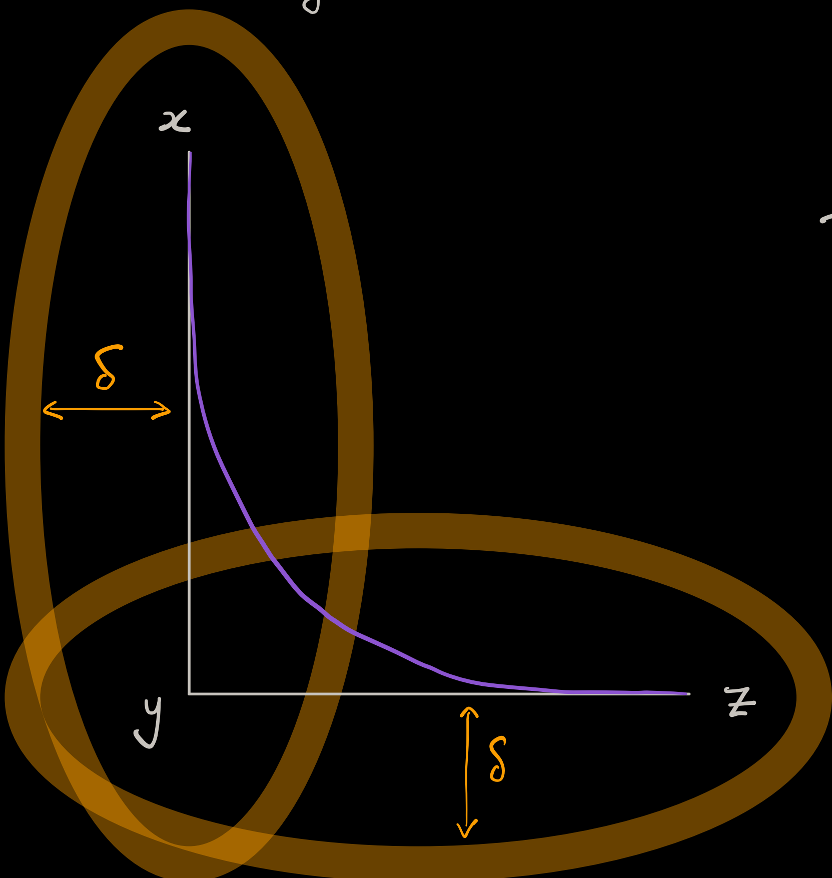
Random Quotients Preserve Negative Curvature

Thomas Ng (Brandeis)

joint with Abbott, Berlyne, Mangioni, Rasmussen



Theorem (ABMNR)

If G is a *hyperbolic group, then "random quotients" $G/\langle\langle r_1, \dots, r_k \rangle\rangle$ are also *hyperbolic.

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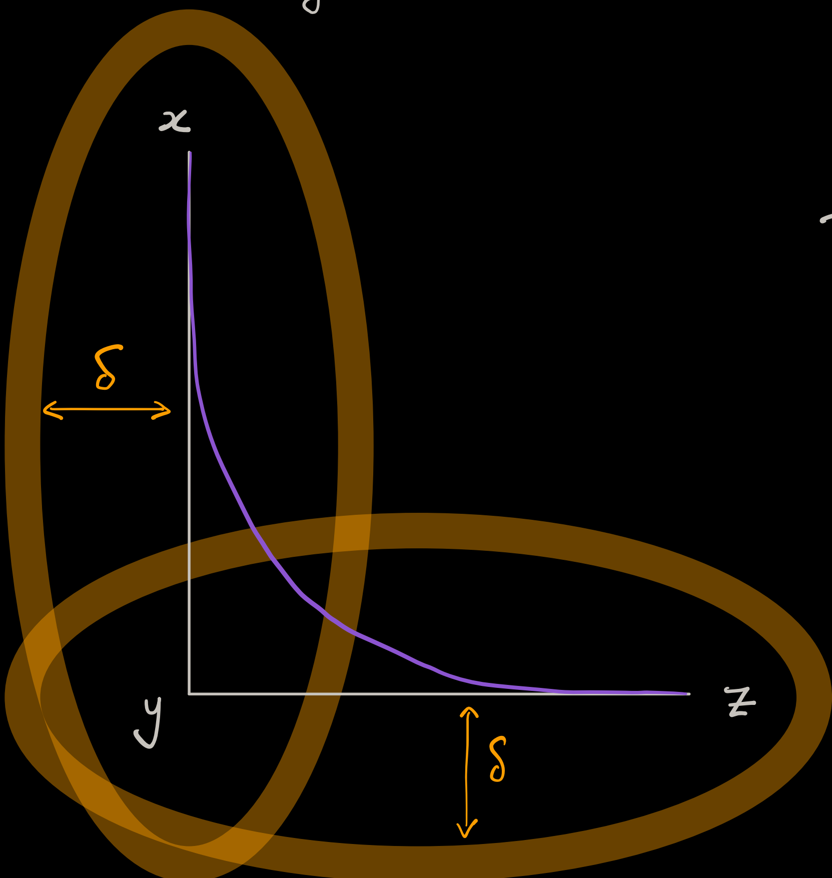

* Rel. hyp.
Hierar. hyp.
Acyl. hyp.

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If G is a *hyperbolic group, then "random quotients" $G/\langle\langle r_1, \dots, r_k \rangle\rangle$ are also *hyperbolic.

$$\langle g r_1 g^{-1}, \dots, g r_k g^{-1} \mid g \in G \rangle$$

Random walks on group a G

Fix $S = S^{-1}$ a finite symmetric generating set.

Let $g_1, \dots, g_n, \dots$ be i.i.d. random variables w.r.t. Uniform(S)

$$(w_n = g_1 g_2 \cdots g_n)_{n \in \mathbb{N}} \text{ is a RANDOM WALK on } G$$

What are generic properties of elements/quotients?

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Def : A property $P = P(w_n)$ holds a.a.s. (asymptotically almost surely)

$$\text{if } \mathbb{P}(G \text{ has } P) \xrightarrow[n \to \infty]{} 1$$

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- w_N is a.a.s. trivial for $N \geq n$ $\rightsquigarrow$ w_n is "recurrent"
- (Maher–Tiozzo) If G is infinite and *hyperbolic

  then a.a.s. $\lim\limits_{n \to \infty} \frac{1}{n} d(w_n x_0, x_0) \cancel{\geq} 0$.

What are generic properties of elements/quotients?

Def: A RANDOM QUOTIENT of G is G/N

where $N = \langle\langle w_{1,n}, \dots, w_{k,n} \rangle\rangle$ and $\{w_{j,n}\}$ are independent.

<u>Idea</u>: $G \curvearrowright X^{\delta\text{-hyp.}} \implies G/N \curvearrowright X/N$

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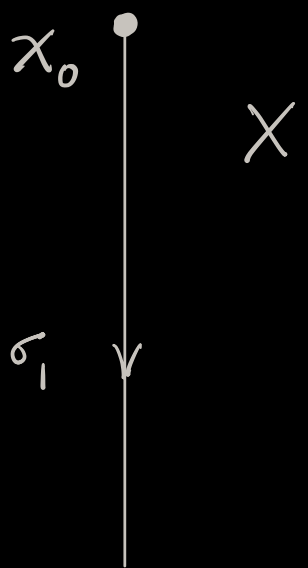
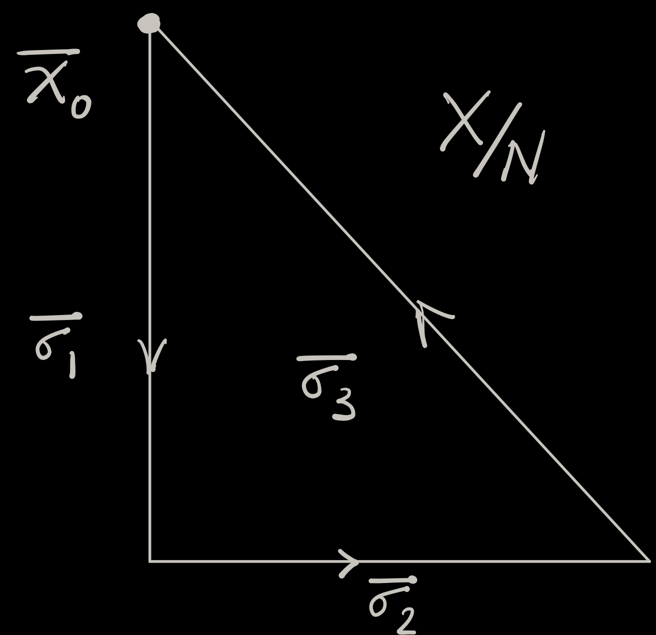


Geodesic
lifts to geodesic



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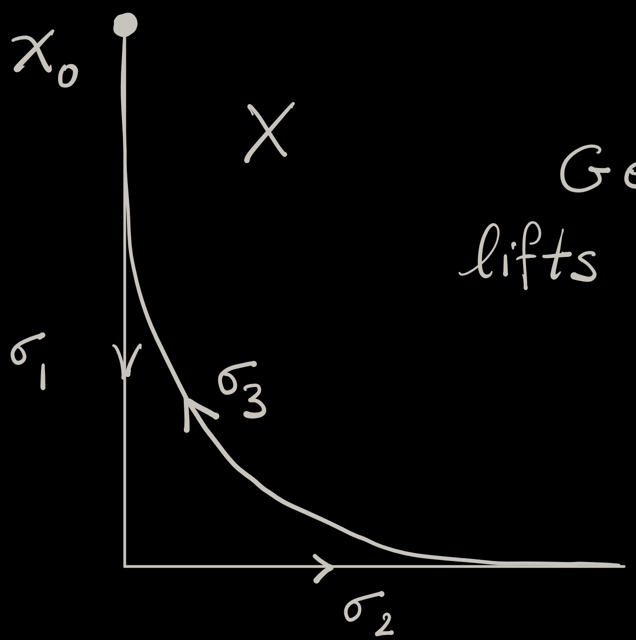

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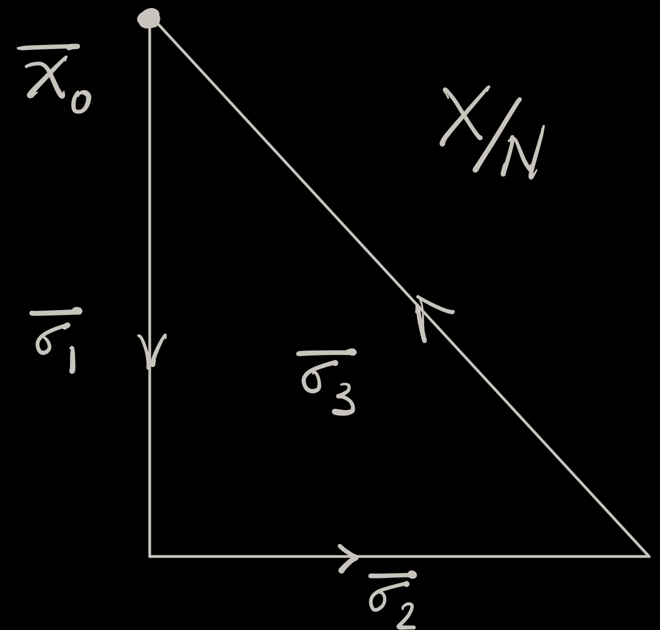

