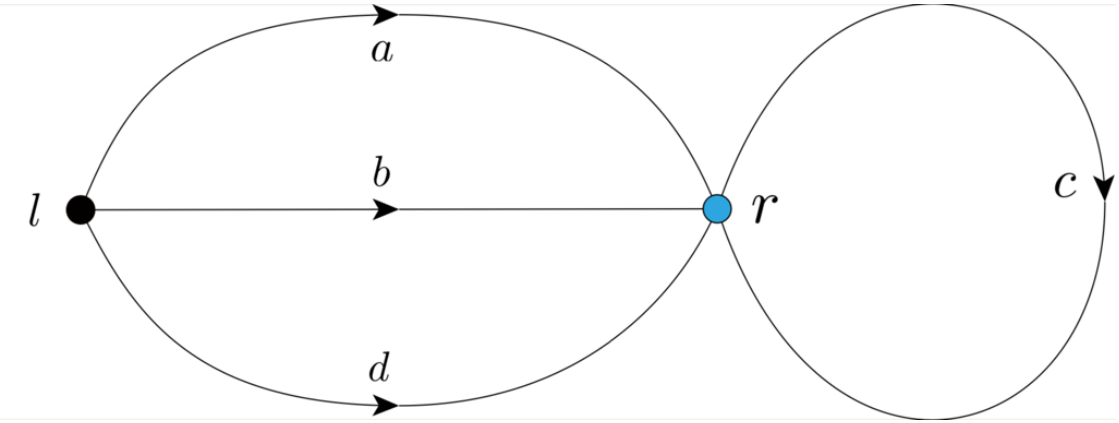



Define a homotopy equivalence $f: \Gamma \to \Gamma$ by

$$f(a) = d,\ f(b) = a,\ f(c) = b^{-1}a,\ f(d) = ba^{-1}db^{-1}ac$$

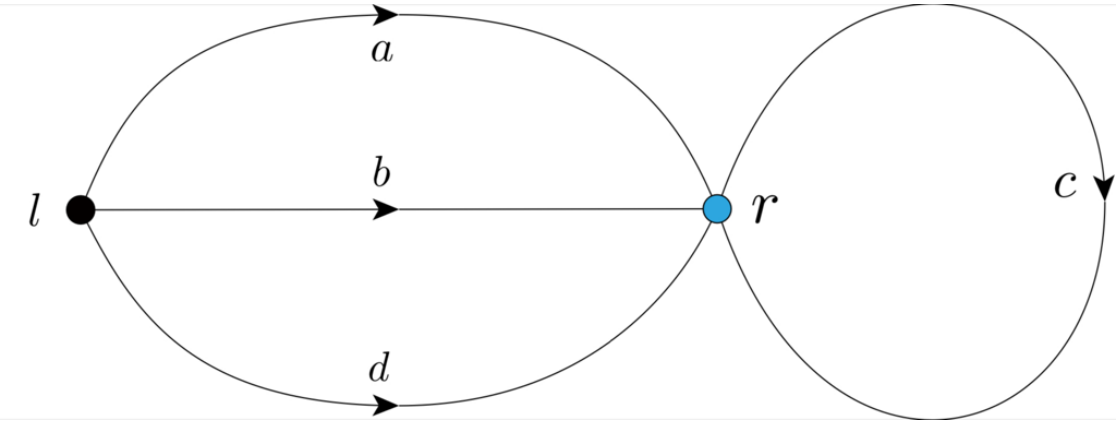
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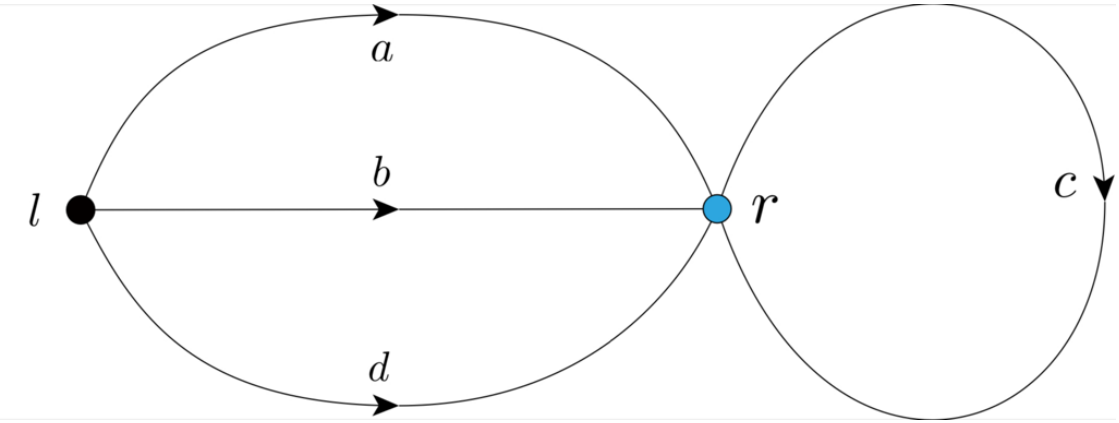



This map is:

irreducible & expanding : transition matrix has a positive power & Perron-Frobenius eigenvalue $> 1$.

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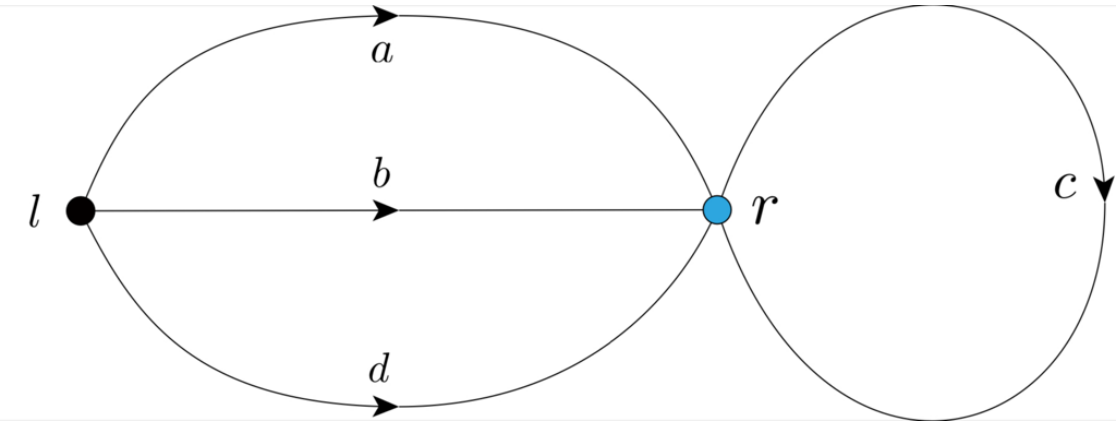

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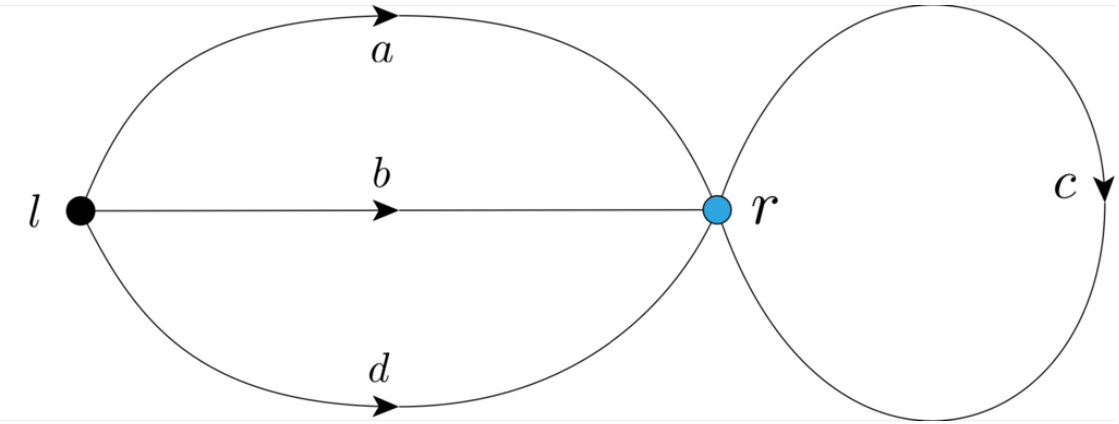
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For example: $a \to d \to ba^{-1}db^{-1}ac \to ad^{-1}ba^{-1}db^{-1}ac\,a^{-1}db^{-1}a \to \cdots$

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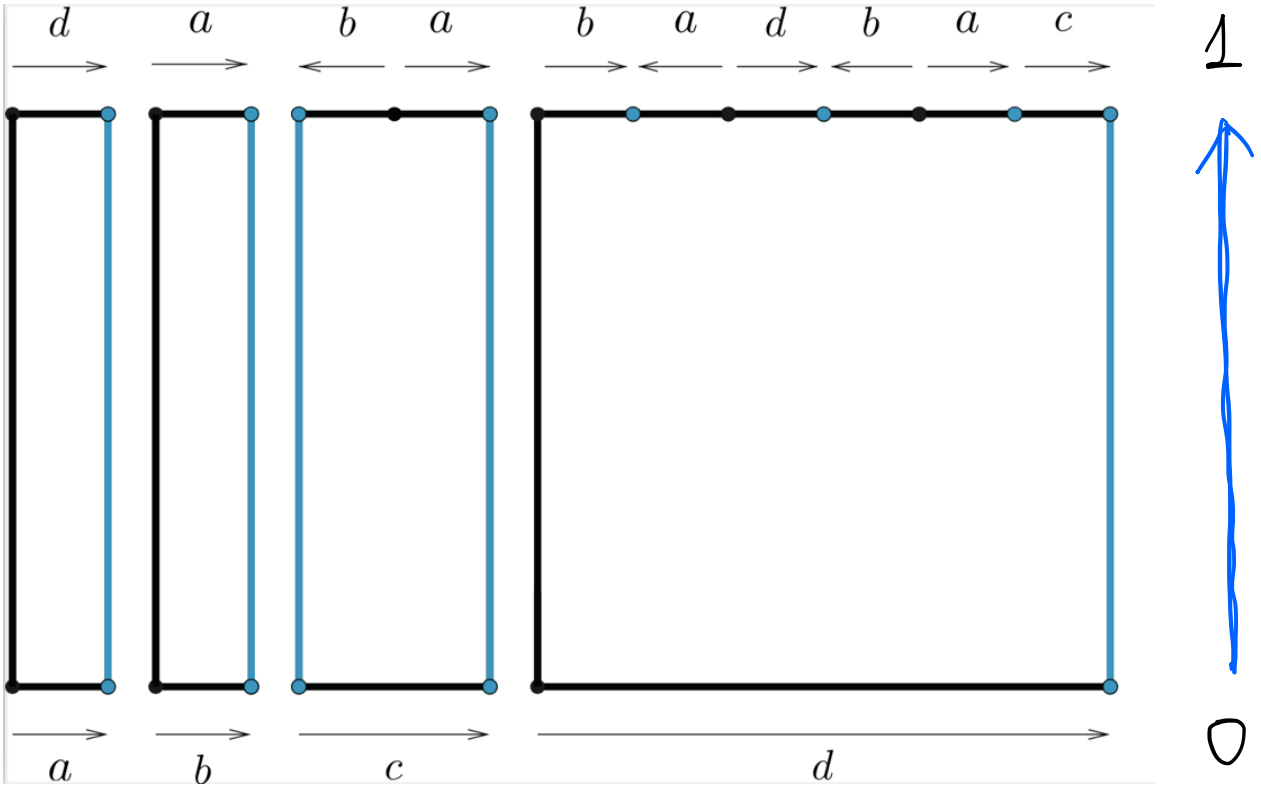
For example: $a \to d \to ba^{-1}db^{-1}ac \to ad^{-1}ba^{-1}db^{-1}ac\,a^{-1}db^{-1}a \to \cdots$

By taking a limit and taking a closure. we obtain the attracting lamination $\Lambda(f)$. which is an $f$-invariant set of immersions $\mathbb{R} \looparrowright \Gamma$. (leaves)

We build a mapping torus of $f$ by

$$Z_f = \Gamma \times [0,1] \Big/ (x,1) \sim (f(x),0).$$

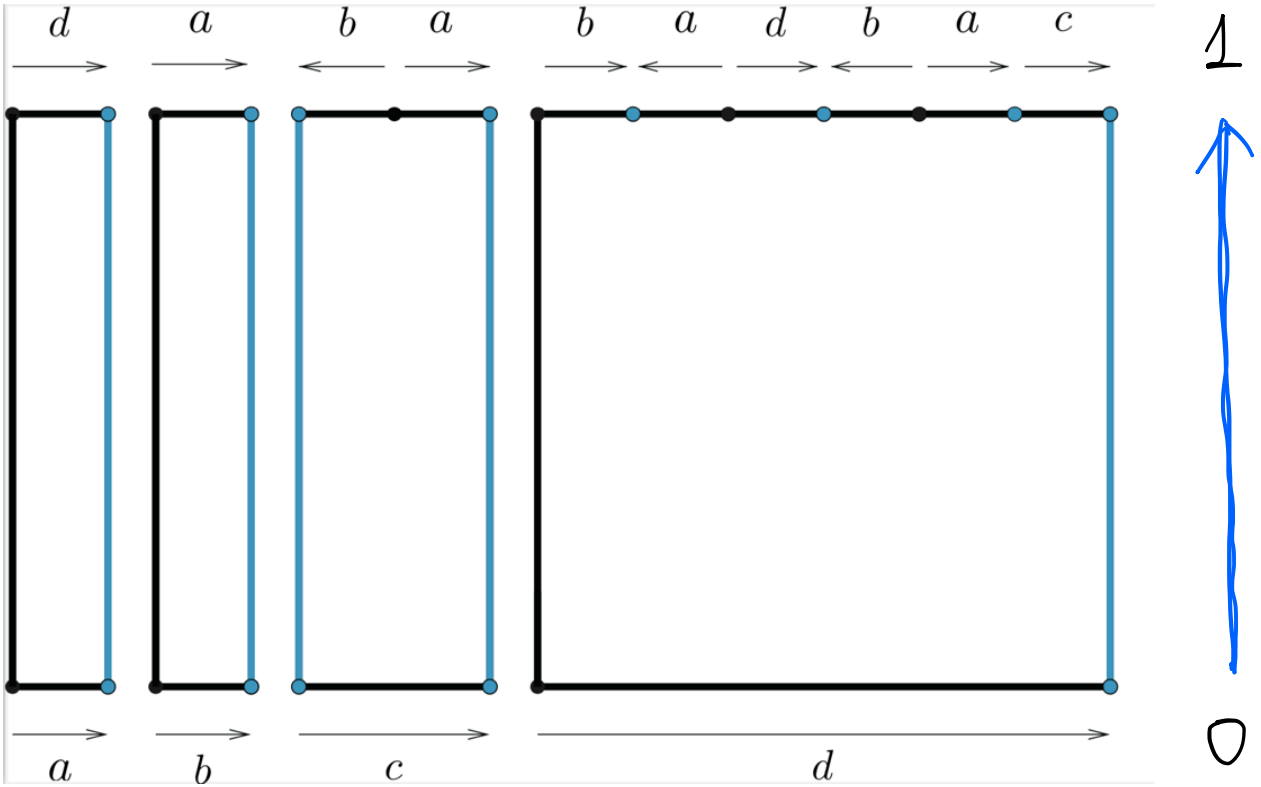
$Z_f$ has a natural CW complex structure. All the 2-cells are rectangles.



