Minimal Volume Entropy in Dimension One

Arianna Zikos

Wesleyan University

GATSBY – November 16, 2024

Volume Entropy of a Graph

Definition

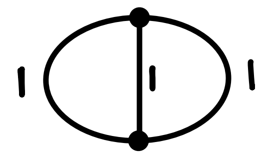
Let X be a finite graph and g be a metric on X so that edges are isometric to segments in $\mathbb{R}$. The volume entropy of (X, g) is

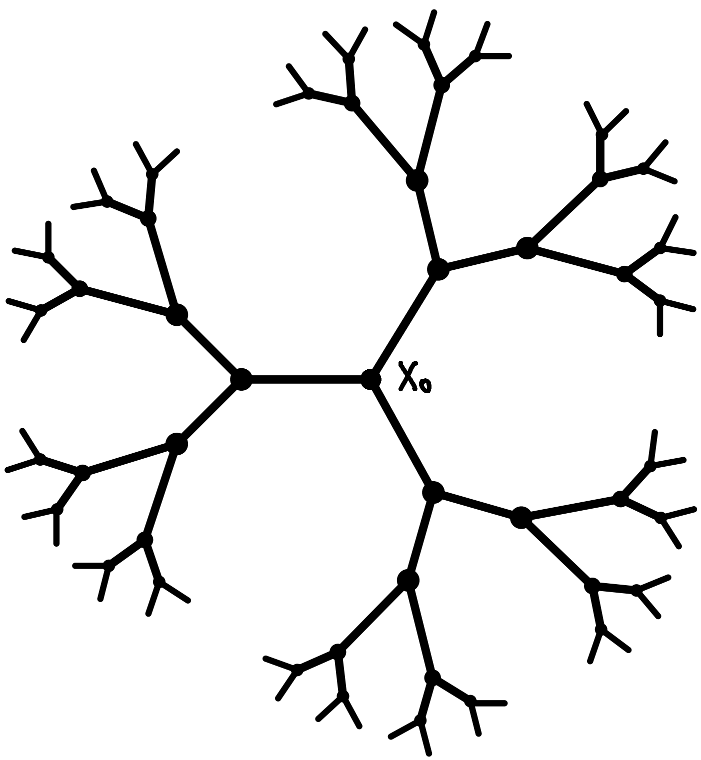
$$ent(X, g) = \lim_{n\to\infty} \frac{\log(\#S(x_0, n))}{n}$$