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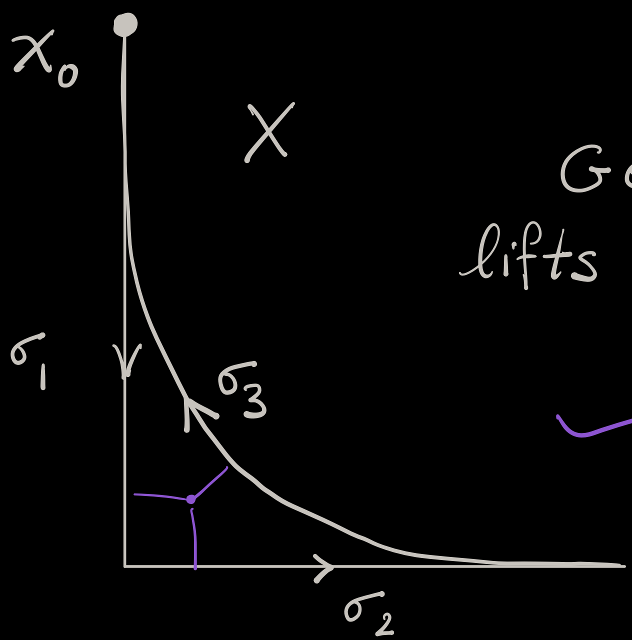
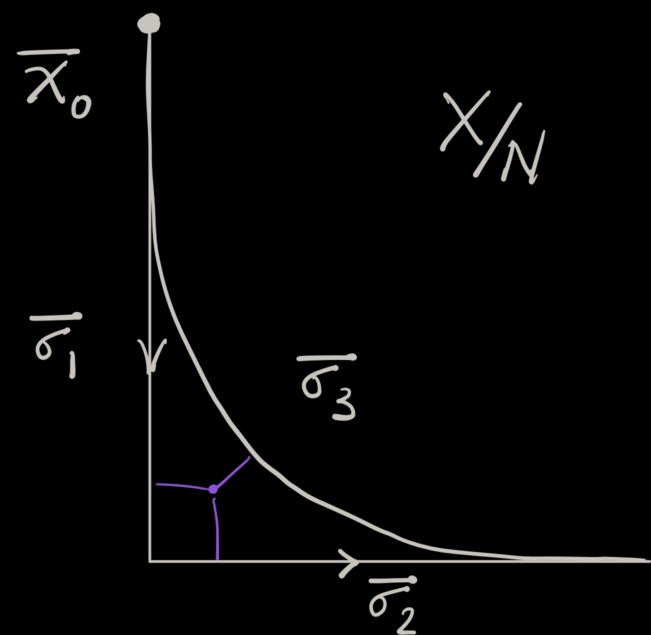


Geodesic lifts to geodesic

δ-thin



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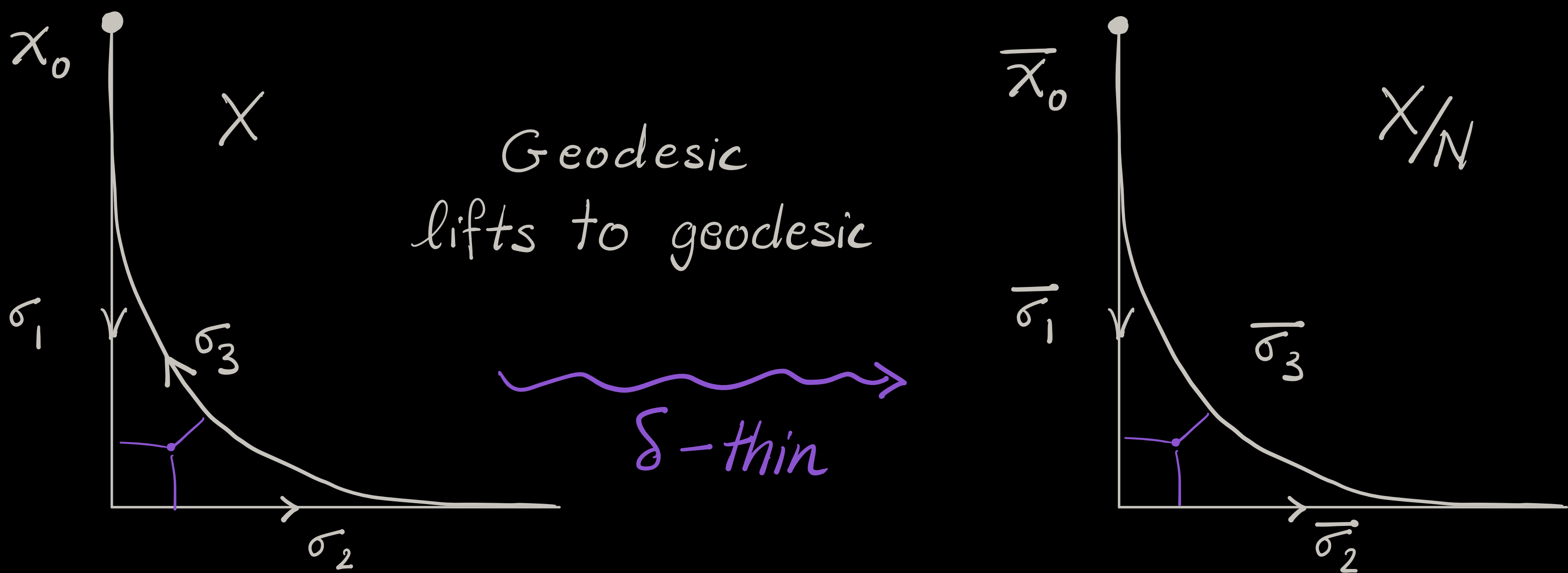
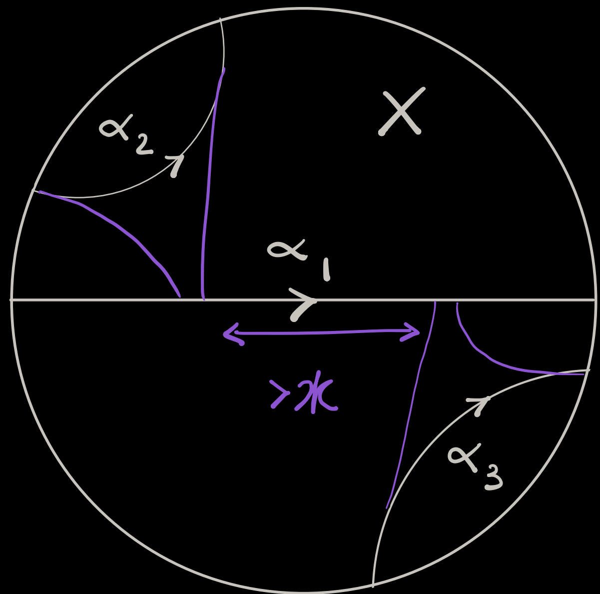
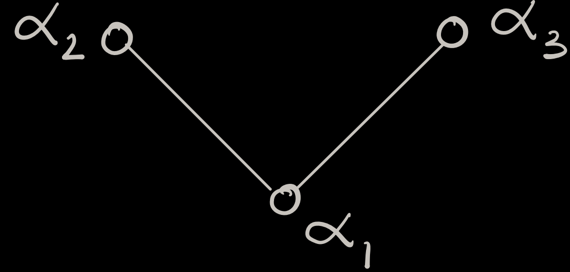


$\mathrm{Cay}(F_n)$ is 0-hyperbolic, but ...
every group is a quotient of a free group!

New tool: Spinning family on Projection complexes

(Maher - Tiozzo) $\Longrightarrow$ Each $w_{j,n}$ is loxodromic on quasigeodesic axis.





$P_\varkappa$ is hyperbolic

(Bestvina, Bromberg, Fujiwara)

Prop: Both $\varkappa$ and L grow linearly in n.
(ABMNR)

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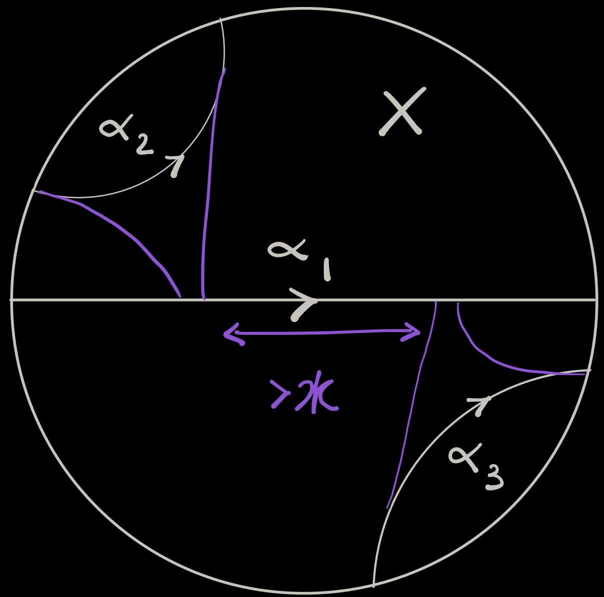

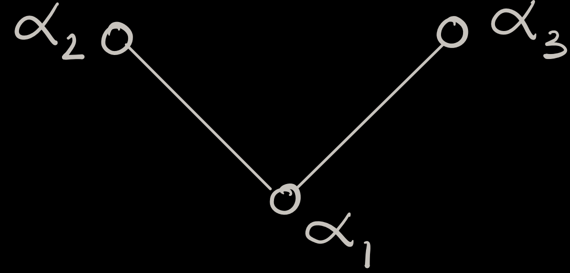

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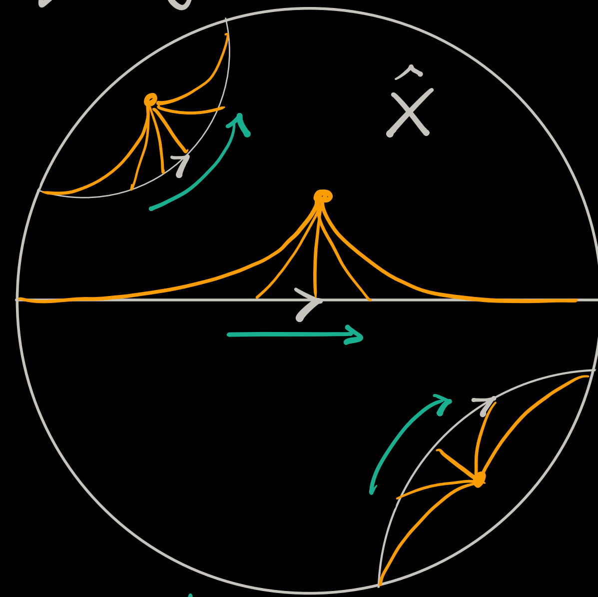
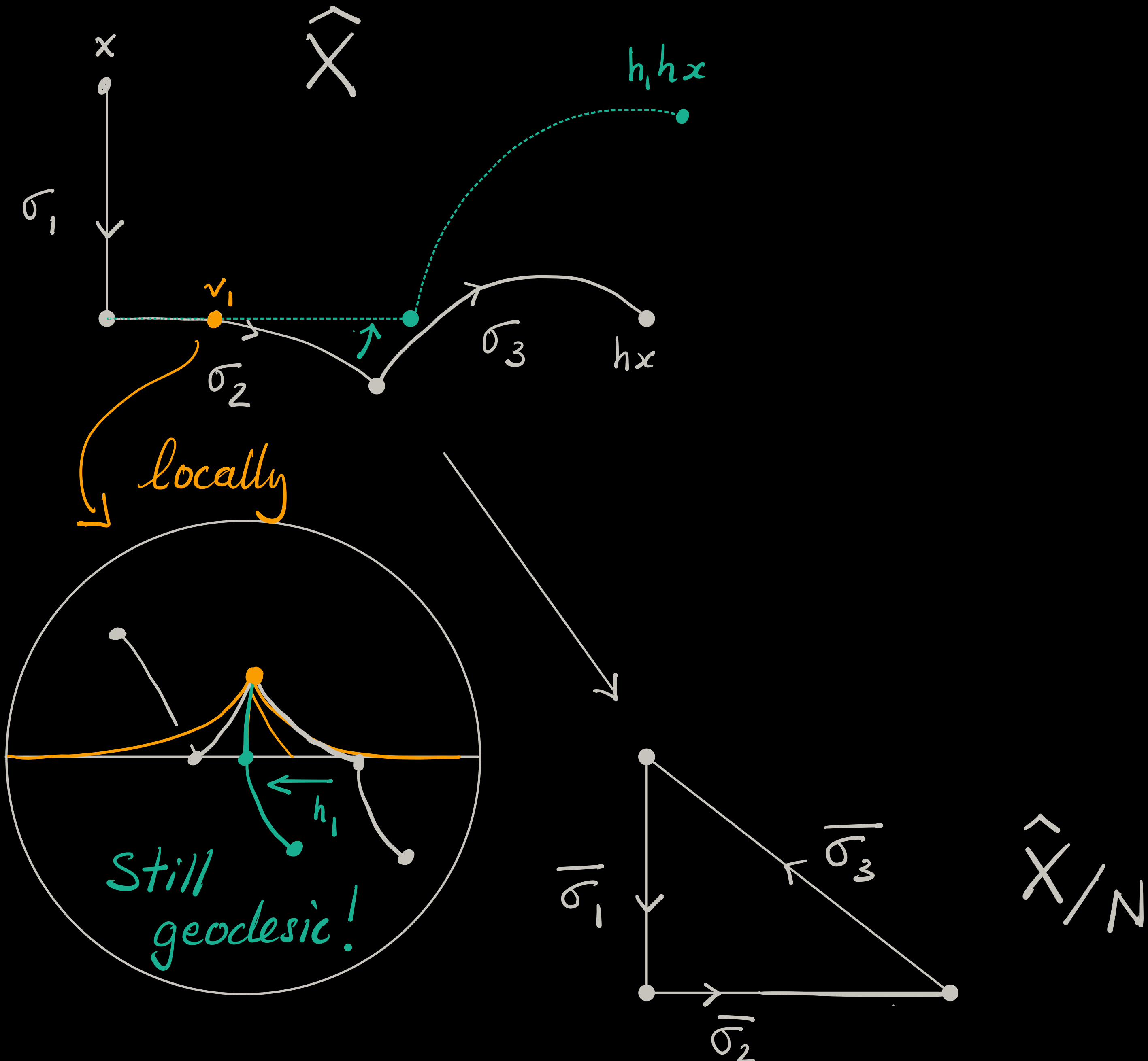


$L \sim$ translation length

(Clay - Mangahas) If $N \curvearrowright P_\varkappa$ is "L-spinning" for $L \gg \varkappa$ geodesic triangles $P_\varkappa/N$ lift to geodesic triangles $P_\varkappa$

Prop: Both $\varkappa$ and L grow linearly in n.
(ABMNR)

Bending geodesics in $\hat{X}$.