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There is a vertical semi-flow $\varphi$ on $Z_f$. (i.e. an $\mathbb{R}_+$-action).

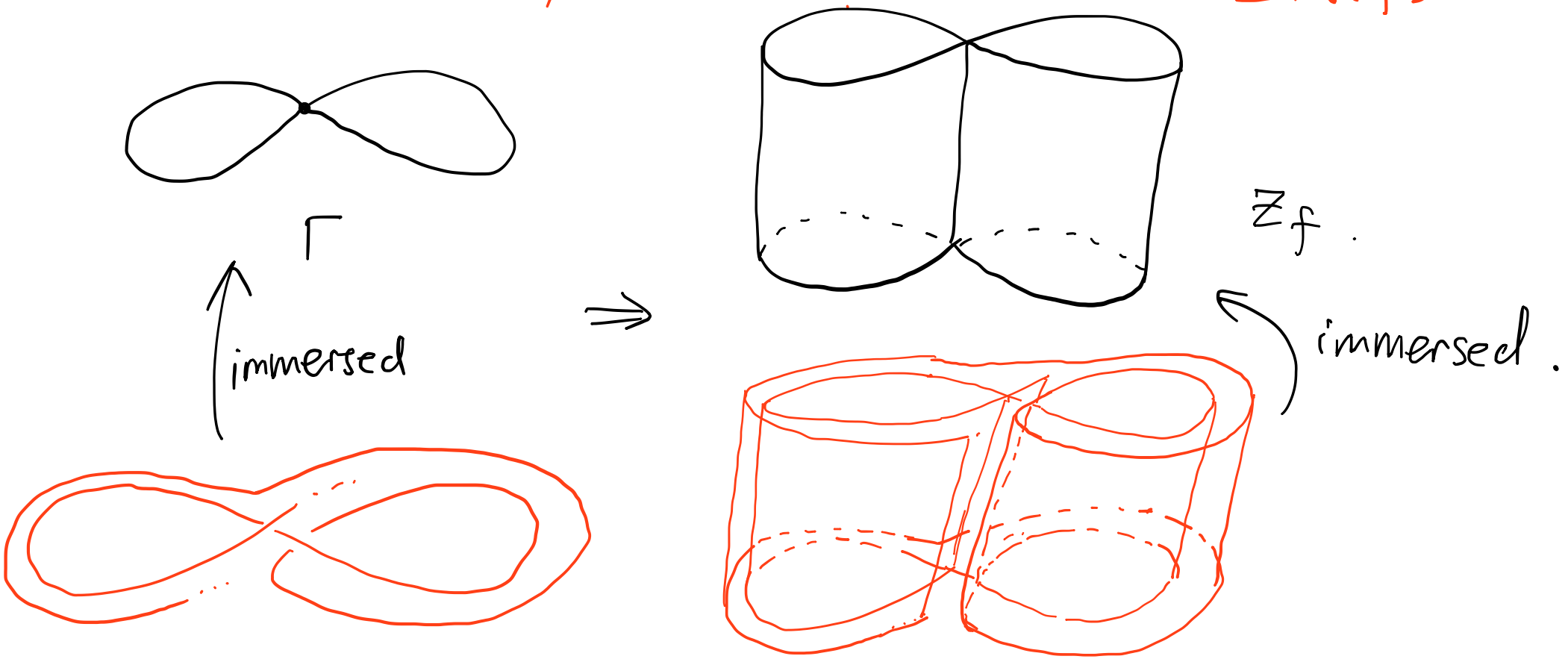
This is just the Suspension semi-flow.

$\Gamma$ is transverse to $\varphi$.

Recall that we have the attracting lamination $\Lambda(f)$. which is an $f$-inv set of immersed lines in $\Gamma$.

Recall that we have the attracting lamination $\Lambda(f)$, which is an $f$-inv set of immersed lines in $\Gamma$.

We can now "suspend" leaves of $\Lambda(f)$ along $\varphi$ to get a set of immersed planes in $Z_f$, which we denote by $\mathcal{Z}\Lambda(f)$

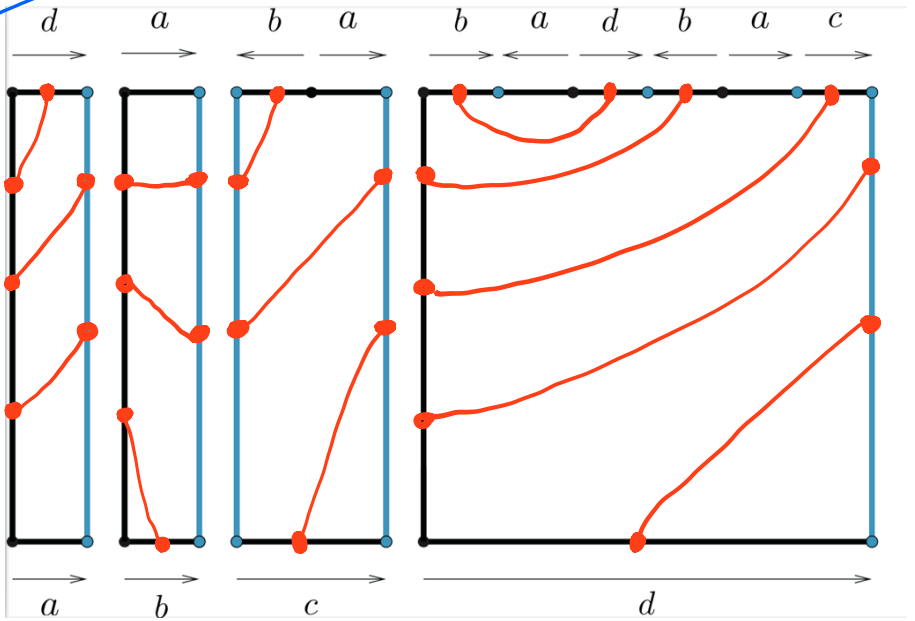


Recall that $\Gamma \subseteq Z_f$ is transverse to $\varphi$.
There are other transverse graphs.

Recall that $\Gamma \subseteq Z_f$ is transverse to $\varphi$. [DKL]: They form a "fiber cone" in $H^1(Z_f, \mathbb{R})$.

There are other transverse graphs.

## ① Cross section



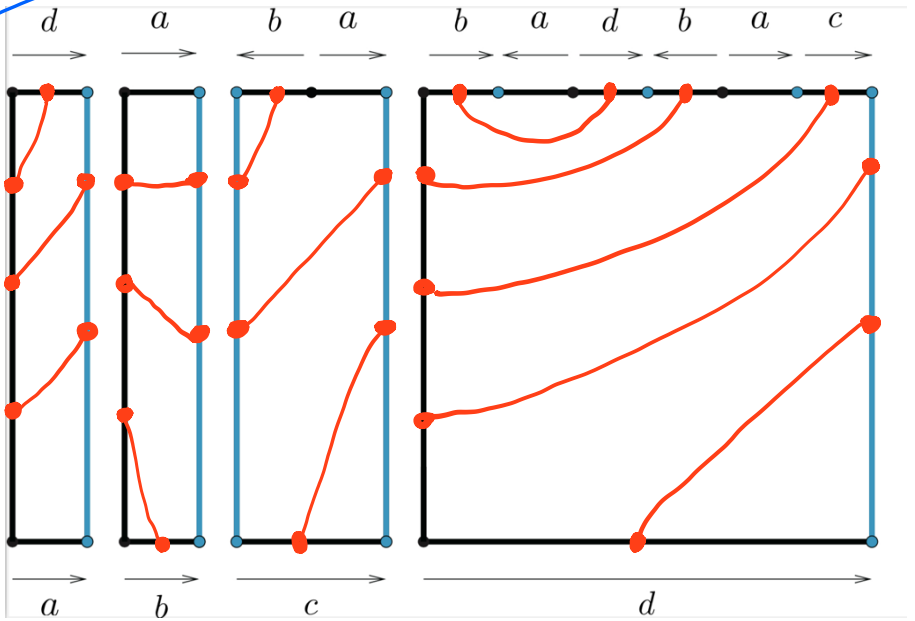

$H$

④ intersects every forward flowline

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## ① Cross section




$\oplus$

$\oplus$ intersects every forward flowline

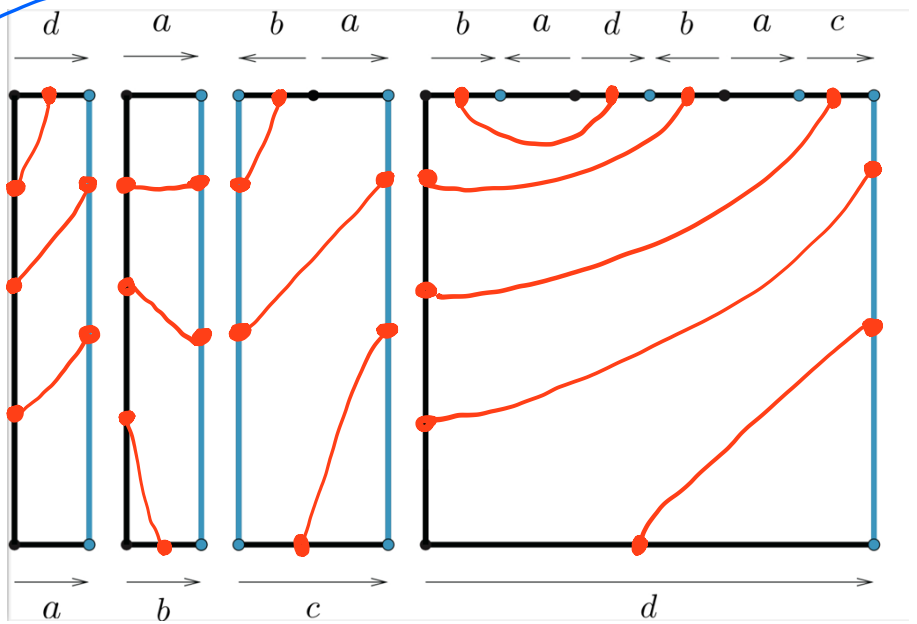
$\Rightarrow$ there is a first return map $g: \oplus \to \oplus$

[DKL]: $g$ is an irreducible, expanding train track map.

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## ① Cross section




$H$ intersects every forward flowline

$\Rightarrow$ there is a first return map $g: H \to H$

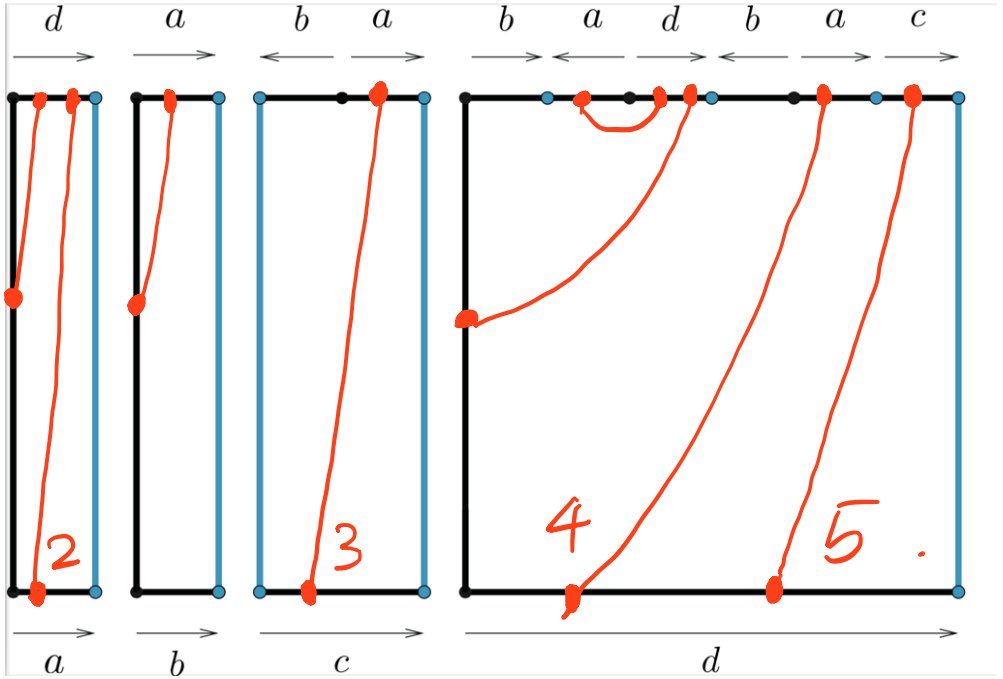
[DKL]: $g$ is an irreducible, expanding train track map.