Cardinality of the sphere of radius n centered at X_0 in universal cover

Exponential growth rate of $\#S(x_0,n)$

Example

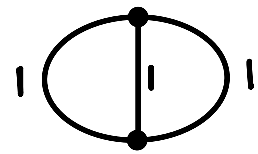
Calculate $ent(X,g)$ for $(X,g) =$ 

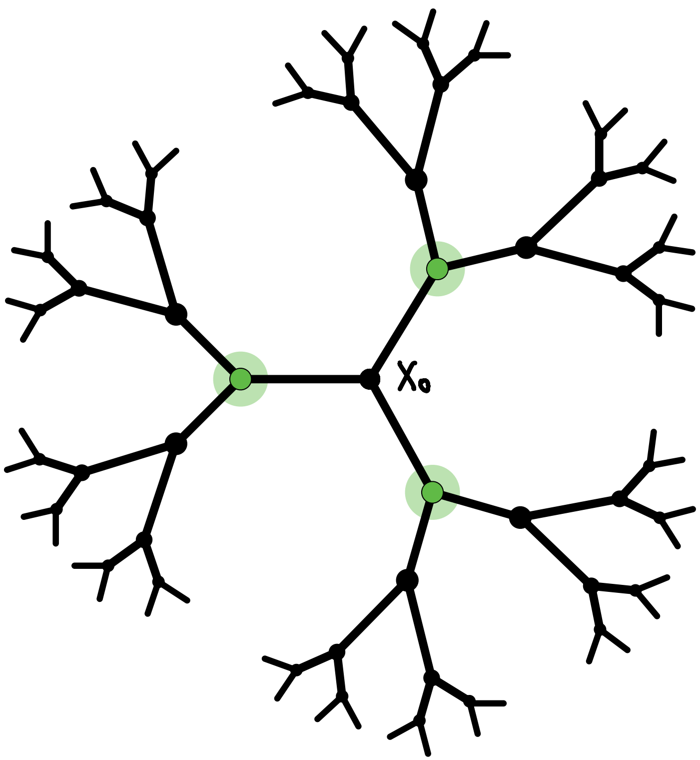


n	#S(x0, n)
1	
2	
3	
n	

$$ent(X,g) = \lim_{n\to\infty} \frac{\log(\#S(x_0,n))}{n}$$

Example

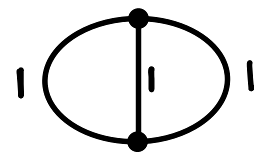
Calculate $ent(X, g)$ for $(X, g) =$ 

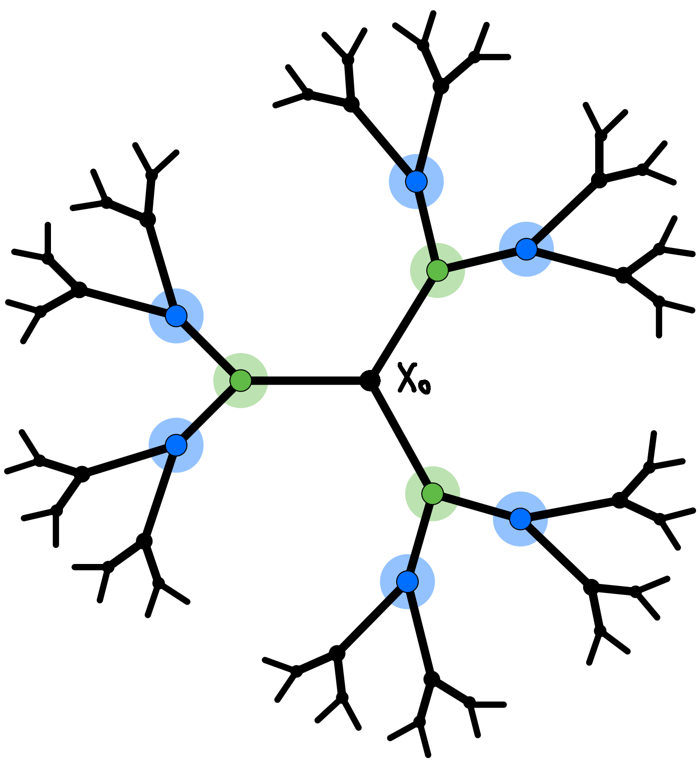


n	$\#S(x_0, n)$
1	$\#S(x_0, 1) = 3$
2	
3	
n	