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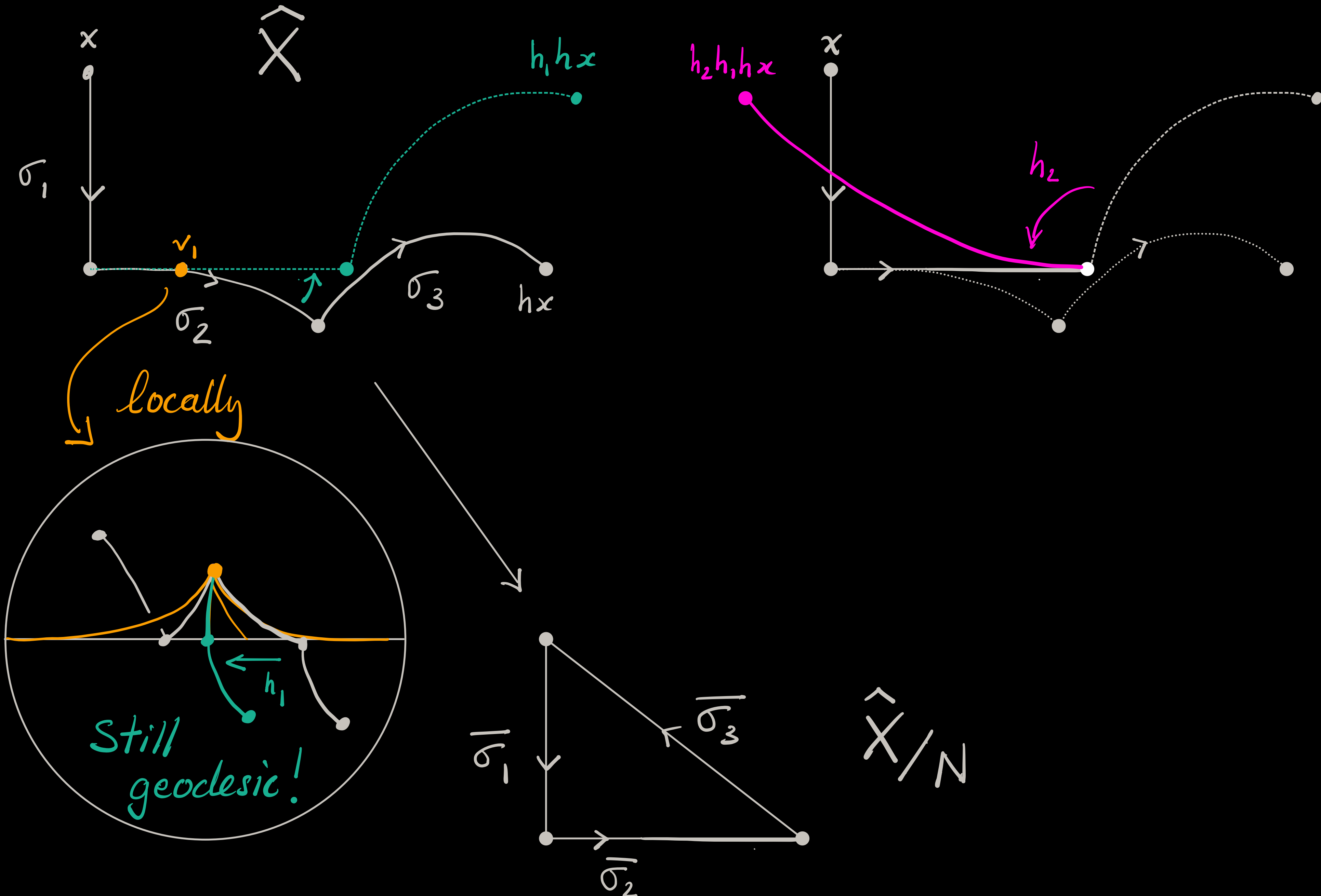

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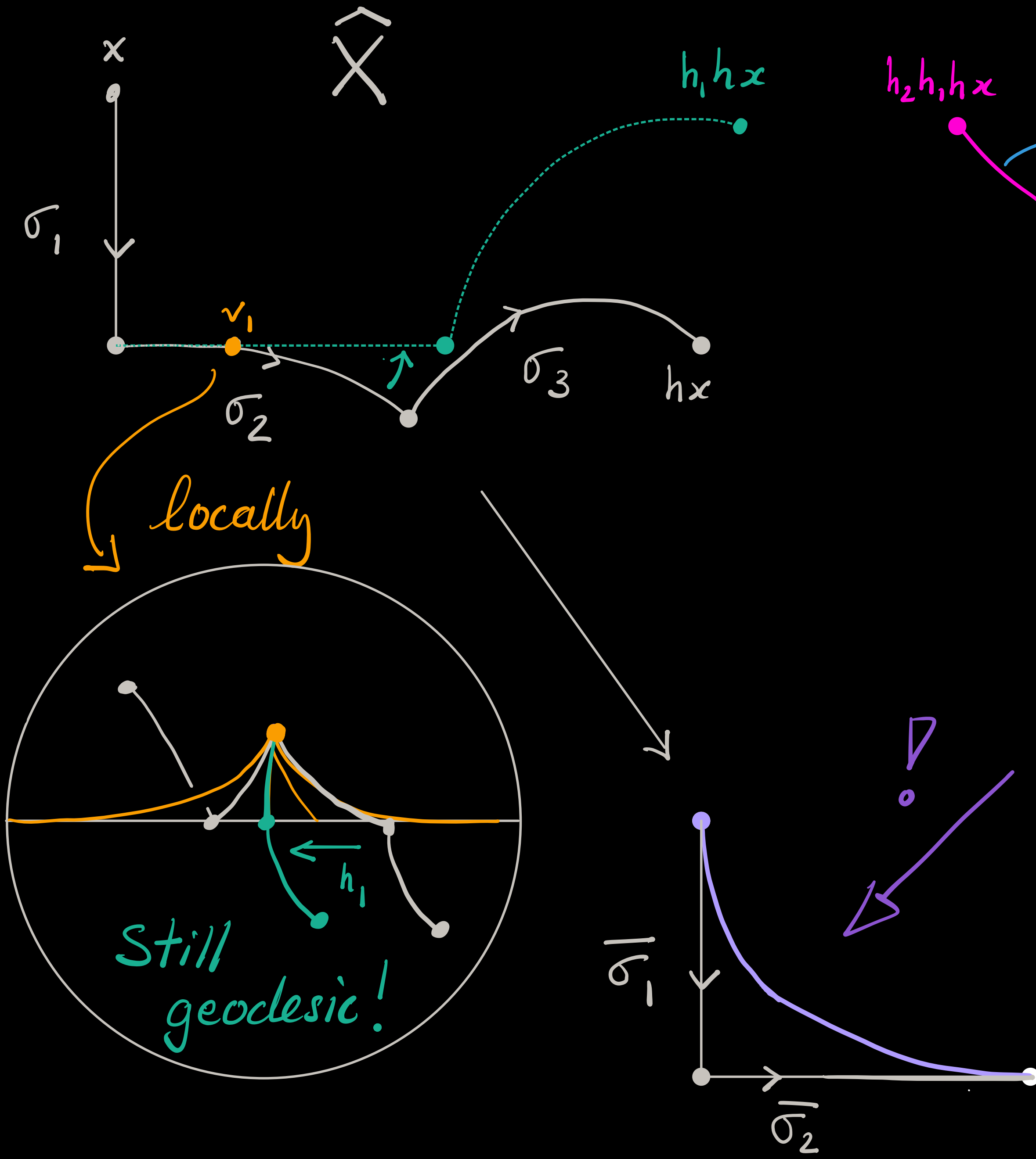
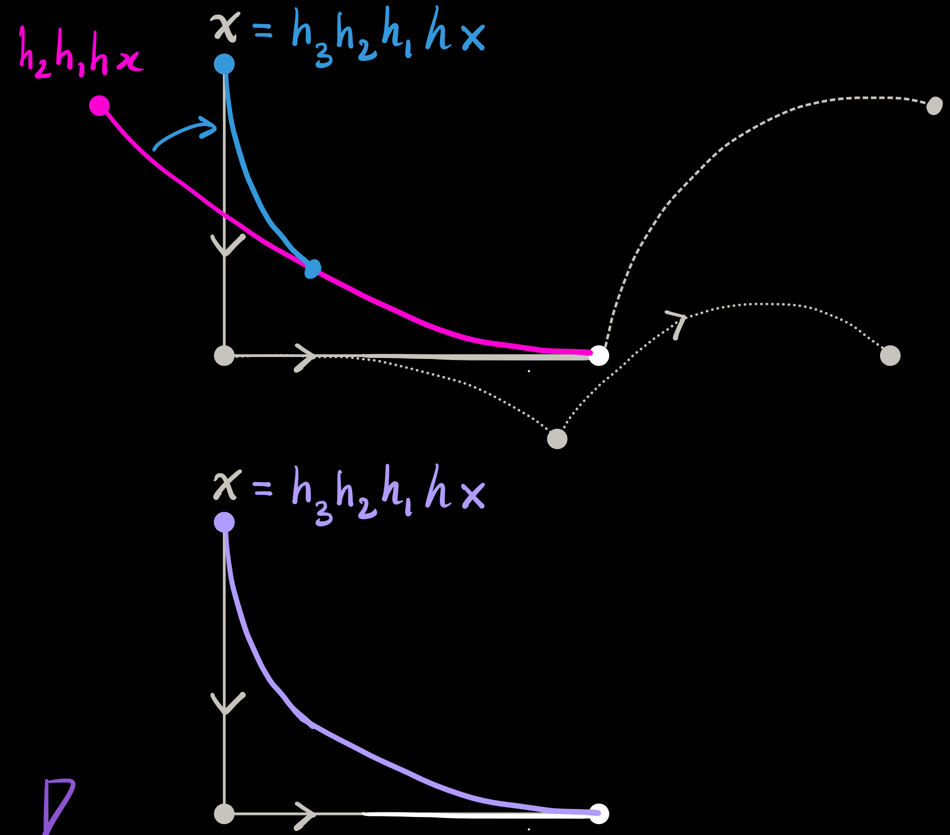