On the other hand, the intersection of $H$ with $Z\Lambda(f)$, the immersed planes, gives a lamination $\Lambda_H$ on $H$.

**Thm A (H-Le):** $\Lambda_H$ = the attracting lamination of $g$.

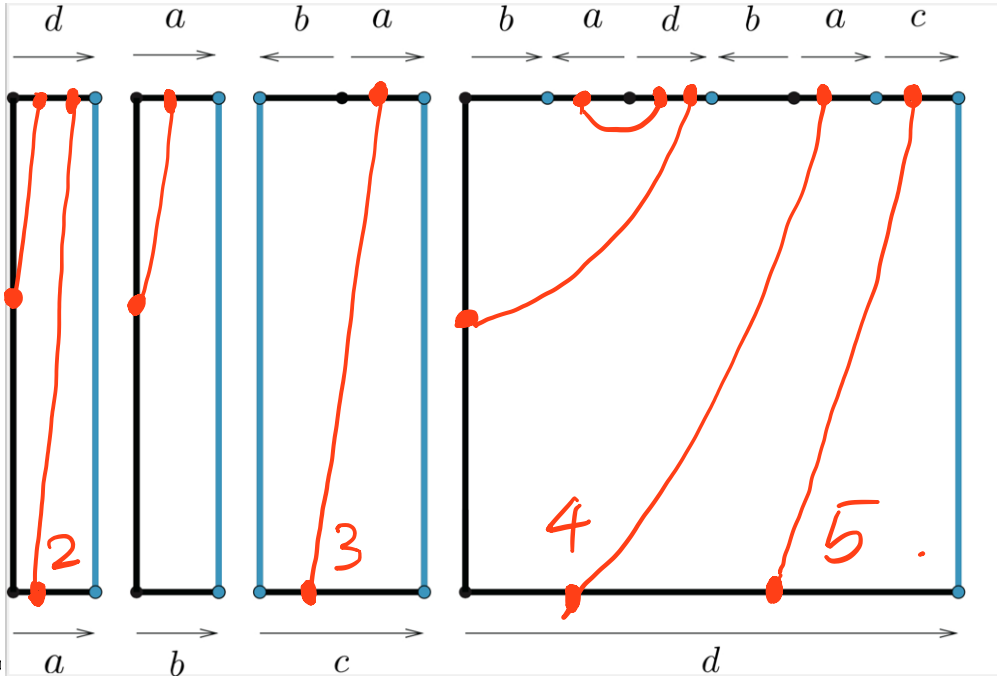
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They lie on the boundary of the "fiber cone" in $H^1(Z_f;\mathbb{R})$



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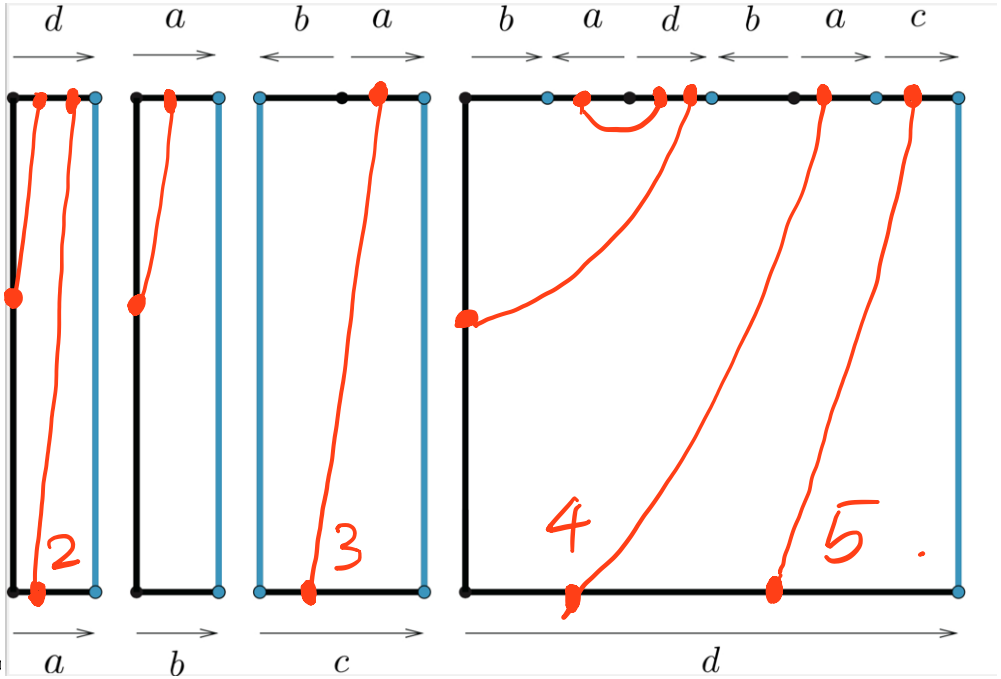
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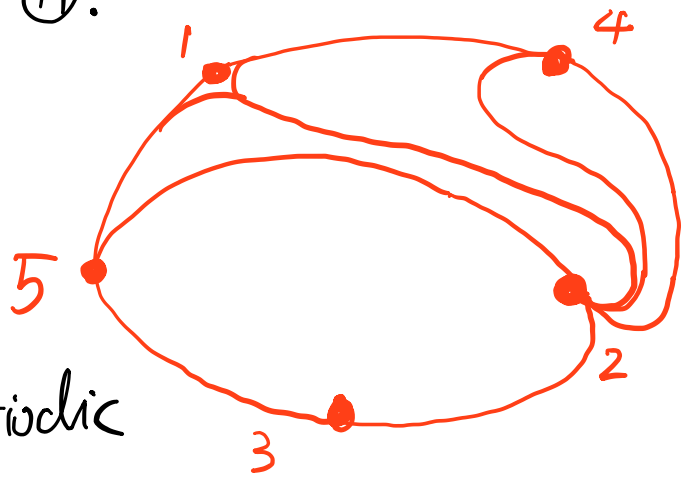

Again, the intersection of ⓗ with the immersed planes in $Z\Lambda(f)$ gives a lamination $\Lambda_{ⓗ}$ on ⓗ.

## ②. Non-section

They lie on the boundary of the "fiber cone" in $H^1(Z_f;\mathbb{R})$




Again, the intersection of ⊕ with the immersed planes in $Z\Lambda(f)$ gives a lamination $\Lambda_{\oplus}$ on ⊕.




Thm B (H-Le): $\Lambda_{\oplus}$ is a spiraling lamination. i.e. every leaf (an immersed line) either is periodic or eventually become periodic on both ends.

THANK You!

# Tame braids and a Birman exact sequence for infinite type surfaces

Jordi Andres Martinez Alvarez

# Assumptions

$S$ closed orientable surface.

$K \subset S$ homeomorphic to a closed subset of a Cantor set.

# Notation

$$\mathrm{PHomeo}(S, K) \coloneqq \{g \in \mathrm{Homeo}(S) \mid g \circ \iota = \iota\}$$

$$\mathrm{PMCG}(S, K) \coloneqq \pi_0 \mathrm{PHomeo}(S, K).$$

We equip $\mathrm{PMCG}(S, K)$ with the quotient topology.

# Finite type Birman exact sequence

Let $K \subset S$ be finite.

$$\mathrm{PB}_K(S) \coloneqq \pi_1(\mathrm{Emb}(K, S); \iota_K).$$

$$\mathcal{P} : \mathrm{PB}_K(S) \to \mathrm{PMCG}(S, K)$$

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Comes from a fiber bundle

$$\mathrm{PHomeo}(S,K) \hookrightarrow \mathrm{Homeo}(S) \to \mathrm{Emb}(K,S).$$

# The point-pushing map

Choose an isotopy $(f_t)$ of $S$ that restricts on $K$ to the braid $[\alpha]$.

Define

$$\mathcal{P}[\alpha] = [f_1]$$

# The isotopy extension problem

Braids on infinite $K$ need not extend.

# Tame braid group

A braid is called *tame* if it is the restriction to $K$ of some isotopy of $S$

**Tame braid group:** $\mathrm{PB}_K^T(S)$

# The Birman exact sequence

If S has genus at least 2, we have a short exact sequence

$$1 \to \mathrm{PB}_K^T(S) \xrightarrow{\mathcal{P}} \mathrm{PMCG}(S,K) \xrightarrow{\mathcal{F}} \mathrm{MCG}(S) \to 1$$

# The natural quotient topology

**Relative loop space:**

$$\Omega(X, A; x_0) := \{\gamma : [0, 1] \to X \mid \gamma(0) = x_0 \text{ and } \gamma(1) \in A\}.$$

endowed with the compact-open topology.

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endowed with the compact-open topology.

**Relative fundamental set:**

$$\pi_1(X, A; x_0) \coloneqq \pi_0\Omega(X, A; x_0)$$

with the quotient topology.

# Topological Birman exact sequence

**Proposition.** $\mathrm{PB}_K^T(S)$ is isomorphic to $\pi_1(\mathcal{H}, \mathcal{H}_K; \mathrm{Id}_S)$ as a group.

$$\mathcal{H} = \mathrm{Homeo}(S)$$

$$\mathcal{H}_K = \mathrm{PHomeo}(S, K)$$

# Topological Birman exact sequence

**Proposition.** $\mathrm{PB}_K^T(S)$ is isomorphic to $\pi_1(\mathcal{H}, \mathcal{H}_K; \mathrm{Id}_S)$ as a group.

**Theorem (Martinez Alvarez.)** If the genus of S is at least 2, then $\pi_1(\mathcal{H}, \mathcal{H}_K; \mathrm{Id}_S)$ is isomorphic to $\ker \mathcal{F}$ as topological groups.

# Disjointness of small Julia sets in matings of quadratic rabbit tunings

Eduardo Ventilari Sodré

Brown University

November 14, 2026

# Quadratic dynamics on $\mathbb{C}$

**Holomorphic dynamics:** dynamics of holomorphic maps $f : X \to X$, complex manifold $X$

Simplest case: dynamics of quadratic polynomials on $\mathbb{C}$

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Every quadratic polynomial is affinely conjugate to $f_c(z) := z^2 + c$, for unique $c \in \mathbb{C}$

Dynamics is (partially) determined by the orbit $P = \{f_c^n(0) : n \in \mathbb{N}\}$ of the critical point $c_0 = 0$

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*Filled Julia set*:

$$K_c := \{z \in \mathbb{C} : |f_c^n(z)| \not\to \infty\}$$

# The Basilica Julia set

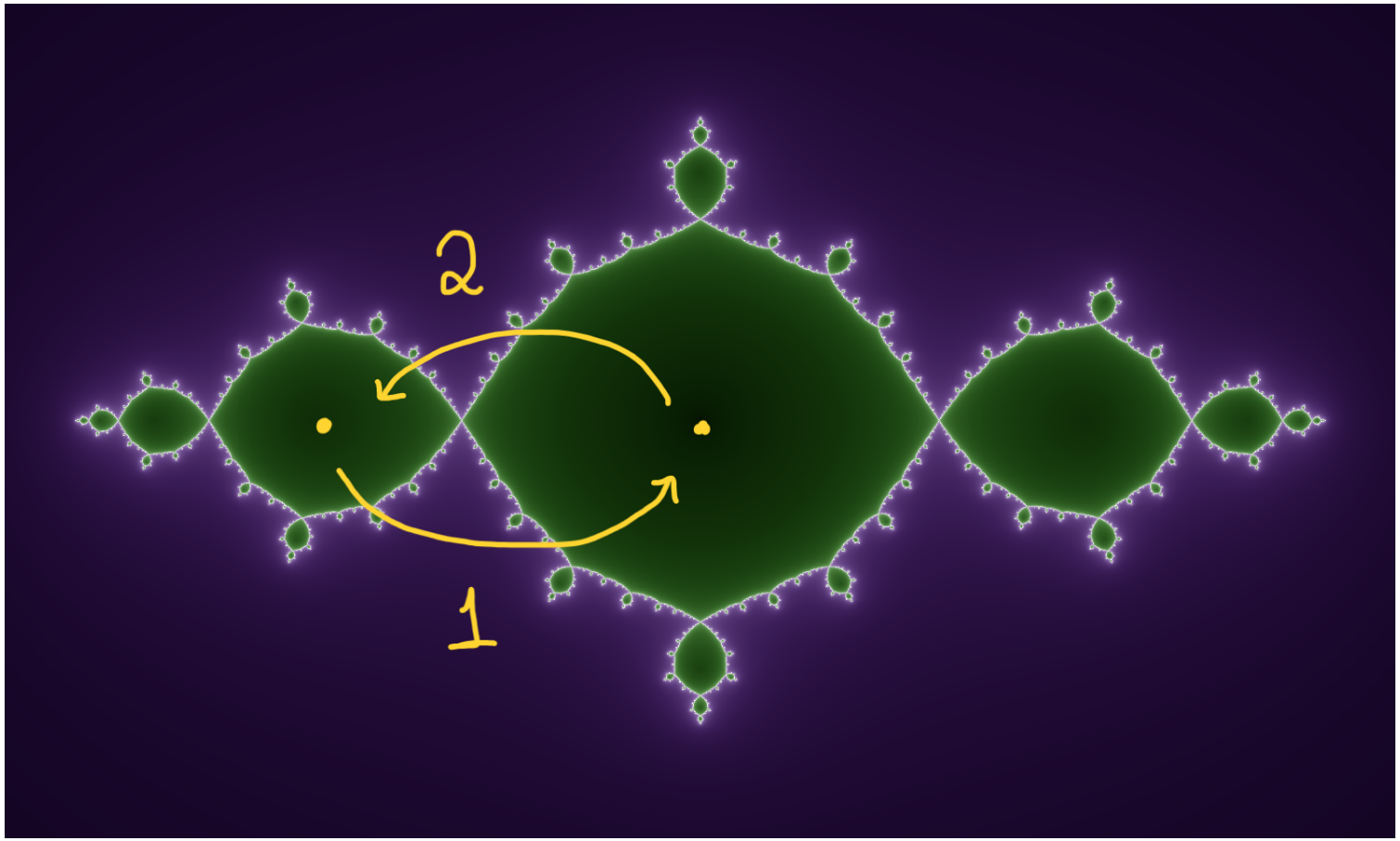



Figure: Filled Julia set for $z^2 - 1$. The orbit of the critical point is periodic of period 2.