$$ent(X, g) = \lim_{n\to\infty} \frac{\log(\#S(x_0, n))}{n}$$

Example

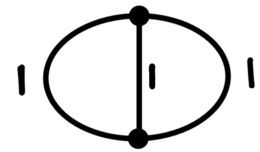
Calculate $ent(X,g)$ for $(X,g) =$ 

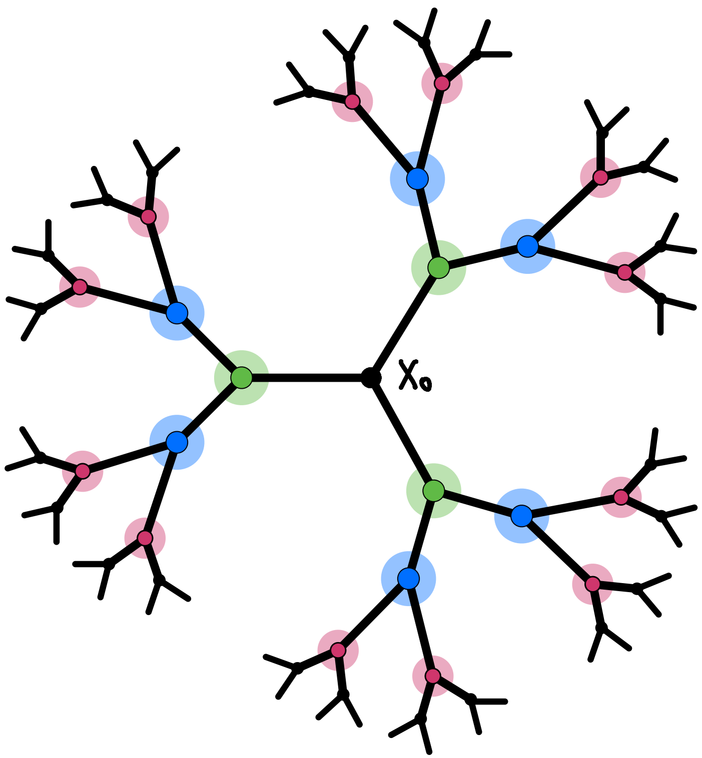


n	$\#S(x_0,n)$
1	$\#S(x_0,1)=3$
2	$\#S(x_0,2)=3\cdot 2$
3	
n	

$$ent(X,g) = \lim_{n\to\infty} \frac{\log(\#S(x_0,n))}{n}$$

Example

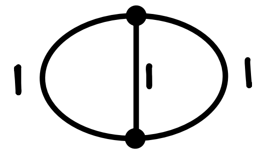
Calculate $ent(X,g)$ for $(X,g) =$ 

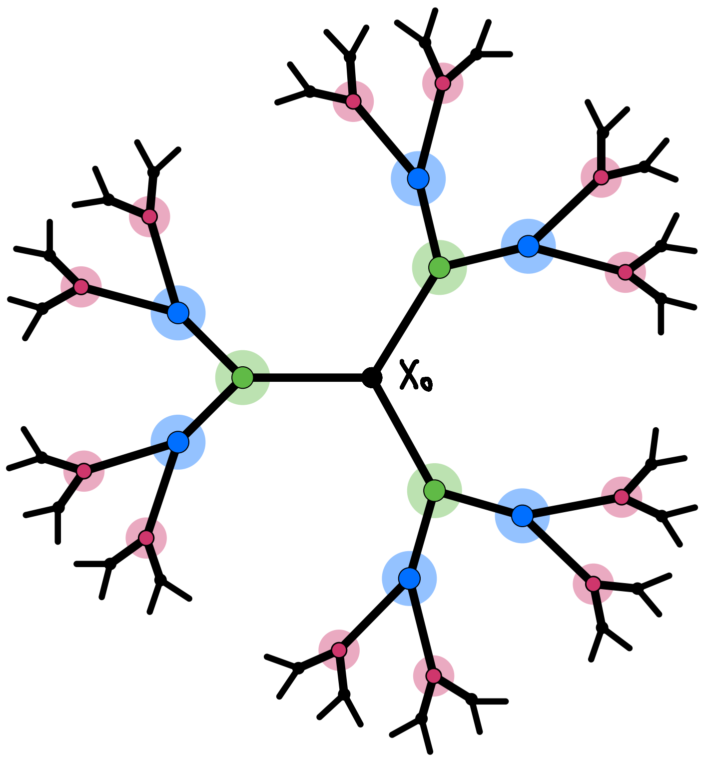


n	$\#S(x_0,n)$
1	$\#S(x_0,1) = 3$
2	$\#S(x_0,2) = 3\cdot 2$
3	$\#S(x_0,3) = 3\cdot 2^2$
n	