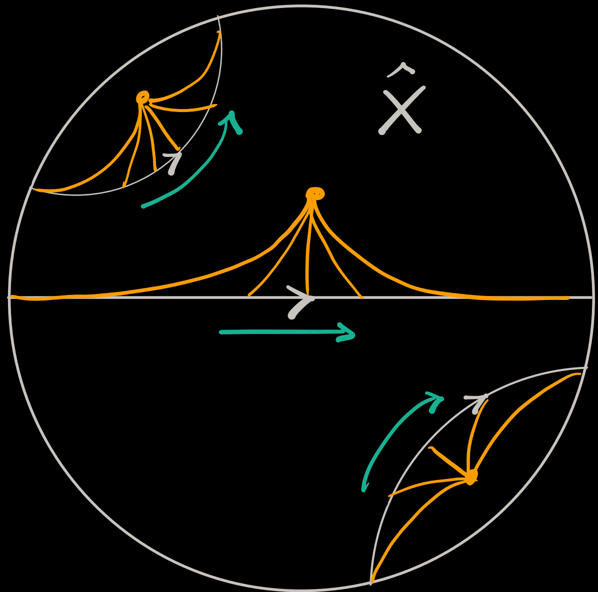


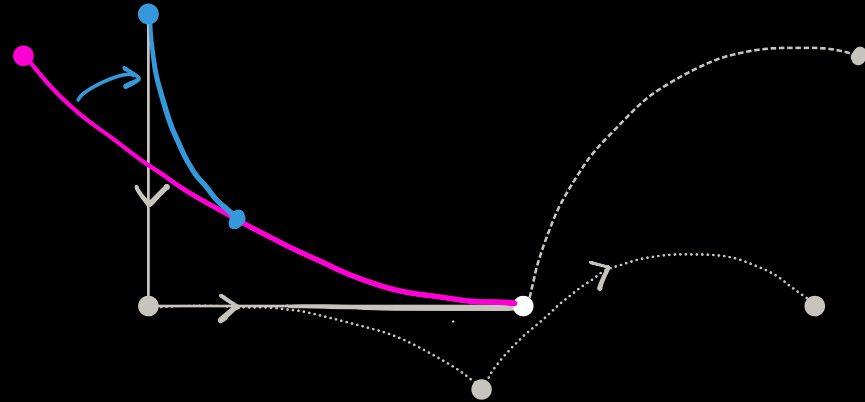
$$\hat{X}/N \simeq_{QI} X/N$$

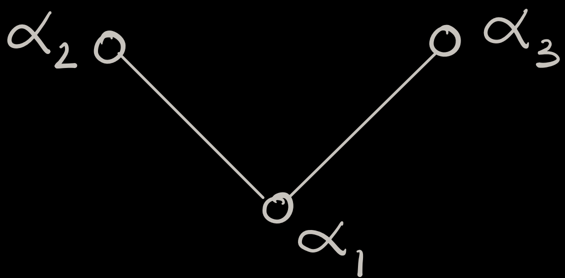
Theorem (ABMNR)

If G is (neg. curved) group,
then "random quotients"
$G/\langle\langle w_{1,n}, \dots, w_{k,n} \rangle\rangle$ are also (neg. curved).

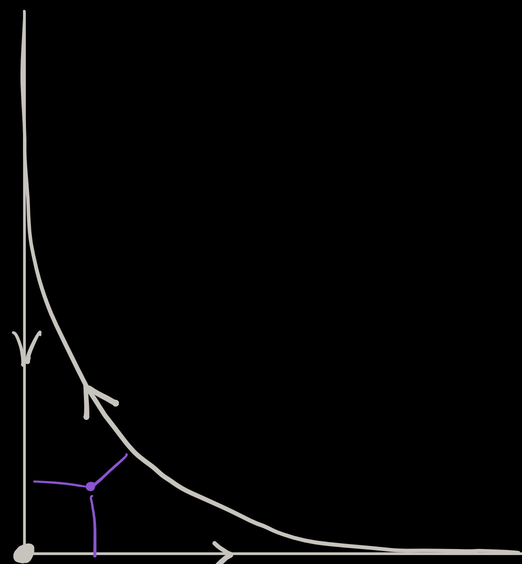
$\left\{ \begin{array}{l} \text{hyperbolic} \\ \text{Rel. hyp.} \\ \text{Hierar. hyp.} \\ \text{Acyl. hyp.} \end{array} \right.$







Thanks!



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#Sample/Drawing

Geometry of transverse surfaces to pseudo-Anosov flows

Joint work with Sam Taylor

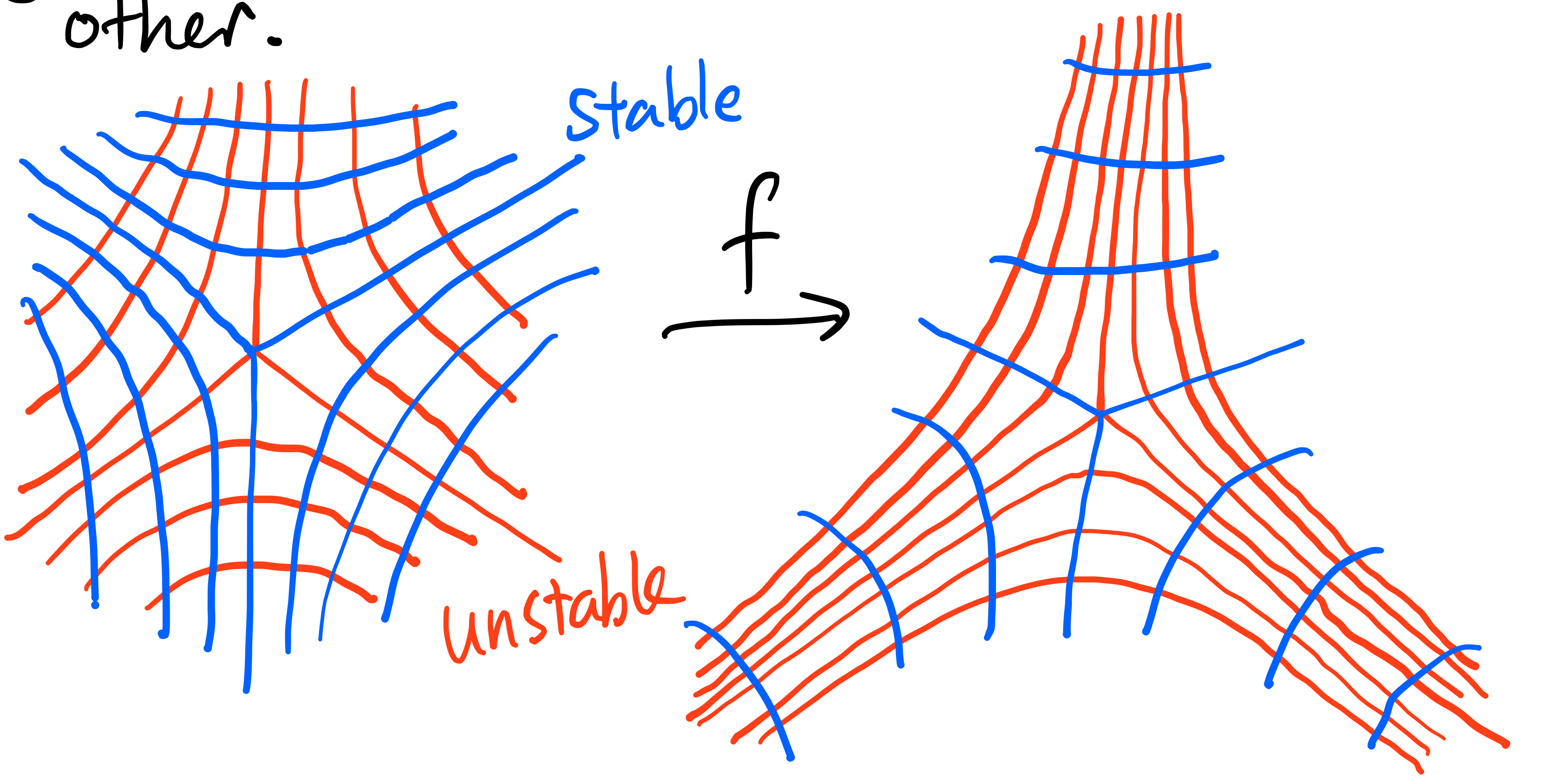
Junzhi Huang (Yale)

GATSBY Fall 2025

Let S_g be a closed surface of genus $g \geq 2$.

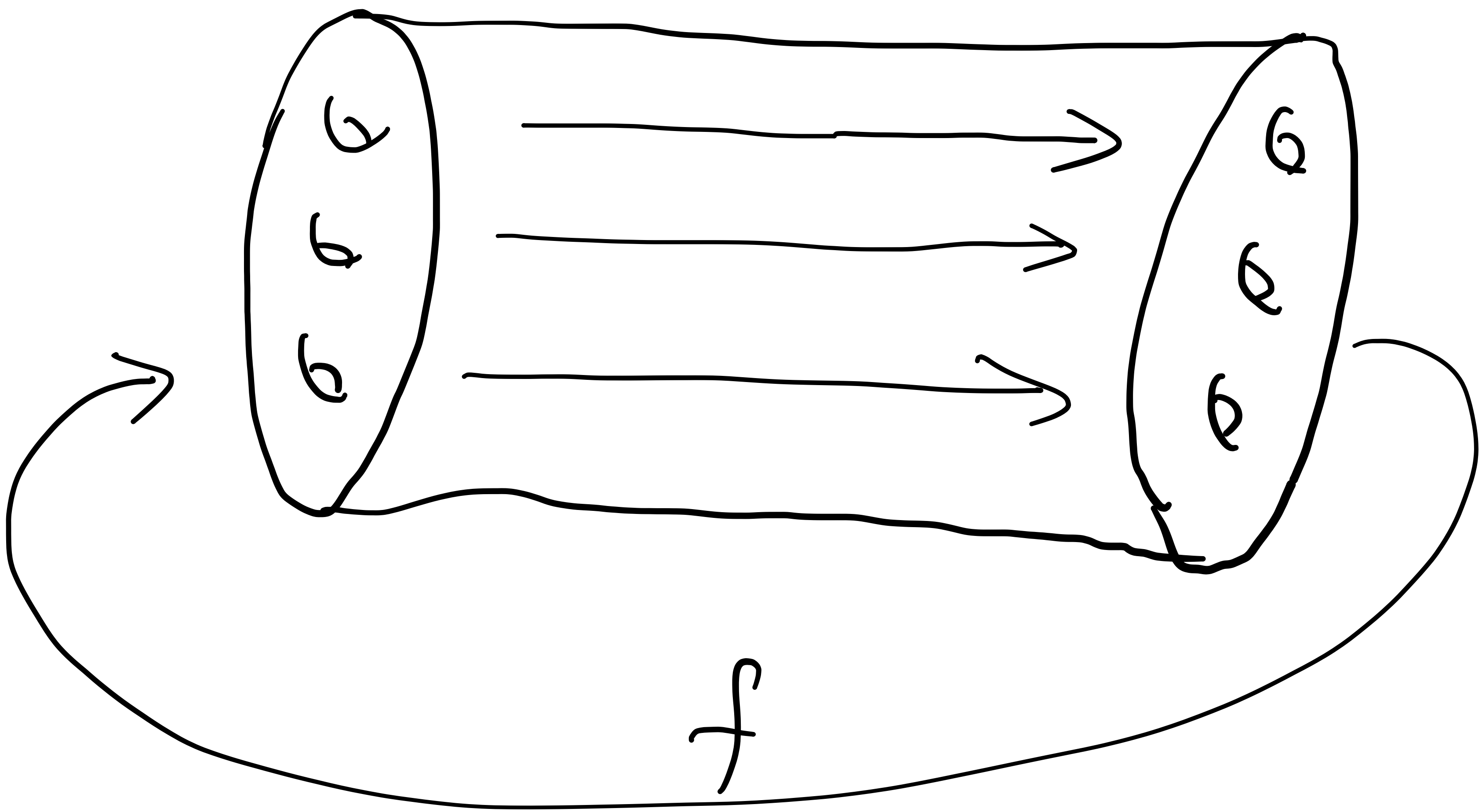
A homeomorphism $f: S_g \to S_g$ is pseudo-Anosov if no power of f preserves a simple closed curve on S_g up to isotopy.

A pseudo-Anosov homeomorphism f will preserve two singular foliations on S_g, contracting one and expanding the other.