# Airplane Julia set

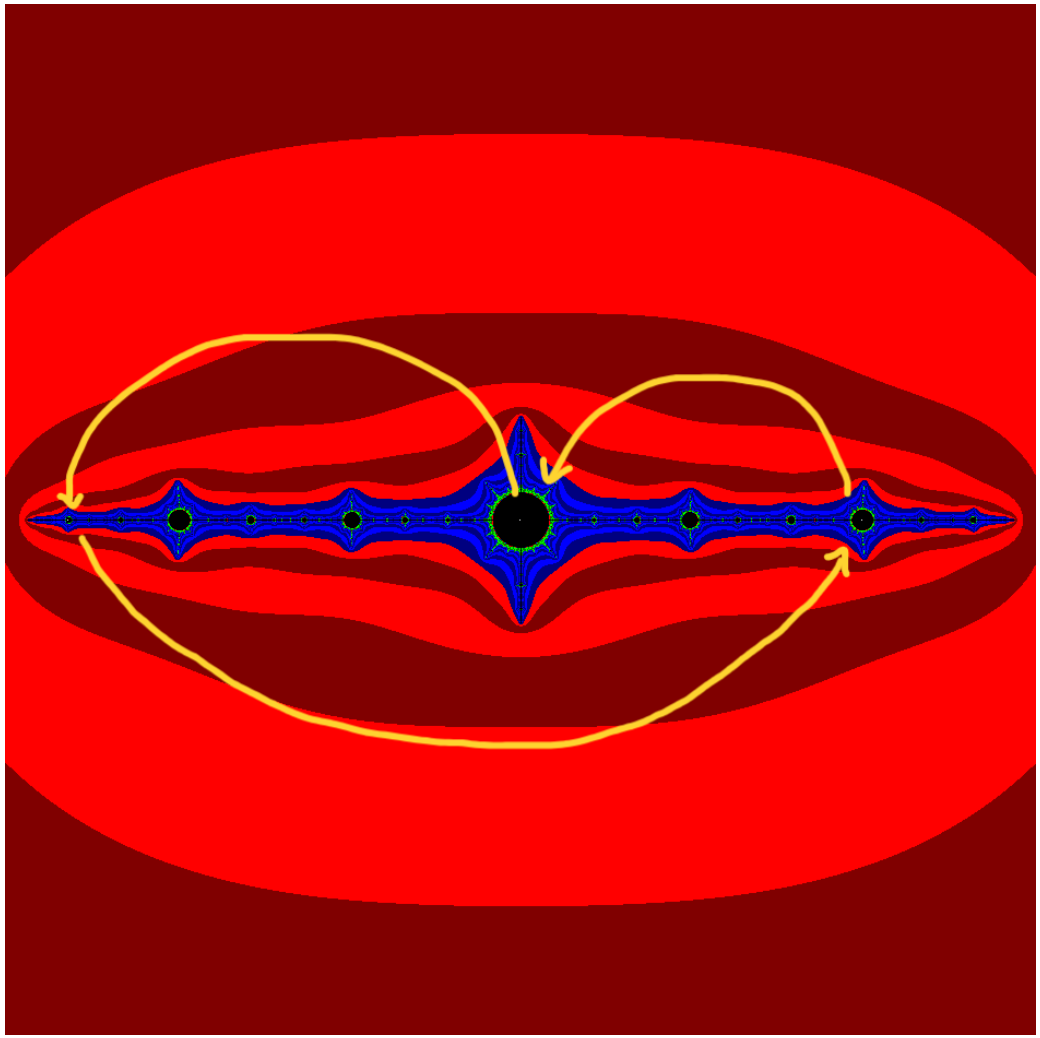


Figure: $c \approx -1.7548$: the airplane polynomial

# 1/3-rabbit polynomial

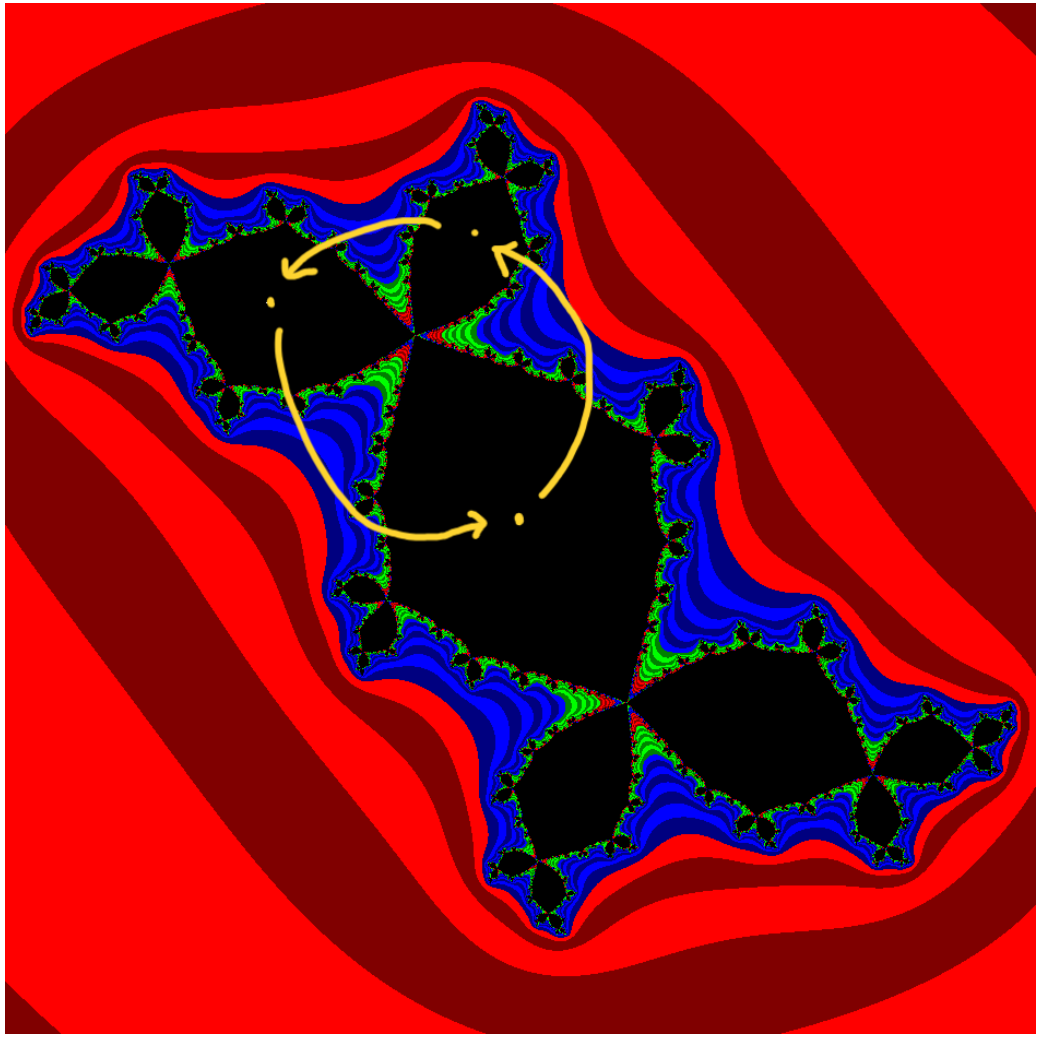


Figure: Critical dynamics has combinatorial rotation number 1/3.

# Tuning

Suppose $f_c$ has periodic critical point. Induces a cycle of open Jordan domains

$$U_0 \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_{p-1} \longrightarrow U_0$$

# Tuning

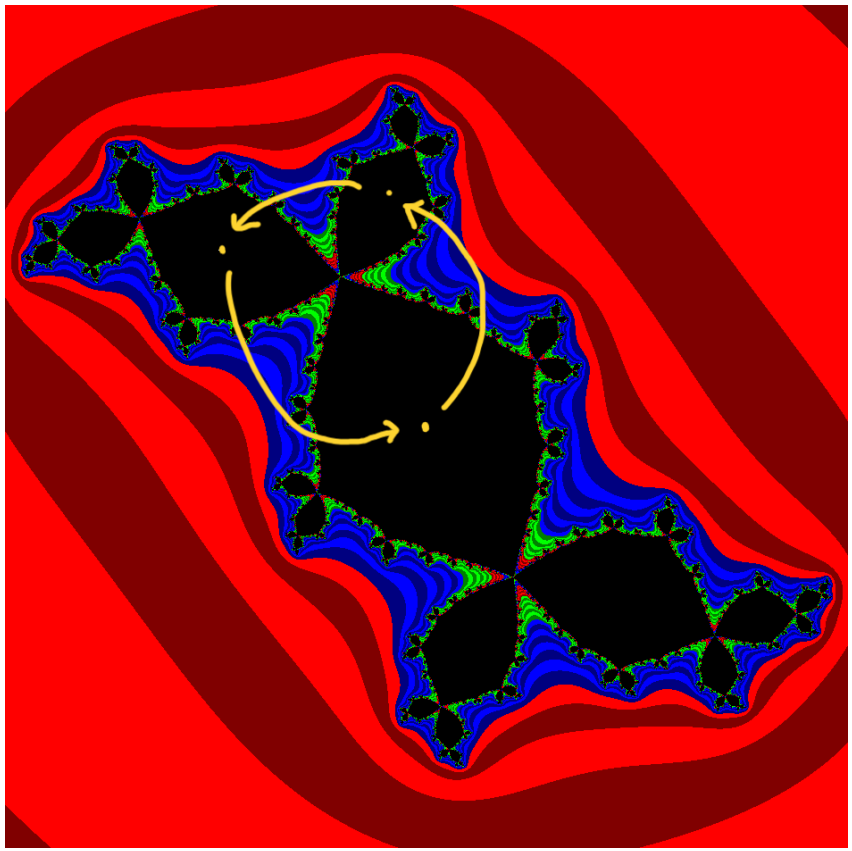
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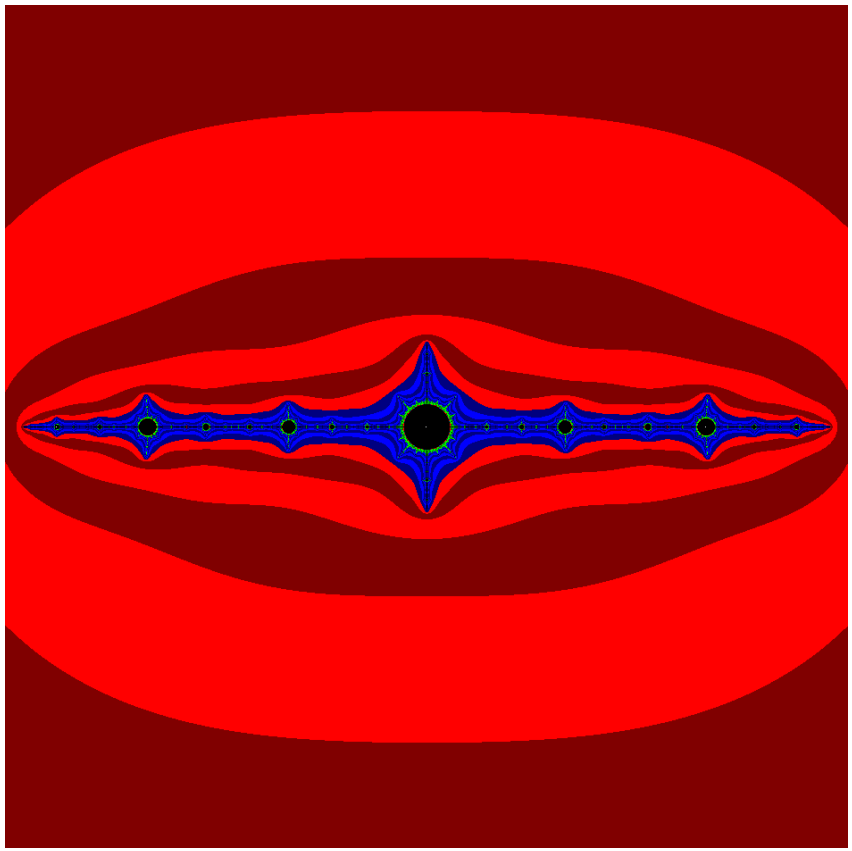
$$U_0 \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_{p-1} \longrightarrow U_0$$

Can replace each component by the filled Julia set $K_{c'}$ of another polynomial $f_{c'}$: produces the *tuning* $c * c'$