$$ent(X,g) = \lim_{n\to\infty} \frac{\log(\#S(x_0,n))}{n}$$

Example

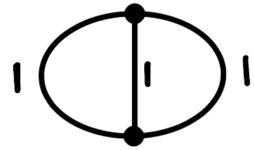
Calculate $ent(X,g)$ for $(X,g) =$ 

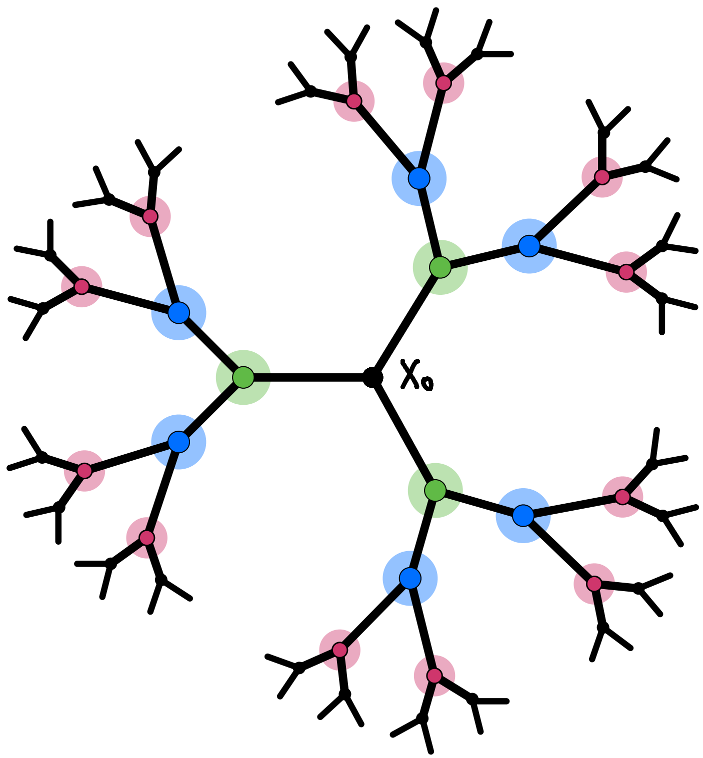


n	$\#S(x_0,n)$
1	$\#S(x_0,1)=3$
2	$\#S(x_0,2)=3\cdot 2$
3	$\#S(x_0,3)=3\cdot 2^2$
n	$\#S(x_0,n)=3\cdot 2^{n-1}$

$$ent(X,g) = \lim_{n\to\infty} \frac{\log(\#S(x_0,n))}{n}$$

Example

Calculate $ent(X,g)$ for $(X,g) =$ 



n	$\#S(x_0,n)$
1	$\#S(x_0,1)=3$
2	$\#S(x_0,2)=3\cdot 2$
3	$\#S(x_0,3)=3\cdot 2^2$
n	$\#S(x_0,n)=3\cdot 2^{n-1}$

$$ent(X,g) = \lim_{n\to\infty} \frac{\log(\#S(x_0,n))}{n} = \lim_{n\to\infty} \frac{\log(3\cdot 2^{n-1})}{n} = \log(2)$$

Definition

If X is a finite graph, then the minimal volume entropy of X is

$$ent(X) = \inf\{ent(X,g)Vol(X,g) \mid g \text{ is a metric on } X\}.$$

makes ent(X) scale invariant

Definition

If X is a finite graph, then the minimal volume entropy of X is

$$ent(X) = \inf\{ent(X, g)Vol(X, g) \mid g \text{ is a metric on } X\}.$$

Theorem

Lim calculates the minimal volume entropy of every finite graph.

Definition

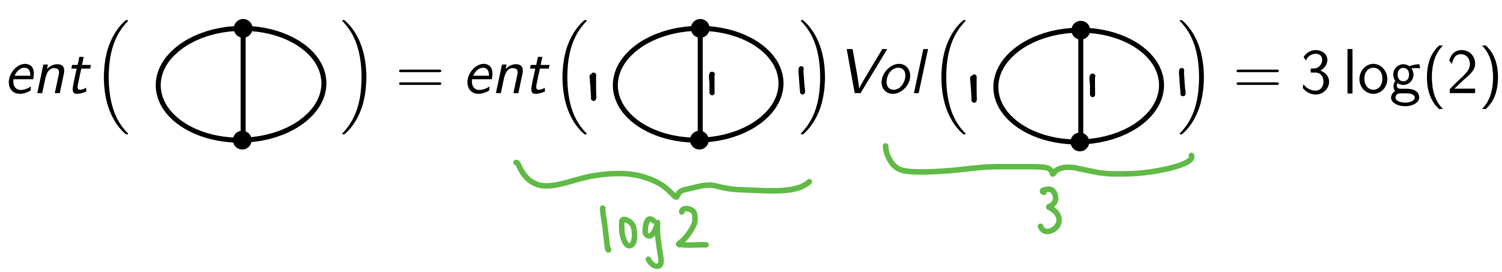
If X is a finite graph, then the minimal volume entropy of X is

$$ent(X) = \inf\{ent(X, g) Vol(X, g) \mid g \text{ is a metric on } X\}.$$

Theorem

Lim calculates the minimal volume entropy of every finite graph.