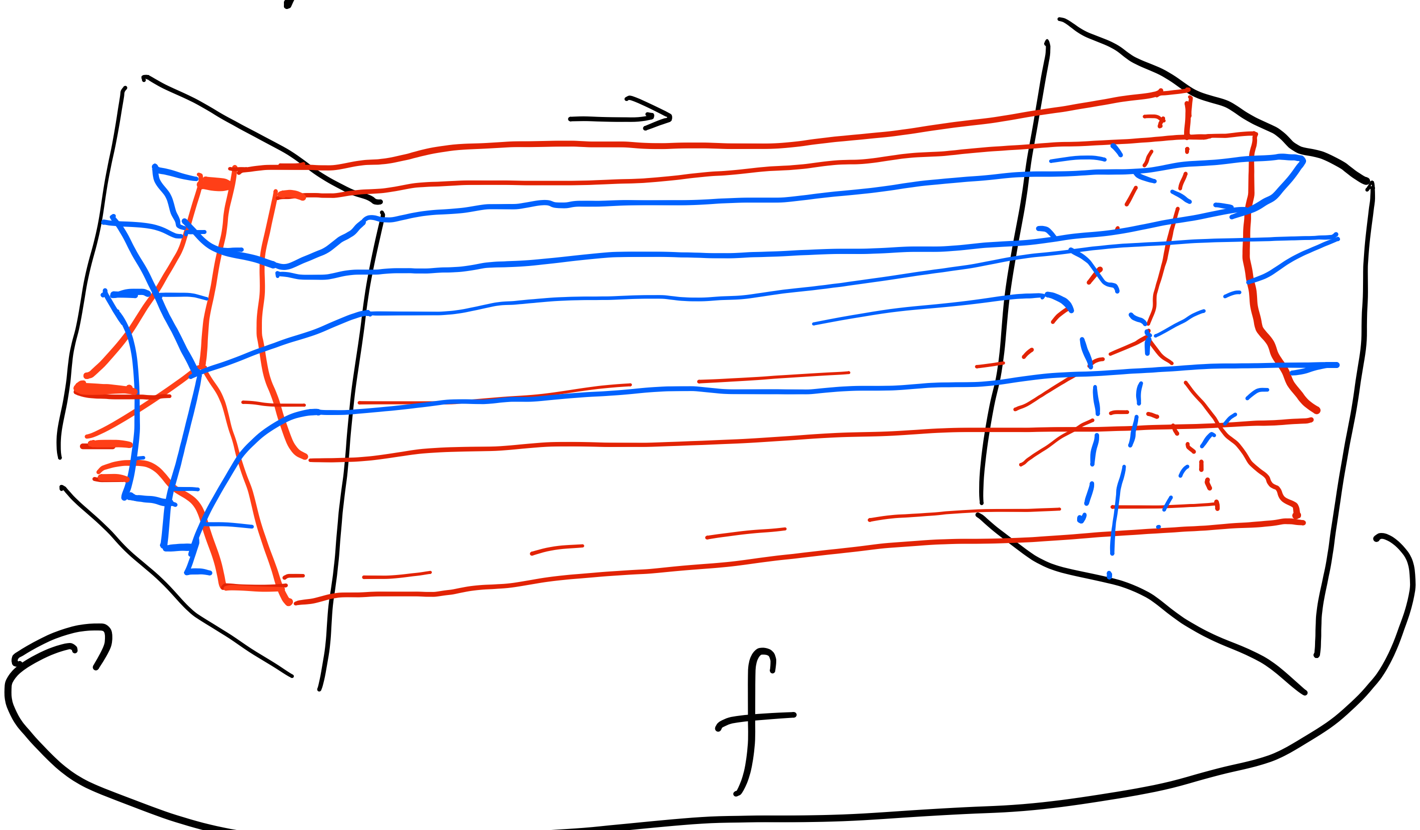


The mapping torus M_f is defined

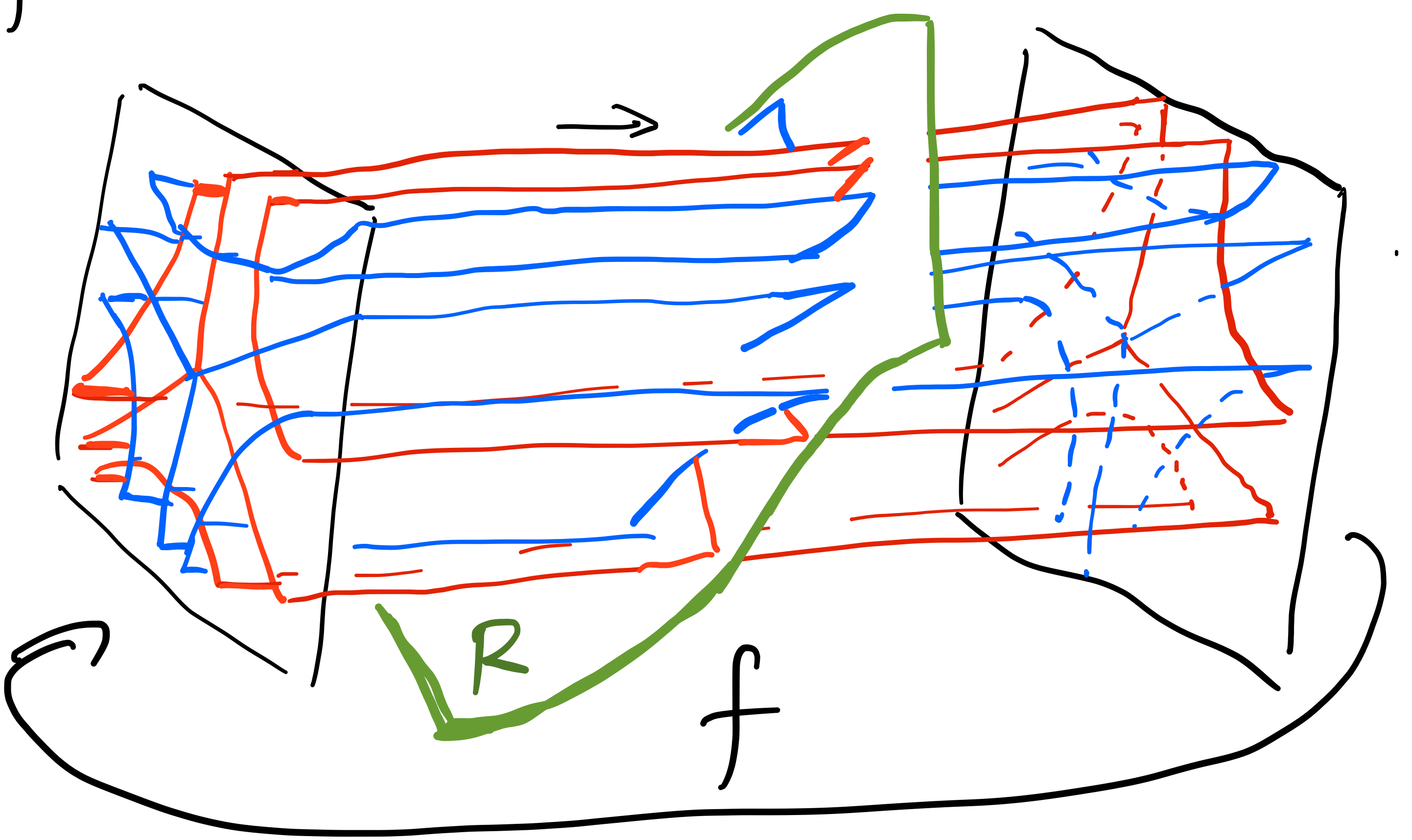
as $\quad S_g \times [0.1] \Big/ (x.1) \sim (f(x).0)$



Since f preserves the two foliations on S_g, we can suspend the foliations to obtain a pair of two dimensional foliations $\mathcal{F}^u$ & $\mathcal{F}^s$ in M:



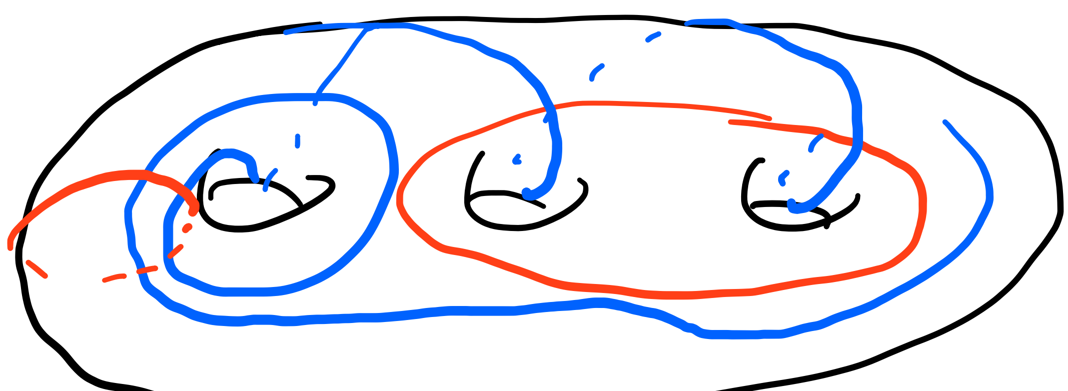
If we have another surface $R \subseteq M_f$ transverse to the $[0,1]$ direction, the intersection $\mathcal{F}^u / \mathcal{F}^s \cap R$ gives two foliations on R:



M_f has a unique hyperbolic structure,

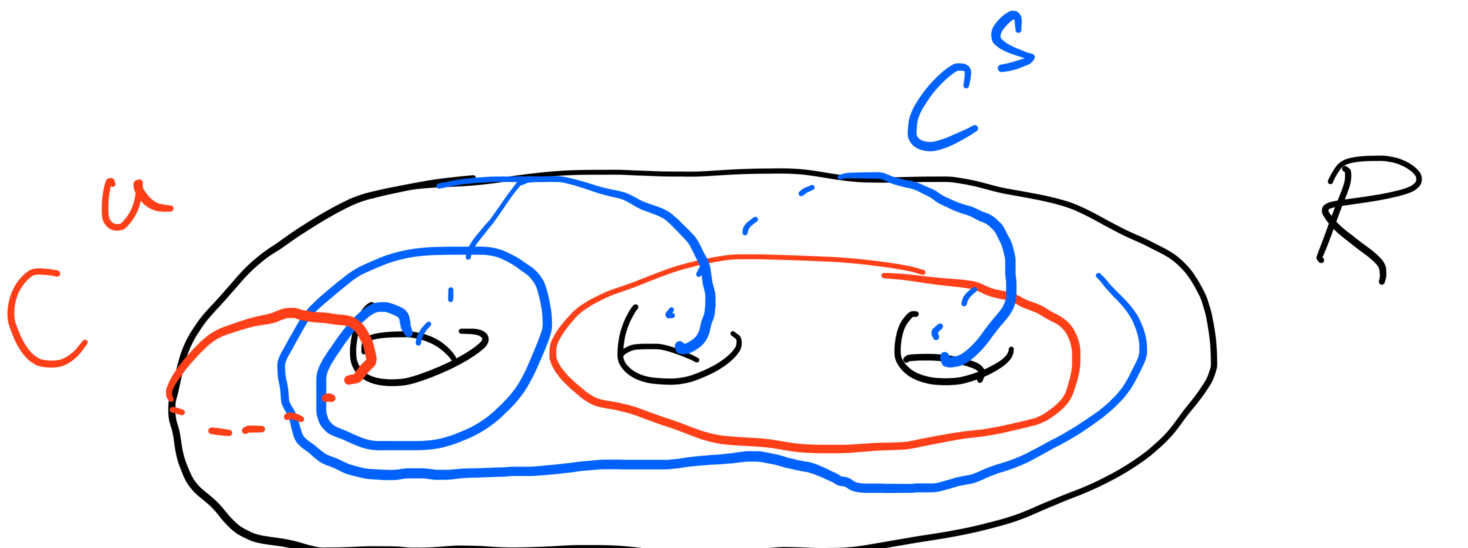
<u>Thm</u> (Cooper-Long-Reid)

For any such transverse surface R, R is quasi-Fuchsian $\Leftrightarrow$ the intersecting foliations on R have closed leaves.



R.

When R is quasi-Fuchsian, let C^s / C^u be the closed multi-curves leaves in $\mathcal{F}^s / \mathcal{F}^u \cap M$.



<u>Q</u>: What can we learn about the geometry of M from C^s and C^u?

Thm A (H.–Taylor) If C^s and C^u look very different, then R is "thick"/has many geom.

With constants depending on $\chi(R)$ and the "flow complexity" of $M \setminus\!\setminus R$,

① Volume bound

$$d_{e(R)}(C^u, C^s) < \mathrm{Vol}(M)$$

② Circumference bound

any closed geodesic γ in M intersecting R essentially satisfies $l(\gamma) > d_{C(R)}(C^u, C^s)$.

<u>Thm B</u>. (H.- Taylor) If c^s and c^u look very different on certain subsurface, then the boundary of the subsurface is short.