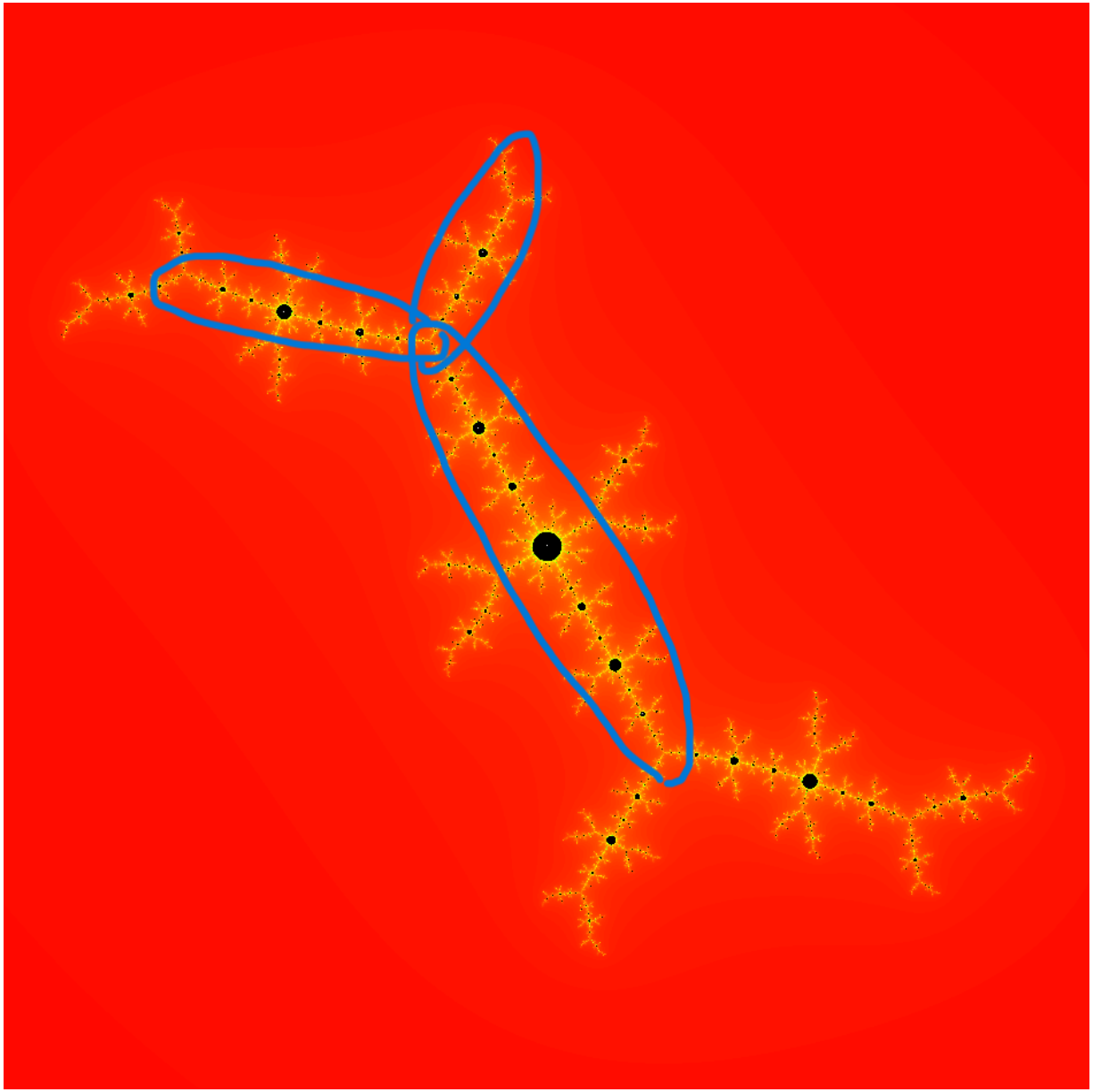
Each $U_i$ turns into a *small filled Julia set* $K_i$ homeomorphic to $K_{c'}$

# Tuning of 1/3-rabbit by the airplane





# Tuning of 1/3-rabbit by the airplane

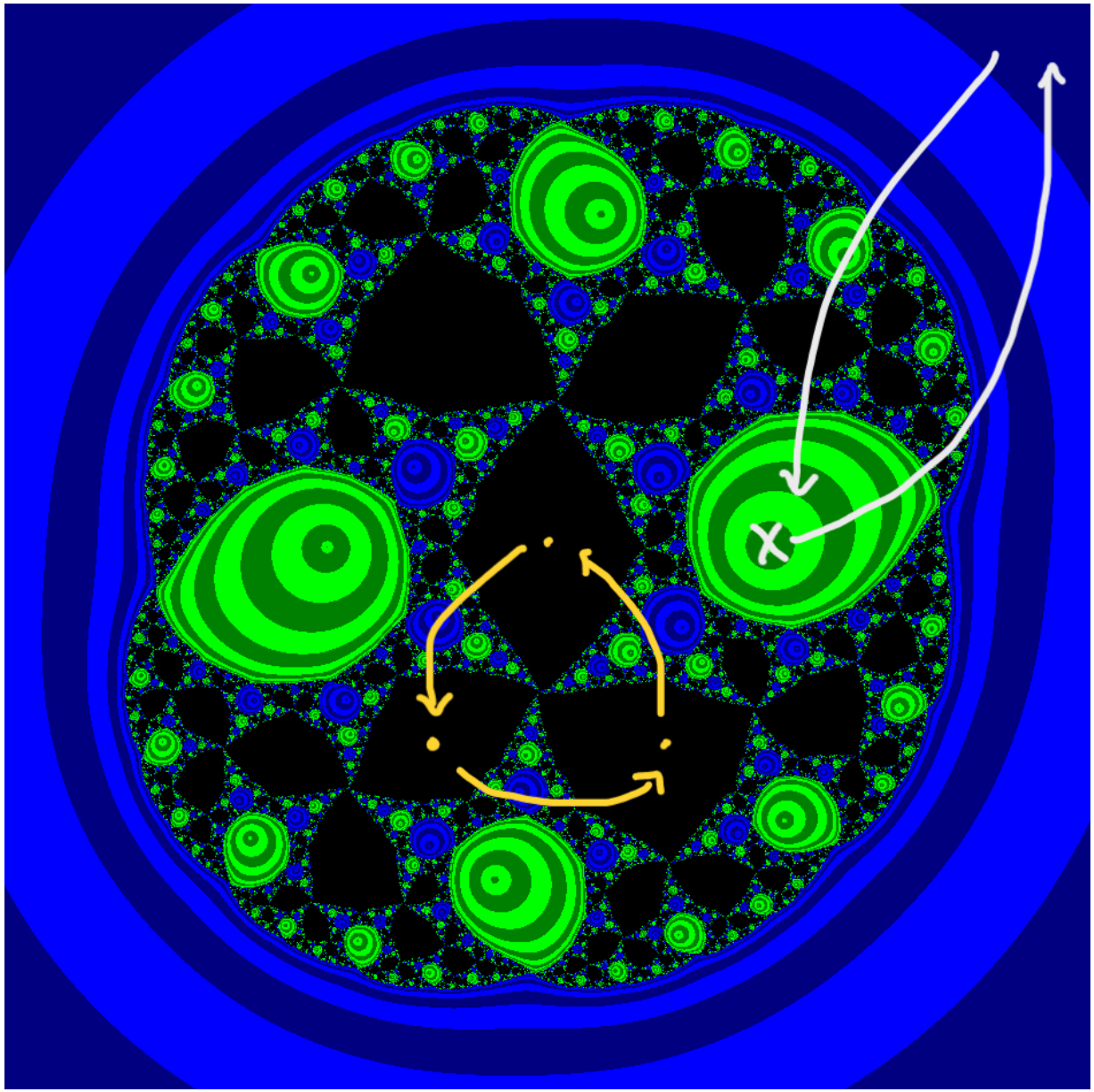


# Matings of polynomials

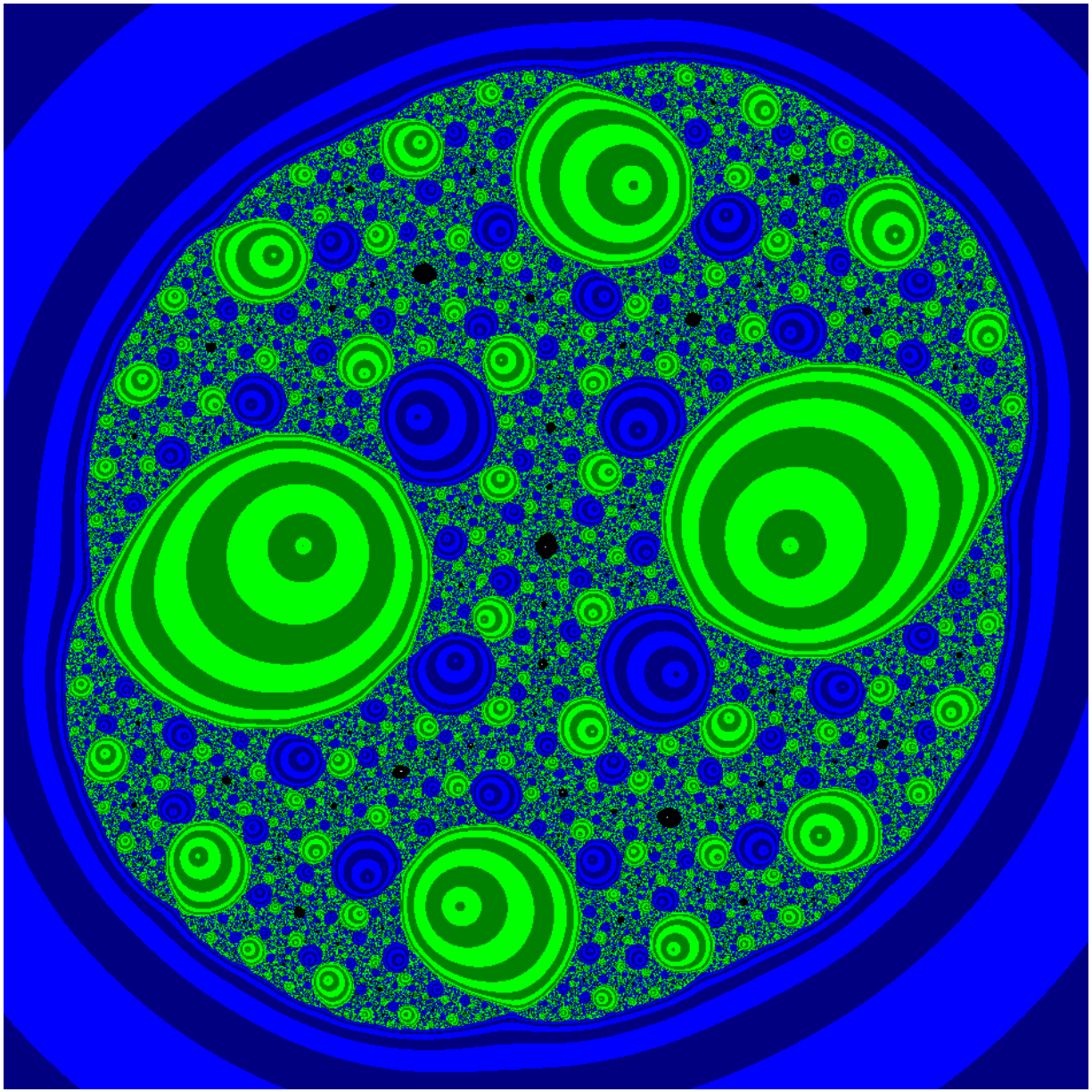
*Mating:* way of combining two polynomials $f$ and $g$ into a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, having the dynamics of both

Glue the two Julia sets $K_f$ and $K_g$ along their boundaries: if in luck, forms a topological 2-sphere with dynamics

# Mating of 1/3-rabbit with Basilica



# Mating of (rabbit tuned by airplane) with basilica



# Disjointness of the small Julia sets

## Theorem (V.S.)

*Let $c_1$ and $c_2$ be two tunings of rabbits having locally connected Julia sets. Then the small Julia sets of $f_{c_1}$ do not touch the small Julia sets of $f_{c_2}$ in the mating $f_{c_1} \sqcup f_{c_2}$.*

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Partially answers the question:

For a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, when are the closures of the immediate basins of attraction disjoint?

Thank you for the attention!

*Quadratic Hubbard trees, automata, and ray connections.* Arxiv. https://doi.org/10.48550/arXiv.2611.04190

# Finite group actions on nonpositively curved manifolds

Joint work with Bena Tshishiku

Byron Zou

November 2026

# Different types of group actions on manifolds

$M^n$ – a closed manifold, and $G$ – a finite group.

1. Isometric actions: $G \hookrightarrow \mathrm{Isom}(M, g)$ where $g$ is a Riemannian metric on $M$ (usually satisfying some special properties).
2. Diffeomorphism actions: $G \hookrightarrow \mathrm{Diff}(M)$ when $M$ has a smooth structure.
3. Homeomorphism actions: $G \hookrightarrow \mathrm{Homeo}(M)$.
4. Actions on homotopy classes: $G \hookrightarrow \mathrm{Out}(\pi_1(M))$.

Action of type $(i)$ induces an action of type $(i+1)$ via natural maps

$$\mathrm{Isom}(M, g) \to \mathrm{Diff}(M) \to \mathrm{Homeo}(M) \to \mathrm{Out}(\pi_1(M)).$$

# Nielsen realization problems

What can we say about the reversed implications?

## Question (Nielsen realization)

Let $M$ be a closed $n$-manifold and $G$ a finite subgroup of $\mathrm{Out}(\pi_1(M))$. Then, does there exist a Riemannian metric $g$ on $M$ and a finite subgroup $H < \mathrm{Isom}(M, g)$ that induces $G$?

We are interested in metrics $g$ that have *nonpositive sectional curvature*.