$\forall \varepsilon>0$, $\exists K>0$ such that

if Y is a subsurface of R

satisfying ① $d_{e(R)}(\partial Y, c^{s/u}) > 1$

② $d_{e(Y)}(c^u, c^s) > K$.

then $\ell_M(\partial Y) \leq \varepsilon$.

Thank you !

$SL_3(\mathbb{R})$ **Hitchin representations from the inside**

Joint work with Joaquín Lejtreger (IMJ-PRJ)

Joaquín Lema

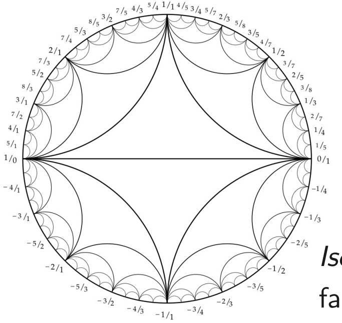
November 8, 2025

Boston College

An enlightening example: ideal triangle group representations

Consider the subgroup Γ generated by reflections along sides of an ideal triangle in $\mathbb{H}^2$.

$$\Gamma \cong W_3 = \langle s_1, s_2, s_3 | s_i^2,\ i = 1, 2, 3 \rangle$$



- We will equip the group with the peripheral system defined by the cusps.
- There is a unique discrete and faithful representation of W_3 in $Isom(\mathbb{H}^2)$ sending the peripheral subgroups to parabolic elements.

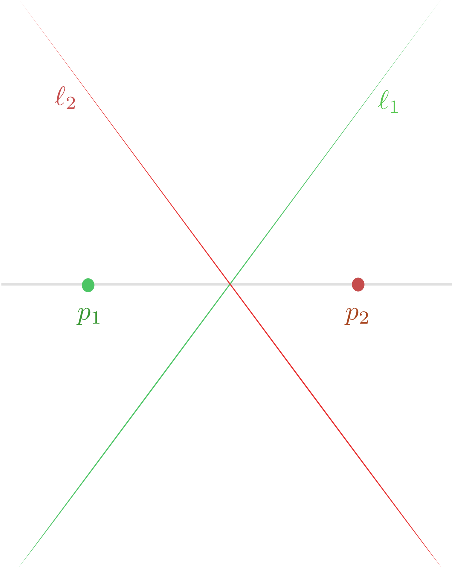
$Isom(\mathbb{H}^2) \cong SO(1, 2) \hookrightarrow SL_3(\mathbb{R})$, there is a one parameter family of representations of W_3 into $SL_3(\mathbb{R})$ sending peripheral subgroups to unipotent matrices.

Constructing the representations (1/2)

We want to find three involutions $\sigma_i \in SL_3(\mathbb{R})$ (i.e., $\sigma_i^2 = Id$) such that

$$\sigma_i \sigma_j \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

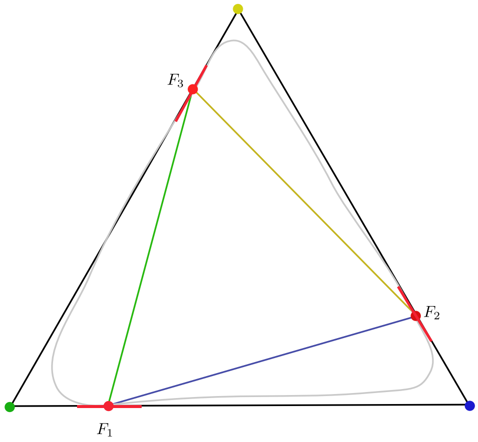
- An involution σ is determined by a pair $(p, \ell) \in \mathbb{RP}^2 \times^{\pitchfork} (\mathbb{RP}^2)^*$.
- $\sigma_i \sigma_j$ is conjugated to a full Jordan block iff:



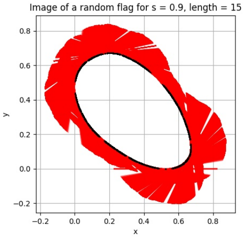
Constructing the representations (2/2)

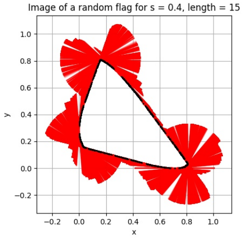
- A triple of flags in general position F_i, $i = 1, 2, 3$ defines a representation $\rho : W_3 \to SL_3(\mathbb{R})$ sending peripheral elements to unipotents.
- The relative character variety is isomorphic to the space of $SL_3(\mathbb{R})$ orbits of triples of flags parametrized by:

$$TR(F_1, F_2, F_3) = \frac{\alpha_1(v_2)\alpha_2(v_3)\alpha_3(v_1)}{\alpha_1(v_3)\alpha_2(v_1)\alpha_3(v_2)}$$



When the triple ratio is positive...





- (Vinberg) $\rho(W_3)$ acts properly discontinuosly on a strictly convex $\Omega \subset \mathbb{RP}^2$.
- When $TR(F_1, F_2, F_3) = 1$, the triple of flags is tangent to a unique ellipsoid.

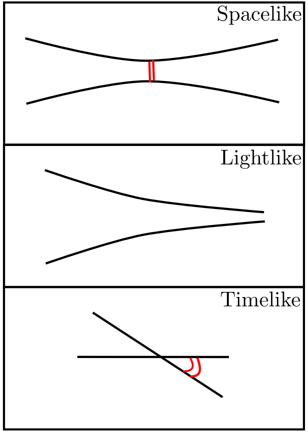
- ρ is relatively Anosov.

Vague goal

Understand the action of ρ in the symmetric space of $SL_3(\mathbb{R})$.

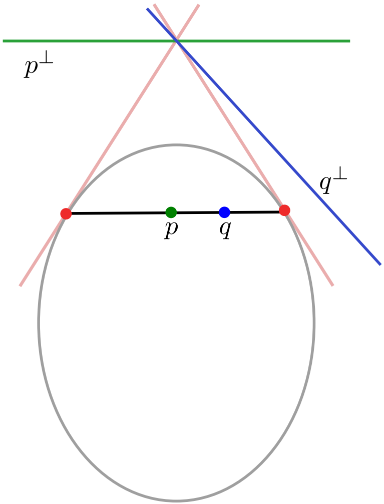
The submanifold fixed by an involutions acting on the symmetric space

- Let X be the space of unit determinant inner products (5-dimensional). The group $SL_3(\mathbb{R})$ acts on X, stabilizer of points are maximal compact subgroup.
- X is a non-positively curved manifold.
- Given $\sigma \in SL_3(\mathbb{R})$ an involution, it fixes a totally geodesic submanifold in X (parallel set). It consists of inner products for which p is orthogonal to ℓ (isometric to $\mathbb{H}^2 \times \mathbb{R}$).
- There is a correspondence between involutions σ, pairs (p, ℓ), and parallel sets.
- The space of parallel sets $\mathcal{P}$ is a pseudo-Riemannian symmetric space.



Spacelike geodesics on the space of parallel sets

- Spacelike geodesics on $\mathcal{P}$ are all isometric.
- These correspond to a one-parameter family of disjoint parallel sets on the symmetric space X.
- We call the resulting (4 manifold) in X a **wall**, it separates the symmetric space into two components.



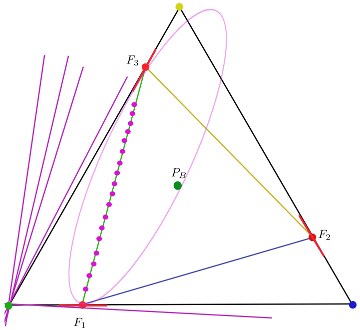
Fundamental domains for the W_3 action on X

Theorem

Let $\rho : W_3 \to SL_3(\mathbb{R})$ be a representation associated to a triple (F_1, F_2, F_3) with triple ratio τ.

- *The walls W_i bound a fundamental domain $\mathcal{D} \subset X$.*
- *$\mathcal{D}/\rho(W_3)$ fibers over $\mathbb{H}^2/\Gamma$, with parallel sets as fibers.*
- *The injectivity radius around the barycenter is:*

$$inj(X/\rho(W_3), q_B) = arccosh\left(\sqrt{\frac{\tau^{\frac{1}{3}} + 2\tau^{\frac{1}{6}} + 2 + \frac{2}{\tau^{\frac{1}{6}}} + \frac{1}{\tau^{\frac{1}{3}}}}{6}}\right)$$



Final remarks

- When $\tau = 1$, we recover *arccosh*$(\frac{2}{\sqrt{3}})$ the inradius of an ideal triangle in $\mathbb{H}^2$.
- For τ big, *inj*$(X/\rho(W_3), q_3) \sim \frac{2}{3}\log(\tau)$, a bit more work shows $h_X(\rho) \sim \frac{3}{2\log\tau}$.
- We can construct Lipschitz maps $F_{\tau_1,\tau_2} : X \to X$ that are $(\rho_{\tau_1}, \rho_{\tau_2})-$equivariant acting isometric on the fibers of the fibration.
- The entire construction depends only on a positive triple of flags. Choosing an ideal triangulation of the surface S_g, and a Hitchin representation $\rho : \pi_1(S_g) \to SL_3(\mathbb{R})$ we can decompose $X/\rho(\pi_1(S_g))$ into foliated regions like before. We can quantify the distance between the fibers in terms of the Fock-Goncharov coordinates of ρ with respect to this triangulation.

Thanks!

Sectional Curvature in Semisimple Lie Groups

Michael Horzepa

Wesleyan University

GATSBY Lightning Talk Fall 2025

Sectional Curvature

Let $\mathcal{G}$ be semisimple, canonical metric ρ

$$K(X, Y) = \frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}$$

- Reduces fully into a polynomial in the entries of X and Y
- Plays a major role in computations of *volume* in $\mathcal{G}$

How so:

Theorem (Günther)

(Loosely) The volume of a ball in $\mathcal{G}$ is bounded below by the volume of a ball with the same radius in $\mathcal{M}_k$, a manifold with constant sectional curvature

$$k = \sup\{K(\Pi) | \Pi \in \mathrm{Gr}_2(\mathfrak{g})\}$$

Computational Techniques

This k has been calculated via root systems of $\mathcal{G}$
Leaves a lot of volume on the table ...

- Since sectional curvature expressed as a polynomial, can compute large dataset quickly
- Can use this to better understand exactly how sectional curvature is distributed

Application: Minimal Covolumes for Lattices in $\mathcal{G}$

- Key result of H.C. Wang that $\exists R_G \in \mathbb{R}$ such that

$$q : G \to G/\Gamma$$

is injective on $B(e, R_G/2)$ for all Γ

How We Generate Data

Key Point: Sectional curvature simplifies for orthonormal generators of Π

1. Begin with a unit vector $v \in \mathfrak{g}$
2. Using Gram-Schmidt find a decomposition $\mathfrak{g} = v \oplus v^{\perp}$
3. Iterate through $v^{\perp}$, then choose a new v and repeat

Using this estimation technique, have improved previous bounds on minimal covolume by

10 Orders of Magnitude

Alternative Method

What if we instead consider the action $G \curvearrowright G/\Gamma$...

- The action is strong mixing, which translate to matrix coefficients vanishing for the restricted representation:

$$\rho^0 : G \to \mathrm{GL}(L^2_0(G/\Gamma))$$

Can derive the following expression:

$$\mathrm{Vol}(G/\Gamma) = \frac{\mathrm{Vol}(B_{R_G})}{(\langle \chi_B, g \cdot \chi_B \rangle - \langle \chi_B^0, g \cdot \chi_B^0 \rangle)^{1/2}}$$

Use uniform matrix coefficient decay for real rank ≥ 2...

$$\mathrm{Vol}(G/\Gamma) \geq \frac{\mathrm{Vol}(B_R)}{\sqrt{1 - C_G A_G^{-\delta_G}}}$$

Persistently foliar (1,1) non-L-space knots

Qingfeng Lyu

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Nov 8, 2025

The L-space knot conjecture

We first recall the famous L-space conjecture:

Conjecture (Boyer-Gordon-Watson, Juhasz)

Let M be a closed, orientable, and irreducible 3-manifold. Then the following statements are equivalent:

- *M is not an L-space ("NLS");*
- *M admits a co-orientable taut foliation ("CTF");*
- *$\pi_1(M)$ is left-orderable ("LO").*

We also recall that a knot $K \subset Y$ in an L-space is called an **L-space knot** if it admits non-trivial L-space surgeries.