- Closed flat manifolds: $T^n$.
- Closed hyperbolic manifolds.
- Product manifolds: $S_g \times S^1$ for $g \geq 1$.
- Symmetric space metrics: $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

# History of Nielsen realization

**Question [Schoen–Yau '79]**

Let $M^n$ be closed, nonpositively curved, and let $G < \mathrm{Out}(\pi_1(M))$ finite. Does there exist a nonpositively curved metric $g$ on $M$ such that $G$ is induced by a finite subgroup of $\mathrm{Isom}(M, g)$?

- $n = 2$: **YES** by Kerckhoff '83.
- $n \geq 7$: **NO**. Farrell–Jones '90 constructed negatively curved $M$ using exotic spheres and finite-order outer automorphisms not induced by *any* diffeomorphisms of $M$.
- $n = 3$: this talk.

# Results in dimension 3

For $n = 3$, Zimmermann '79, '82, and Zieschang–Zimmermann '79 showed that $G < \mathrm{Out}(\pi_1(M))$ is always induced by a subgroup of $\mathrm{Diff}(M)$.

## Question (Isometric realization)

Let $M$ be a closed nonpositively curved 3-manifold, and let $G < \mathrm{Diff}(M)$ finite. Does $M$ admit a *$G$-invariant* nonpositively curved metric?

- $M$ is *geometric* (modeled by $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, or $\mathbb{E}^3$): **YES** by Meeks–Scott '86 and Dinkelback–Leeb '09.
- $M$ has a *nontrivial JSJ decomposition*, and $G$ acts *freely*: **YES** by Kapovich–Leeb '96 using CAT(0) geometry.

# Results in dimension 3

**Theorem (Tshishiku–Z. 2026)**

Let $M$ be a closed, *orientable* nonpositively curved 3-manifold. Every smooth, *orientation-preserving* action of a finite group on $M$ preserves a nonpositively curved metric.

- Consider the quotient orbifold $\mathcal{M} = M/G$.
- Cut $\mathcal{M}$ along its Euclidean JSJ 2-orbifolds.
- Choose compatible flat boundary metrics and extend them over the pieces.
- Glue the metrics and lift to $M$.

**Theorem (Tshishiku–Z. in progress)**

Let $M$ be a closed, nonpositively curved 3-manifold. Every smooth action of a finite group on $M$ preserves a nonpositively curved metric

# Patterns of Horocycles and Rigidity for Kleinian Groups

GATSBY 2026

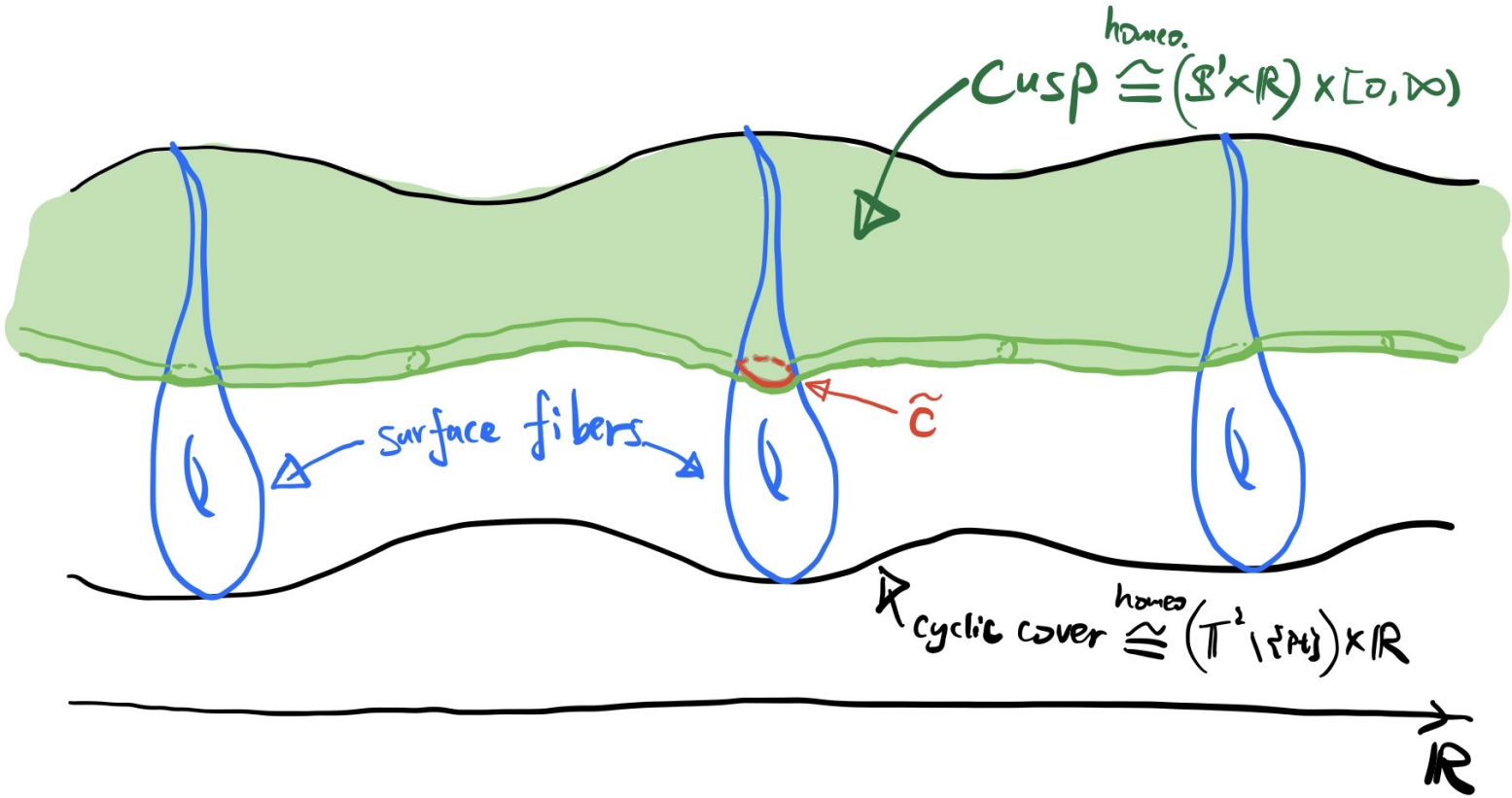
**Yushan Jiang**

CUNY Graduate Center

November 14, 2026

# Example of a pattern of horocycles

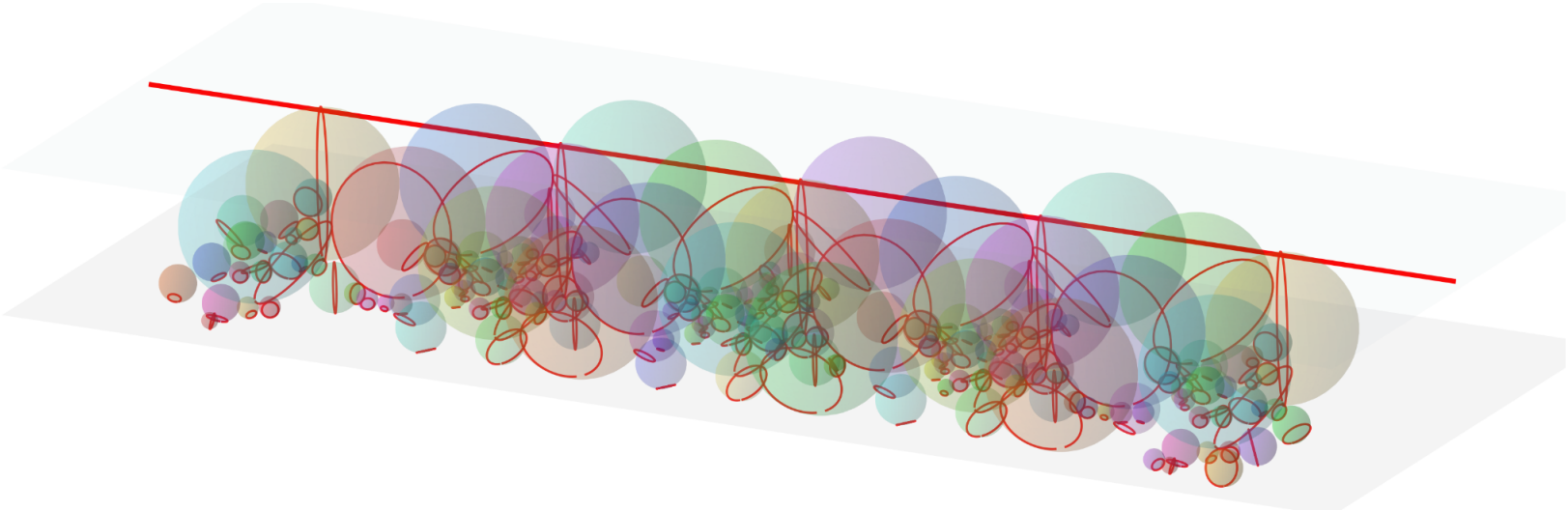
Let $N = \mathbb{H}^3/\Gamma$ be the figure-eight knot cyclic fiber cover. $\Gamma$ is isomorphic to a surface group $\pi_1(\mathbb{T}^2 - \{pt\})$. Choose a closed horocycle $\widetilde{c} \subset N$



# Example of a pattern of horocycles

**Lifting to the universal cover:** Let $N = \mathbb{H}^3/\Gamma$ be the figure-eight knot cyclic fiber cover. Choose a closed horocycle $\widetilde{c} \subset N$ and one lift $C_0$.

$$\mathcal{P} = \{\gamma C_0 : \gamma \in \Gamma\}.$$



red: lifts of $\widetilde{c}$ transparent: lifts of cusp horosphere

Exactly one horocycle per lifted cusp horosphere.

## Sparseness

Even the union of the corresponding horodisks is **not coarsely dense** in $\mathbb{H}^3$ (i.e. every neighborhood with finite diameter can not cover $\mathbb{H}^3$).

# Ambient rigidity in $\mathbb{H}^3$

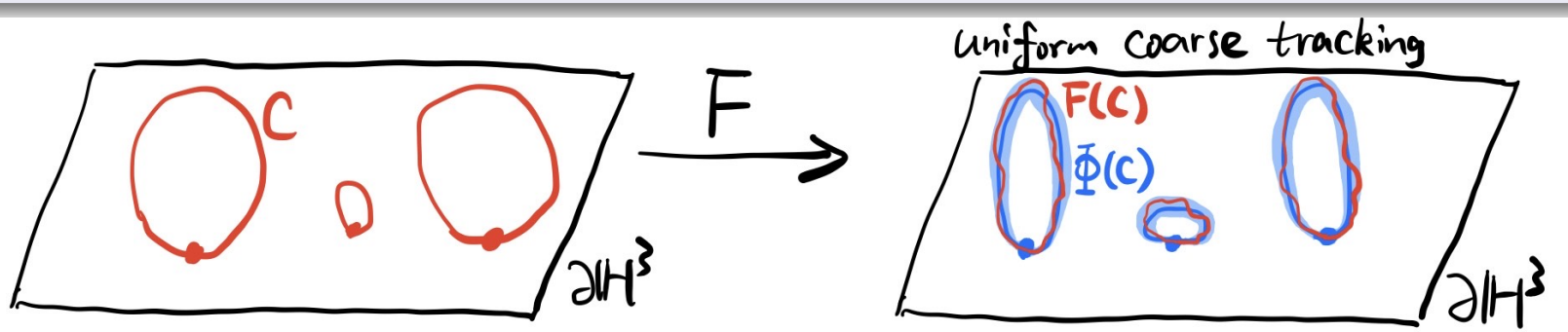
A $\Gamma$-symmetric horocycle pattern $\mathcal{P} \subset 2^{\mathbb{H}^3}$ lifts finitely many closed horocycles from $\mathbb{H}^3/\Gamma$.
$\mathcal{P}$: **$\Gamma$-invariant, finitely many $\Gamma$ orbits, cocompact component stablizers**.

## Theorem (J., in preparation)

Let $\Gamma_i < \mathrm{Isom}(\mathbb{H}^3)$ be non-elementary discrete groups, and let $\mathcal{P}_i$ be $\Gamma_i$-symmetric horocycle patterns. Assume $\mathbb{H}^3/\Gamma_1$ has **Liouville conservative** geodesic flow ($\iff$ the conical limit set $\Lambda_c$ has full Lebesgue measure on $\mathbb{S}^2$). If a quasi-isometry $F : \mathbb{H}^3 \to \mathbb{H}^3$ and a bijection $\Phi : \mathcal{P}_1 \to \mathcal{P}_2$ satisfy

$$d_{\mathrm{H}}\big(F(C), \Phi(C)\big) \leq D < \infty \qquad (\forall C \in \mathcal{P}_1),$$

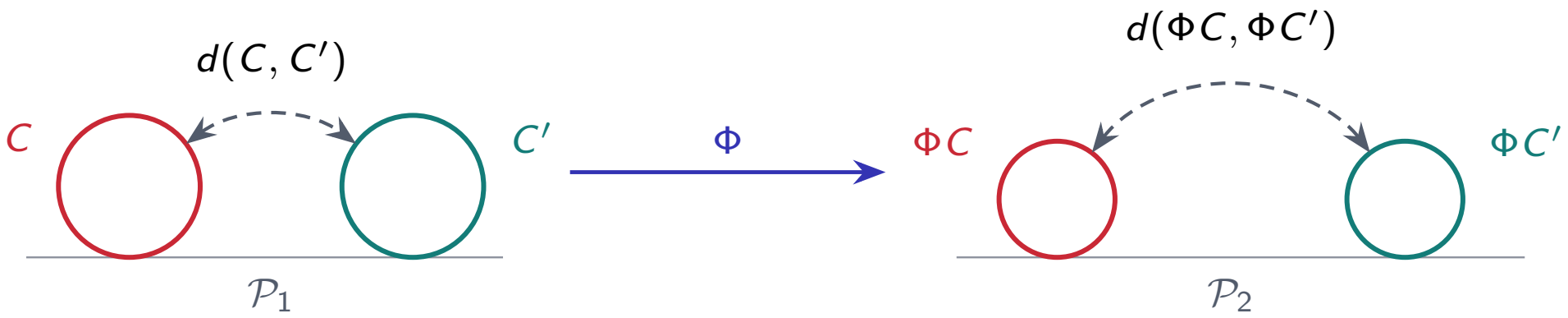
then there is a **unique isometry** $I$ such that $\sup_{x \in \mathbb{H}^3} d(Fx, Ix) < \infty$. Moreover, $I\Gamma_1 I^{-1}$ and $\Gamma_2$ are commensurable.



# Uniformly proper pairings

Pair components using their **infimal distance**:

$$d(C, C') = \inf\{d(x, y) : x \in C,\ y \in C'\}.$$




## Uniform properness

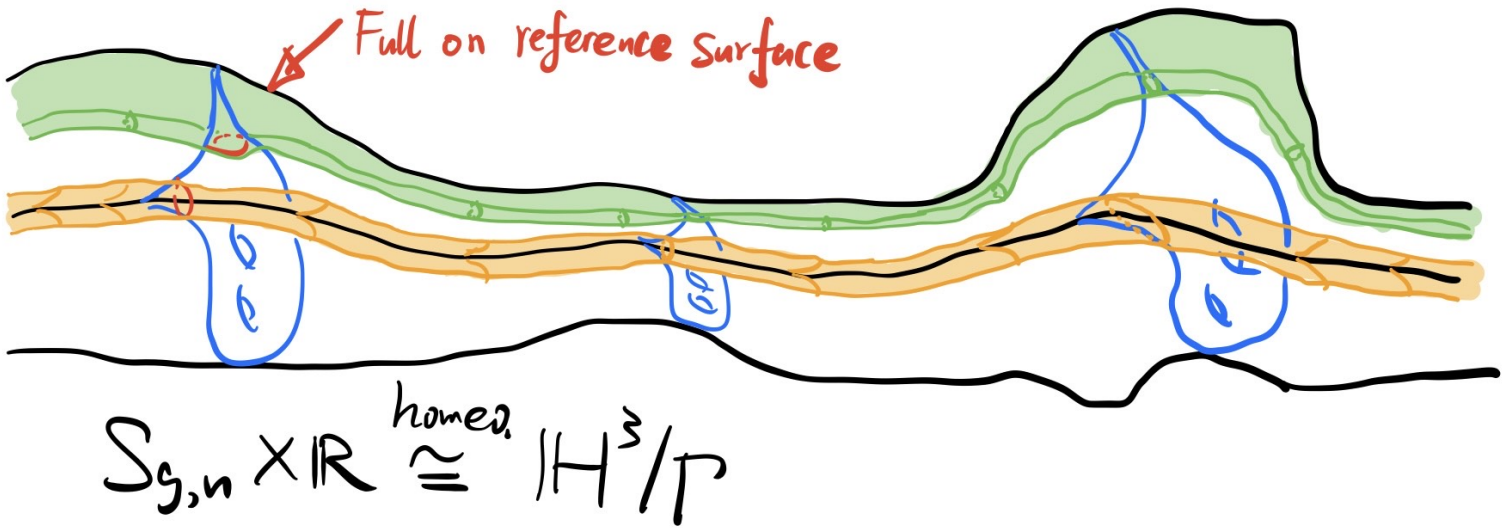
A bijection $\Phi : \mathcal{P}_1 \to \mathcal{P}_2$ has a control function $\omega : \mathbb{N} \to \mathbb{N}$ such that, for all $C, C'$ and $n \in \mathbb{N}$,

$$d(C, C') \leq n \implies d(\Phi C, \Phi C') \leq \omega(n),$$
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The control function need not be linear as ambient quasi-isometrc pairing.

# Rigidity of uniformly proper pairings

Let $\Gamma_i$ be doubly degenerate punctured-hyperbolic surface groups from faithful, strictly type-preserving representations. Let $\mathcal{P}_i$ be **full** $\Gamma_i$-symmetric horocycle patterns. Full: the reference surface truncation is compact.




## Theorem (J., in preparation)

Suppose the bijection $\Phi : \mathcal{P}_1 \to \mathcal{P}_2$ is uniformly proper and the center correspondence $z(C) \longmapsto z\big(\Phi(C)\big)$ is well-defined and extends to a homeomorphism $h : S^2 \to S^2$. Then $h$ is **Möbius or anti-Möbius**. In particular, $h = \partial I$ for an isometry $I$, and $I\Gamma_1 I^{-1}$ and $\Gamma_2$ are commensurable.

# Flexibility of uniformly proper pairings

**For doubly degenerate punctured-surface Kleinian groups:**

## Rigidity with boundary compatibility

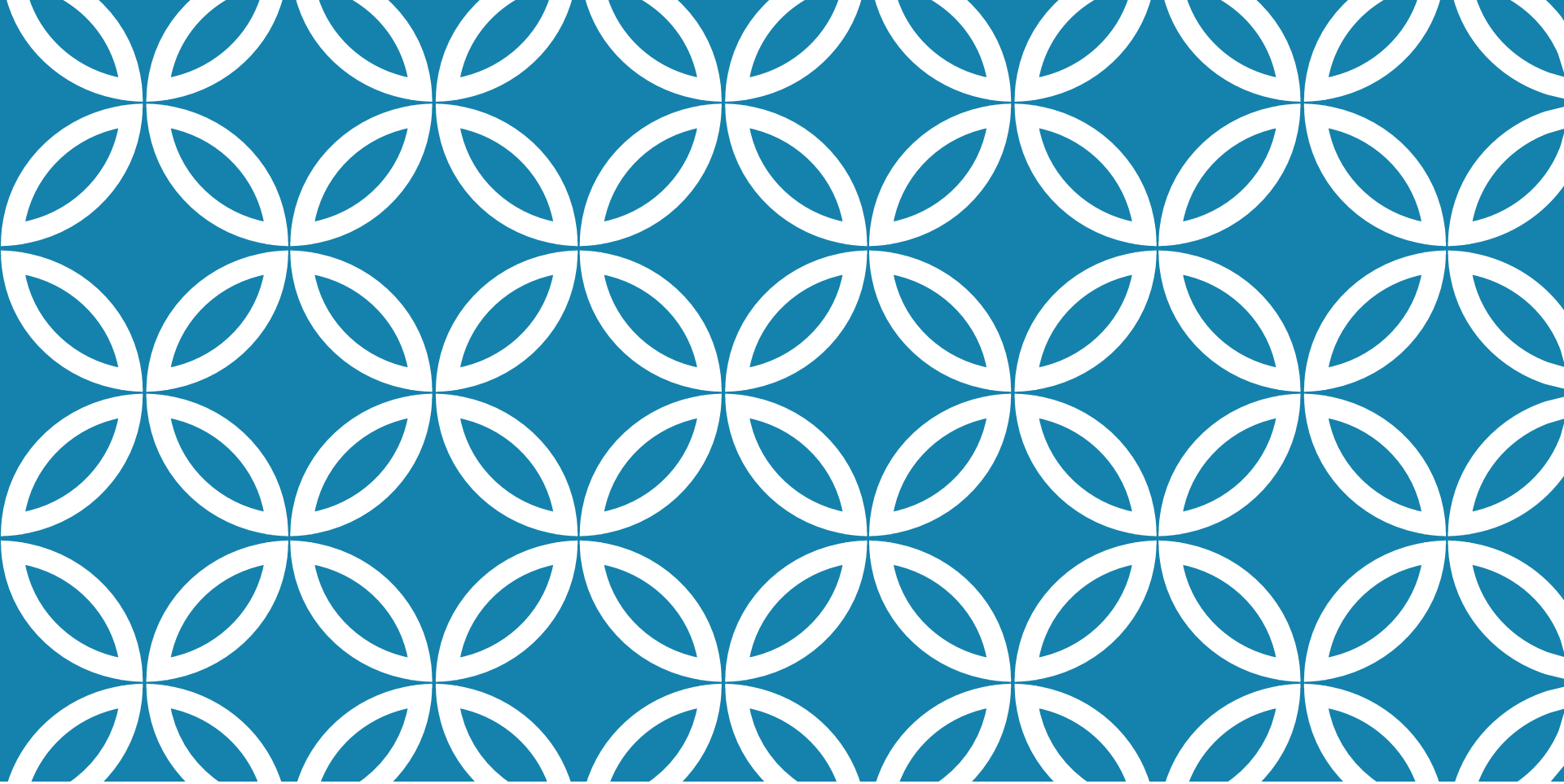
A uniformly proper bijection between full symmetric horocycle patterns whose well-defined center map extends to a sphere homeomorphism forces that homeomorphism to be **Möbius or anti-Möbius**.

## Flexibility without the extension hypothesis (J., in preparation)

There exist uniformly proper bijections between full symmetric horocycle patterns whose well-defined center maps have **no extension** to a homeomorphism of $S^2$. In particular, such bijections do not come from an isometry or quasi-isometry.

Boundary compatibility is essential.

Thank you!



# ARITHMETIC HYPERBOLIC MANIFOLDS WITH BOUNDED RANK

By Matthew Zevenbergen

# INTRODUCTION

**Main theorem (Z.):** There are finitely many closed arithmetic hyperbolic 3-manifolds with rank at most 3.

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**Definition:** The *rank* of a manifold is the minimal size of a generating set for its fundamental group.

# INGREDIENTS

**Theorem (Biringer-Souto):** For all $\varepsilon > 0$, there are finitely many closed arithmetic hyperbolic 3-manifolds with rank at most 3 that contain no closed geodesics with length less than $\varepsilon$.

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$\Longrightarrow$ Main theorem

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**Theorem (classical):** The length of the shortest closed geodesic in an arithmetic closed hyperbolic 3-manifold has a positive lower bound in terms of its invariant trace field degree.

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$\Longrightarrow$ Proposition

# ADDITIONAL RESULTS

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# ADDITIONAL RESULTS

**Theorem (Z.):** For each $r$, there are only finitely many commensurability classes of closed arithmetic hyperbolic 3-manifolds with rank at most $r$.

**Theorem (Z.):** For each $r$, all but finitely many closed arithmetic hyperbolic 3-manifolds with rank $r$ have Heegaard genus $r$.

Thank you!

# Surface subgroups in hyperbolic 3-manifolds with prescribed finite quotients

Zhenghao Rao

Rutgers University–New Brunswick

GATSBY, Nov. 14, 2026

Joint work with

Jean-François Lafont and Hongbin Sun

Let $M$ be a complete, orientable, finite-volume hyperbolic 3-manifold, and set $G = \pi_1(M)$.

**Surface subgroup**

$H < G$ with $H \cong \pi_1(S)$, where $S$ is a closed, connected, orientable surface of genus at least 2.

**Kahn–Marković (2012)**
Nearly geodesic surfaces in closed hyperbolic 3-manifolds.

**Kahn–Wright (2021)**
Nearly geodesic surfaces in cusped hyperbolic 3-manifolds.

Earlier cusped results: Cooper–Long–Reid (1997), Masters–Zhang (2008/09), Baker–Cooper (2015), and Cooper–Futer (2019).

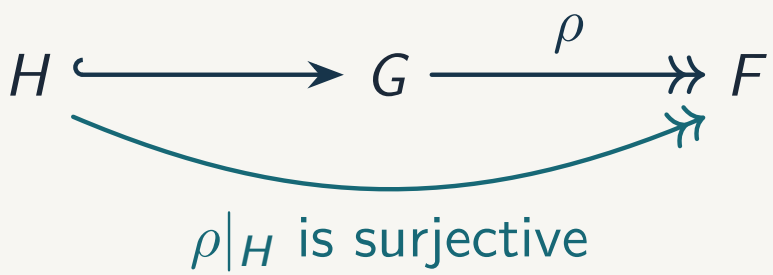
# Prescribing the image in a finite quotient

Keep $G = \pi_1(M)$.

**Theorem (Lafont–R.–Sun, 2026)**

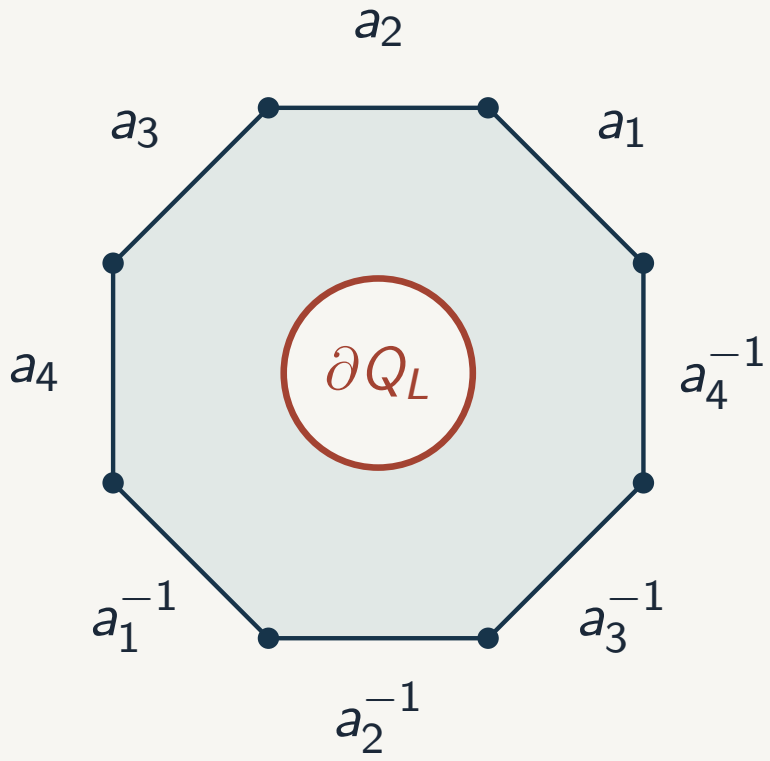
For every epimorphism $\rho : G \twoheadrightarrow F$ onto a finite group, there is a quasi-Fuchsian surface subgroup $H < G$ such that

$$\rho(H) = F.$$




- The surface subgroup may depend on the finite quotient.
- The theorem also holds for finite-volume hyperbolic 3-orbifolds, and every subgroup $A \leq F$ can occur as $\rho(H)$.
- There are infinitely many commensurability classes of such $H$.