Hence if K is not an L-space knot, nontrivial surgeries of K are either reducible or expected to admit co-orientable taut foliations.

The L-space knot conjecture

Definition

A knot in a 3-manifold is called **persistently foliar**, if except for one meridional slope, all boundary slopes of the knot complement are strongly realized by co-oriented taut foliations.
(That is, to each boundary slope there exists a co-oriented taut foliation of the knot complement, which intersects the boundary torus transversely in a *linear* foliation of that slope.)

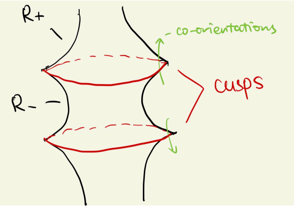
Conjecture (Delman-Roberts)

A knot in an L-space is persistently foliar if and only if it has no non-trivial L-space or reducible surgeries.

Meridional cusps

One of the reasons that we consider persistent foliar-ity of knots is that there is a special strategy to prove it. This classical construction dates back to 1960s-70s, and is usually referred to as the "even meridional cusps" construction.

Suppose we can foliate the knot complement, such that the foliation is transverse to the boundary torus at some meridional annuli (the "cusps"), and tangent elsewhere, with co-orientations as below.



When filling in the solid torus, we can view it as a sutured manifold, where the cusps are the sutures. Then the solid torus is taut sutured unless the sutures bound disks in the solid torus.

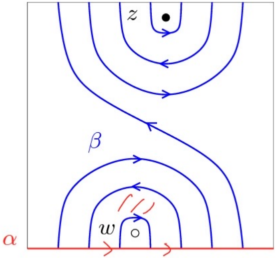
(1,1) non-L-space knots

We use branched surfaces to construct taut foliations. Our construction relies on a topological description of (1,1) L-space knots by Greene-Lewallen-Vafaee.

Theorem (Greene-Lewallen-Vafaee)

A reduced (1,1) diagram represents an L-space knot if and only if it is coherent, that is, all the rainbow arcs are of the same direction.

An incoherent diagram is given below.



Laminations?

For any incoherent reduced (1,1) diagram, we can construct a branched surface, such that

1. The regular neighborhood of the branched surface is homeomorphic to the knot complement.
2. The branched surface has 4 meridional cusps and 2 meridional boundary circles.

Proposition (L.)

If this branched surface fully carries a lamination, then the corresponding knot is persistently foliar.

- We can verify that for (1,1) almost L-space knots our branched surfaces do fully carry laminations.
- We are working on the general case (work in progress).

Invariant Radon measures on $\mathcal{ML}$ for subgroups of mapping class group

Dongryul M. Kim

Yale University

GATSBY 2025 Fall

S : surface (finite-type, Euler char. < 0)

$\mathrm{Mod}(S) =$ Mapping class group of S

$\mathcal{ML} = \{$meas. laminations on $S\}$

$$\mathrm{Mod}(S) \curvearrowright \mathcal{ML}$$

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Theorem (Thurston)

$\exists$ $\mathrm{Mod}(S)$-inv. Radon meas. on $\mathcal{ML}$ of Leb. class.

"Thurston measure μ_{Th}"

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"Thurston measure μ_{Th}*"*

Theorem (Masur, Veech)

μ_{Th} *is* $\mathrm{Mod}(S)$*-ergodic.*

Question

Any other ergodic, invariant Radon measure?

Lindenstrauss–Mirzakhani, Hamenstädt:

$$\text{Classify all } \mathrm{Mod}(S)\text{-inv. Radon meas. on } \mathcal{ML}.$$

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Classify all $\mathrm{Mod}(S)$-inv. Radon meas. on $\mathcal{ML}$.

Main step:

Theorem (Lindenstrauss–Mirzakhani, Hamenstädt)

$\mathrm{Mod}(S) \curvearrowright \mathcal{ML}$ is essentially uniquely ergodic, i.e.,

$$\mu \text{ is erg. Radon meas. on } \mathcal{R}_{\mathrm{Mod}(S)} \implies \mu = \mu_{Th} \cdot \text{const.}$$

where $\mathcal{R}_{\mathrm{Mod}(S)} \subset \mathcal{ML}$ consists of “recurrent” meas. lam., i.e.,

assoc. Teich. geod. rays are recurrent in $\mathrm{Mod}(S) \backslash \mathrm{Teich}(S)$

Major inputs:

- $\mathrm{Vol}(\mathrm{Mod}(S) \backslash \mathrm{Teich}(S)) < +\infty$
- Minsky–Weiss' non-div. of unipotent flows.

Question (Lindenstrauss–Mirzakhani)

Can anything be said for **subgroups** $\Gamma < \mathrm{Mod}(S)$*?*

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Let $\Gamma < \mathrm{Mod}(S)$ *be non-elementary.*

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Moreover,

$\Gamma \curvearrowright (\mathcal{ML}, \mu_\Gamma)$ *is ergodic* $\iff$ *geod. flow on* $\Gamma\backslash \mathrm{Teich}(S)$ *is ergodic*

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Finally,

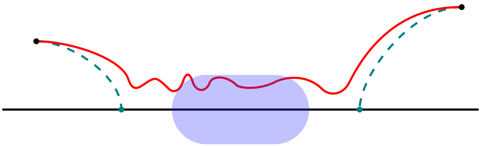
$$\mu \text{ is erg. Radon meas. on } \mathcal{R}_\Gamma \quad \Longrightarrow \quad \mu = \mu_\Gamma \cdot \text{const.}$$

Key in the proof:

Theorem (Minsky's contraction thm)

In $\mathrm{Teich}(S)$, the axis of a pseudo-Anosov mapping class is contracting.

(In fact, contracting holds for more general geodesics)



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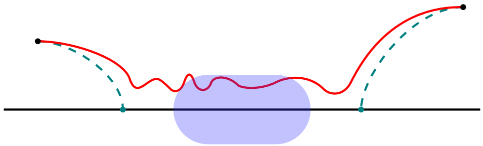
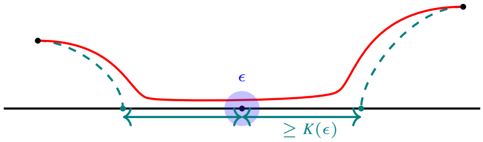


We observe: Actually, something stronger holds... "squeezing"



Corollary (Choi–K.)

Let $\Gamma < \mathrm{Mod}(S)$ *be non-elt. convex cocompact*

If μ *is a* Γ*-inv. erg. Radon meas. on* $\mathcal{ML}$*, then either*

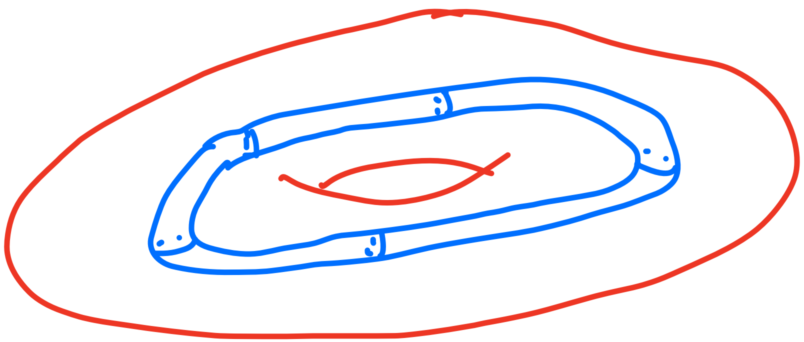
$$\mu = \mu_\Gamma \cdot \textit{const.}$$

or

$$\mu = \textit{const.} \cdot \sum_{\gamma \in \Gamma} \textit{Dirac}(\gamma \cdot \xi) \quad \textit{for some non-recurrent } \xi \in \mathcal{ML}$$

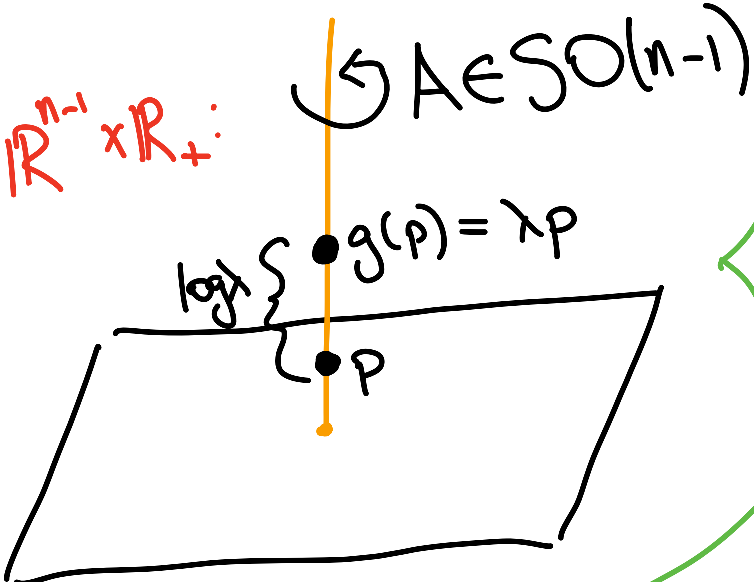
Separating nested Margulis tubes

Matt Zevenbergen (Boston College)



① Loxodromic isometries of $\mathbb{H}^n$ are conjugate to $g(\vec{x},t) = \lambda(A\vec{x},t)$, $A \in SO(n-1)$

$\mathbb{H}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}_+$:



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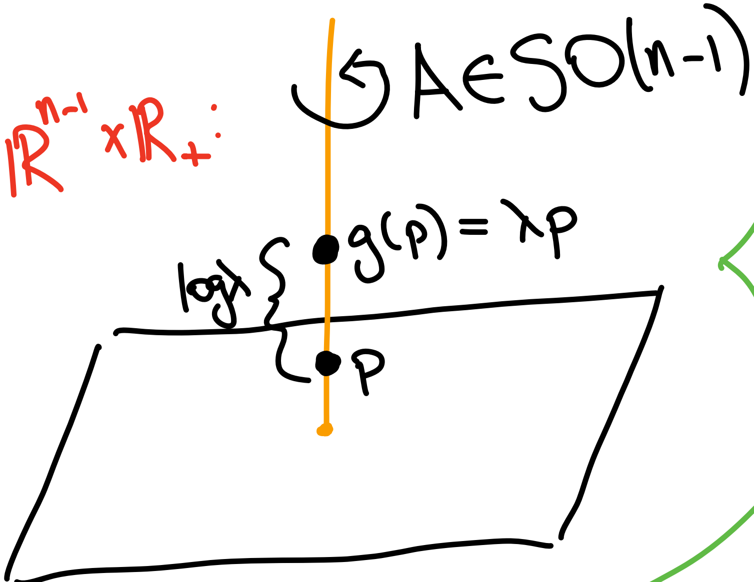

2. For $\varepsilon>0$, the <u>ε- thin part</u> of $\mathbb{H}^n$ for g is

$$\mathrm{Thin}(g,\varepsilon)=\{p\in\mathbb{H}^n \mid d(p,g^k(p))<\varepsilon \text{ for some } k\neq 0\}$$

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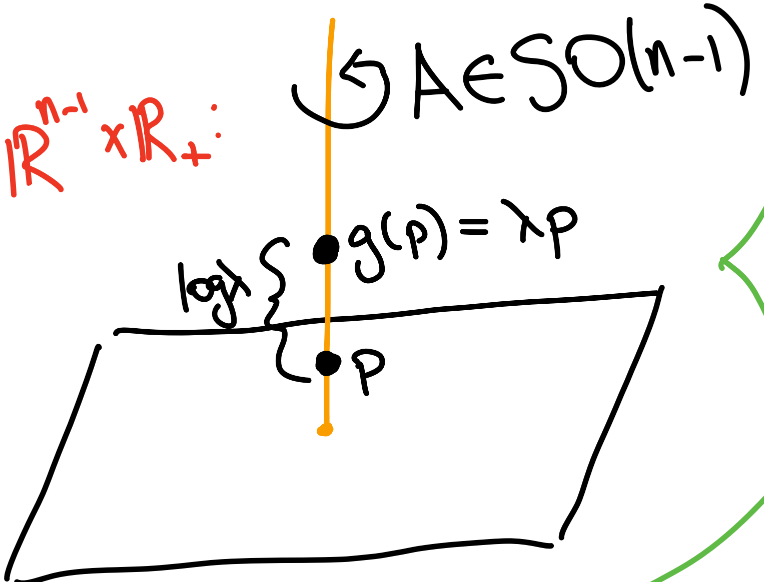

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③ <u>Thm</u>: (Brooks-Matelski, Meyerhoff, Minsky)

For $0<\delta<\varepsilon$, $\exists D=D(\varepsilon,\delta)$ with $D \xrightarrow{\delta\to 0} \infty$ such that every loxodromic $g\in \mathrm{Isom}^+(\mathbb{H}^3)$ with transl. length$(g)<\delta$ satisfies

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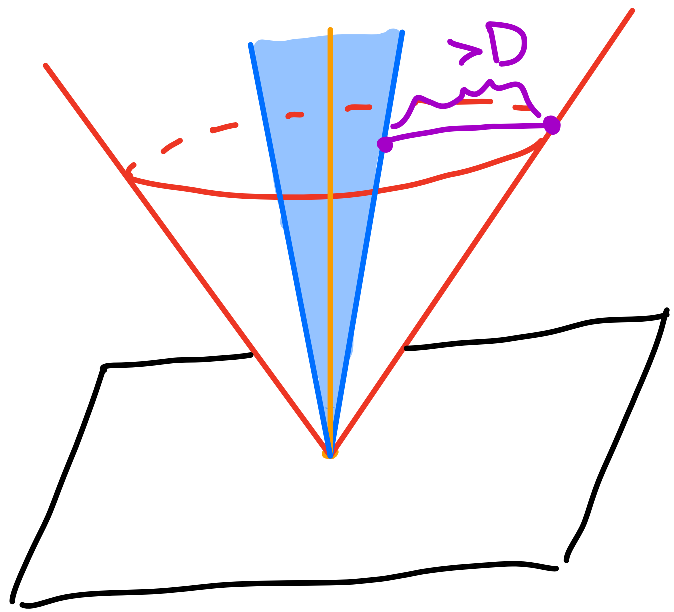

④ Proof Sketch:

- As transl. length$(g) \to 0$, $d(\text{axis}(g), \partial \text{Thin}(g,\varepsilon)) \to \infty$.
- $\text{Thin}(g,\delta)$ and $\text{Thin}(g,\varepsilon)$ are metric neighborhoods of $\text{axis}(g)$.
  - ↳ Shortest path is radial from $\text{axis}(g)$.
- How each g^k moves points on geodesics radial from $\text{axis}(g)$ is controlled.

∎

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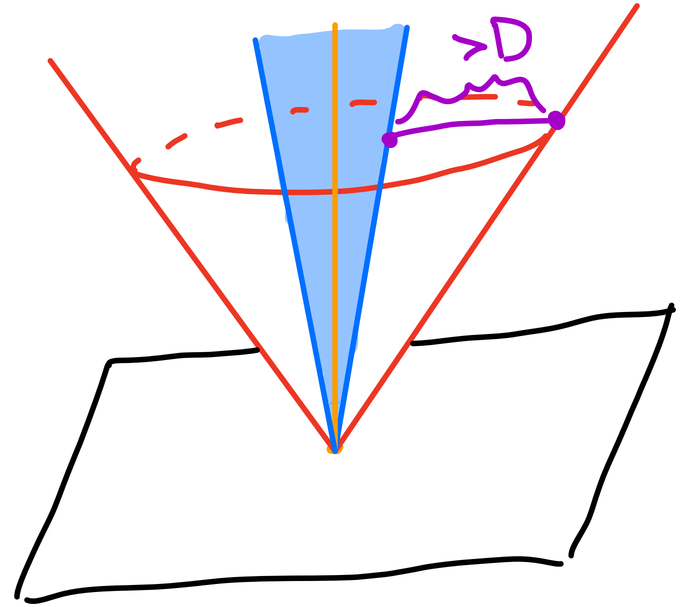

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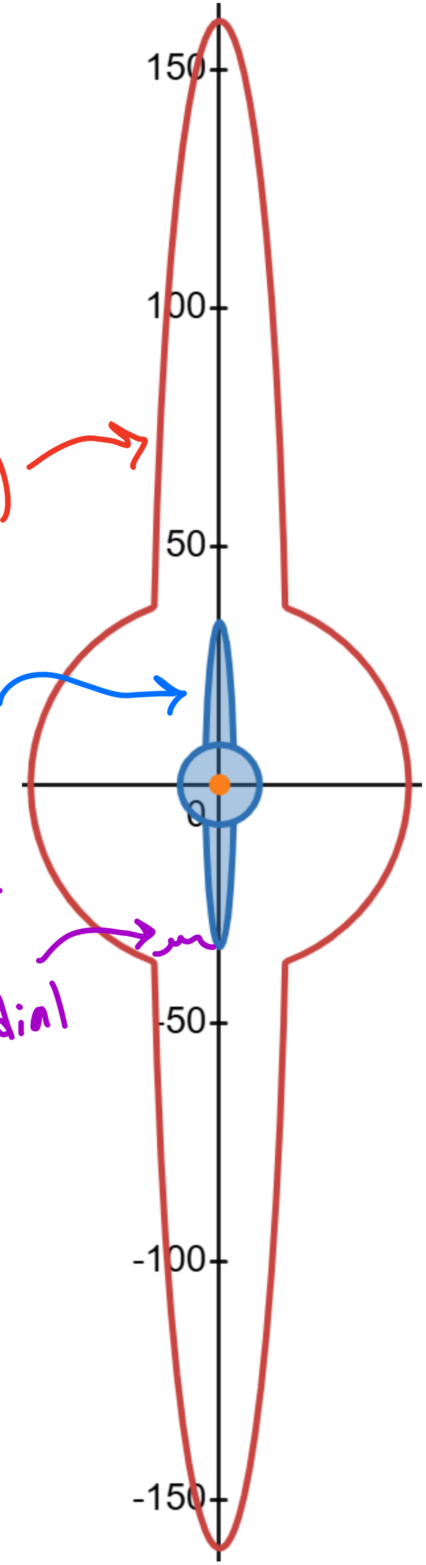
Depicts intersection of $\mathbb{H}^3$ with height-1 horosphere, projected to 2-dimensional subspace.



⑥ <u>Example:</u> $g(\vec{x},t) = e^{1/8}(A\vec{x},t)$

where $A = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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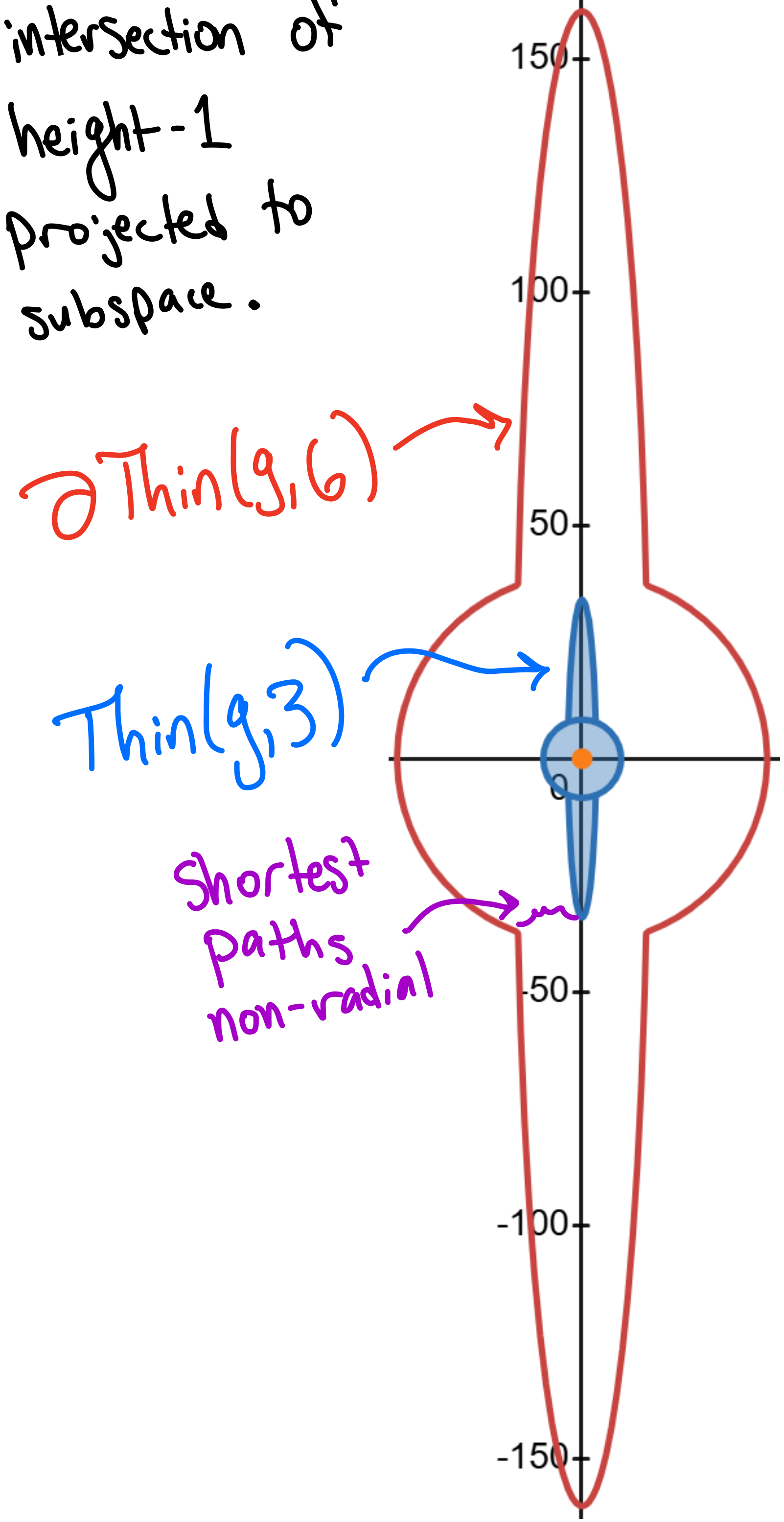

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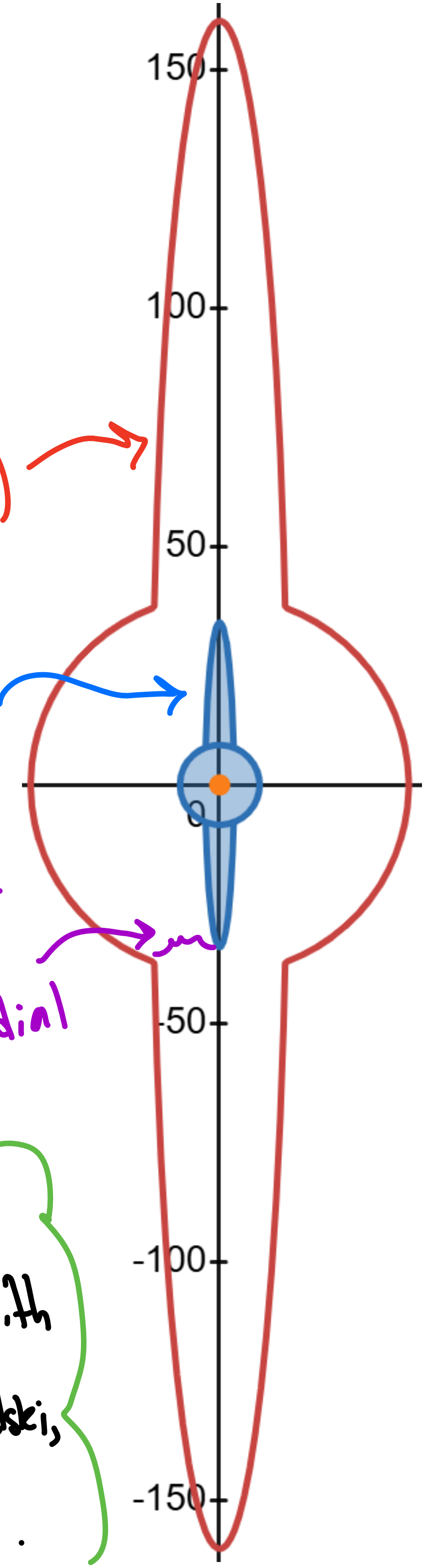

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⑦ Thm: (Z., in progress) Actually can replace 3 with 4 in the Brooks-Matelski, Meyerhoff, Minsky theorem.