# The construction in one slide




Hyperbolic annulus ($r = 2$)

The eight outer sides have length $L$. Glue $a_i$ to $a_i^{-1}$ for $i = 1, \ldots, 4$. All eight vertices become one point.

The quotient $Q_{L,2}$ with genus 2 and one geodesic boundary is called a *rose*. The loops $a_1, \ldots, a_4$ freely generate $\pi_1(Q_{L,2})$.

**Realize and complete the rose**

For sufficiently large $r$, construct a good rose $f : Q_{L,r} \to M$ with $\rho(f_*\pi_1(Q_{L,r})) = F$. Close it up with good pants and hamster wheels to obtain a quasi-Fuchsian surface.

**Corollary**

The union of the quasi-Fuchsian surface subgroups is dense in $G$ in the profinite topology.

**What this means concretely**

For every finite-index normal subgroup $N \triangleleft G$ and every $g \in G$, some quasi-Fuchsian surface subgroup $H < G$ contains an element $h$ with

$$hN = gN.$$

Apply the theorem to the finite quotient $G \twoheadrightarrow G/N$, and we obtain the subgroup $H$ that meets every coset of $N$.

Fix a connected finite cover $p : \overline{M} \to M$.

**Corollary (Lafont–R.–Sun)**

There is a closed quasi-Fuchsian immersion $f : S \to M$ whose entire pullback $\overline{S} = S \times_M \overline{M}$ is connected.

connected

$$\begin{array}{ccc} \overline{S} & \xrightarrow{\overline{f}} & \overline{M} \\ \downarrow & & \downarrow p \\ S & \xrightarrow[f]{} & M \end{array}$$

**Homological consequence**

If $f_*[S] = 0$ in $H_2(M; \mathbb{Z})$, then $\overline{f}_*[\overline{S}] = 0$ in $H_2(\overline{M}; \mathbb{Z})$.

**Kent–Leininger (2024)** construct surface bundles

$$\widehat{S} \longrightarrow E \longrightarrow \Sigma$$

Here the fiber $\widehat{S}$ and base $\Sigma$ are closed oriented surfaces, and $\pi_1(E)$ contains no $\mathbb{Z}^2$.

Their input is a finite acceptable branched cover $p : \widehat{S} \to T$, where $T$ is a torus with three marked points:

- branching occurs only over the marked points;
- every preimage of a marked point has local degree $\geq 2$.

**Corollary (Lafont–R.–Sun, 2026)**

For every such cover, infinitely many base surface subgroups yield surface bundles with $\sigma(E) = 0$.

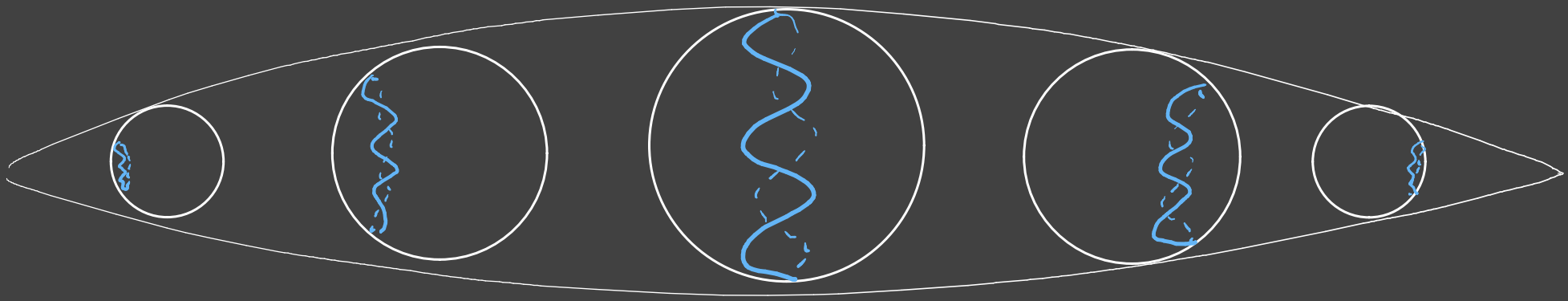
Lafont–Miller–Ruffoni (2024) obtain zero signature for a special family of covers.

# Thank you!

J.-F. Lafont, Z. R. and H. Sun,

*Surface subgroups with prescribed finite quotients.* arXiv:2611.13458.

This work is supported in part by NSF DMS-2603932.



# Conformal Length of Knots

Ethan Dlugie*

(Brown U)

GATSBY

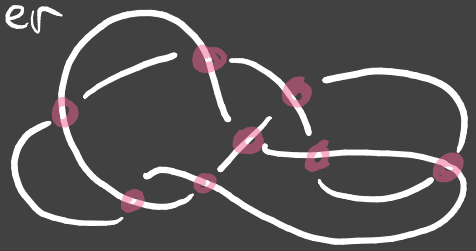
November 14, 2026

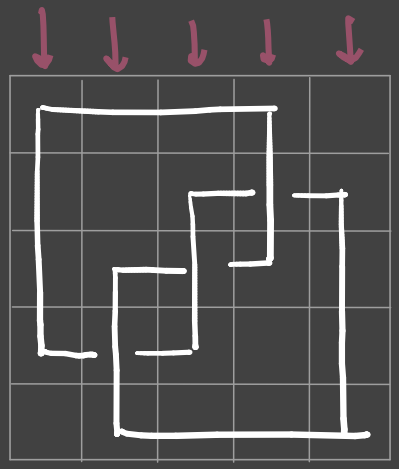
*joint w/ C. Breiner

# Knot invariants via extremization :

## Combinatorics / topology

- crossing number





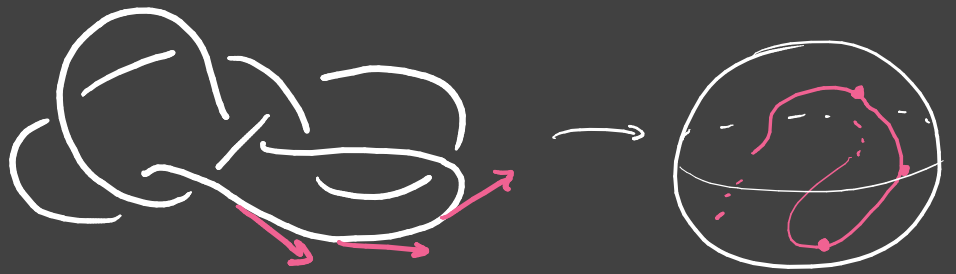
- grid number

- meridional rank

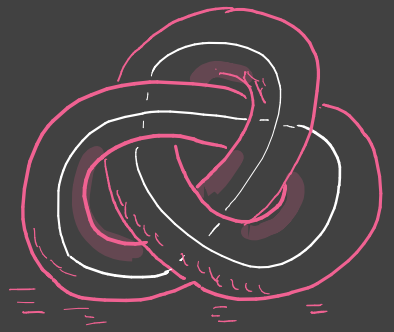


## Geometry

- total curvature



- ropelength



- Möbius energy

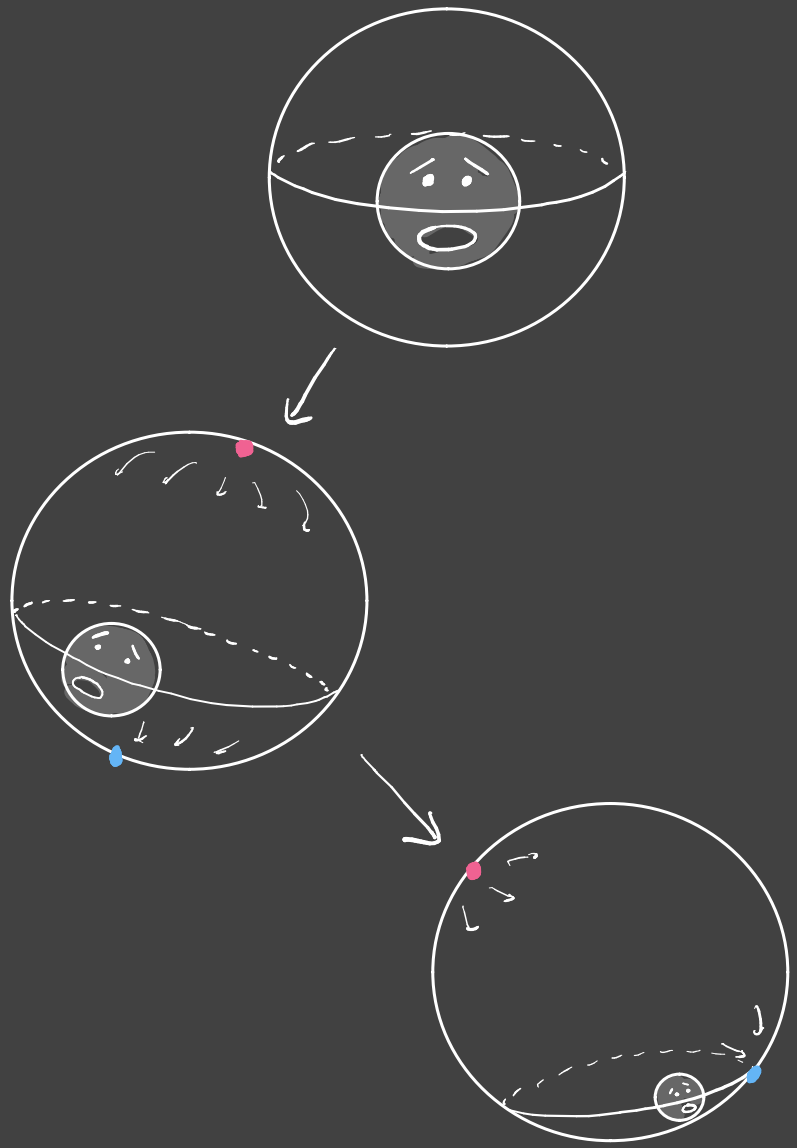
$$G = \text{Möb}(S^3)$$
$$= \langle \text{Isom}(S^3), \text{loxodromics} \rangle$$

**Def** The conformal length of a loop $\gamma : S^1 \longrightarrow S^3$ is

$$L_c(\gamma) = \sup_{\varphi \in G} L(\varphi \circ \gamma)$$

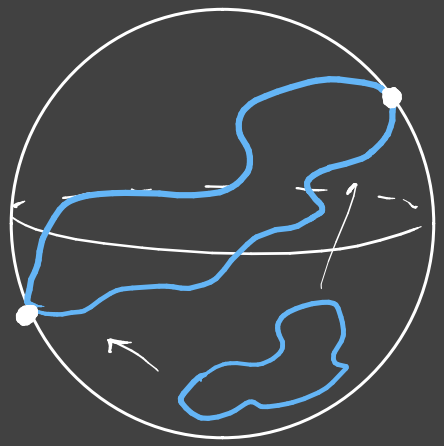
**Def** The conformal length of a knot type $K$ in $S^3$ is

$$L_c(K) = \inf_{\gamma \in K} L_c(\gamma).$$



**Lemma** $L_c(K) \geq 2\pi$.

**Pf** For any $\gamma \in K$, some $\varphi$ takes it to antipodal points.



Thus $L_c(\gamma) \geq 2\pi$. □

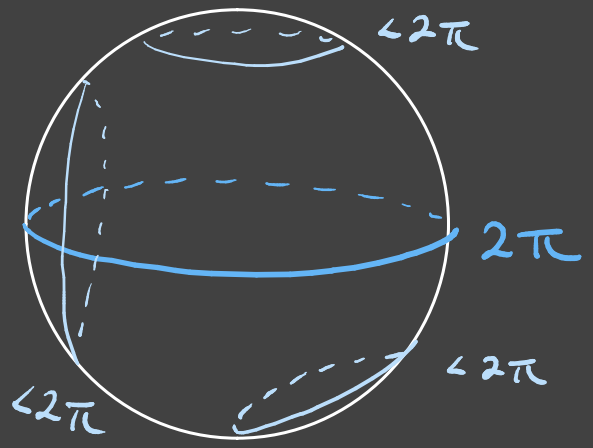
**Prop** $L_c(\text{unknot}) = 2\pi$.

**Pf** Take $\gamma$ a geodesic.

Then $L(\gamma) = 2\pi$ and

$L(\varphi \circ \gamma) \leq 2\pi$. Thus

$L_c(\gamma) = 2\pi \Rightarrow L_c(\text{unknot}) \leq 2\pi$. □



results w/ C. Breiner

(in progress)

If $K \neq$ unknot, then

$$L_c(K) \geq 4\pi.$$

**Prop** For torus knots $T_{p,q}$,

$$L_c(T_{p,q}) \leq 2\pi \min(p,q).$$

**Pf** Pick nice $\gamma \in T_{p,q}$. Some harmonic analysis shows

$$L(\gamma) = L_c(\gamma).$$

And then $\exists\, \gamma_n$ s.t.

$$L_c(\gamma_n) \to 2\pi \min(p,q). \square$$