Example



The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ (set we are indexing over)	Results
X Finite graph		

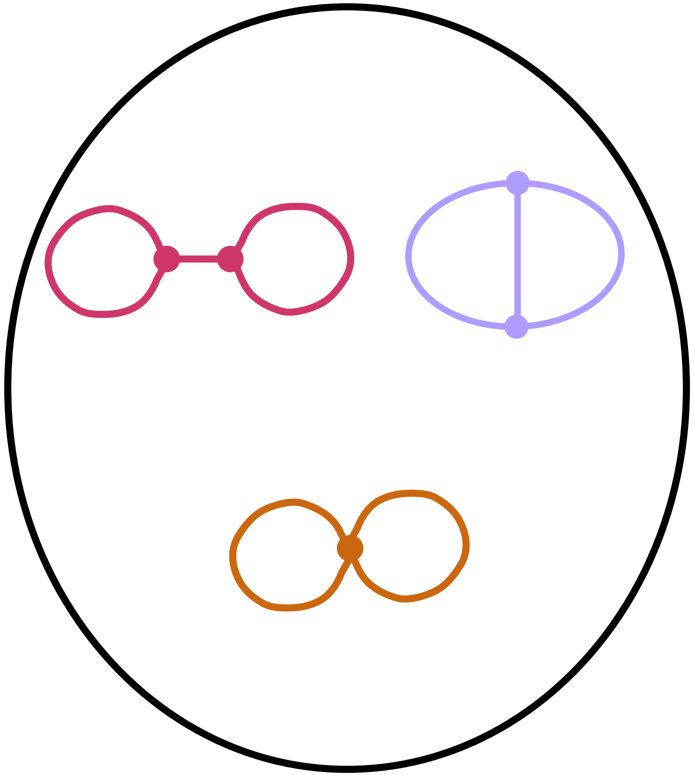
The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \text{——}\}$ set we are indexing over	Results
X Finite graph	g metric on X	

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ set we are indexing over	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \text{\_\_\_\_}\}$ (set we are indexing over)	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group		

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ (set we are indexing over)	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \simeq F_n$, g metric	

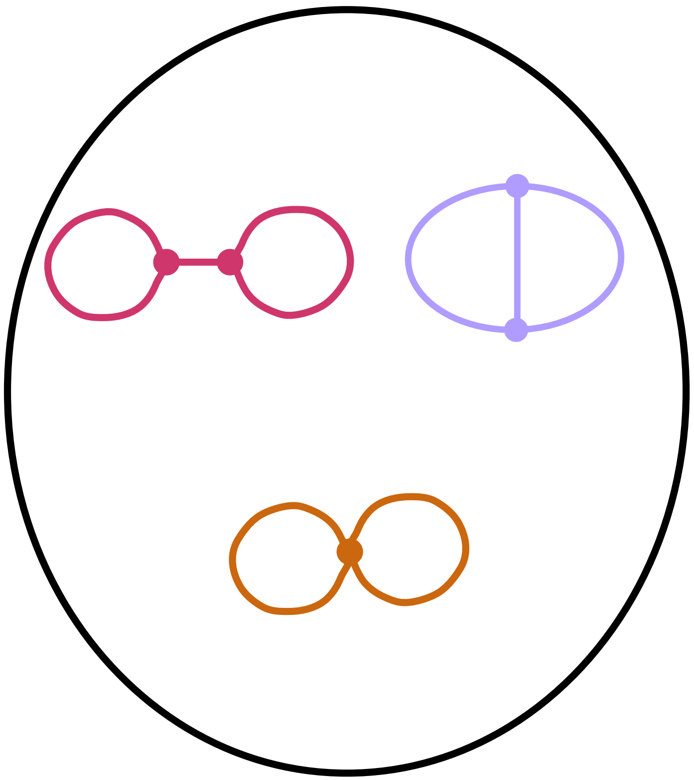
Example F_2



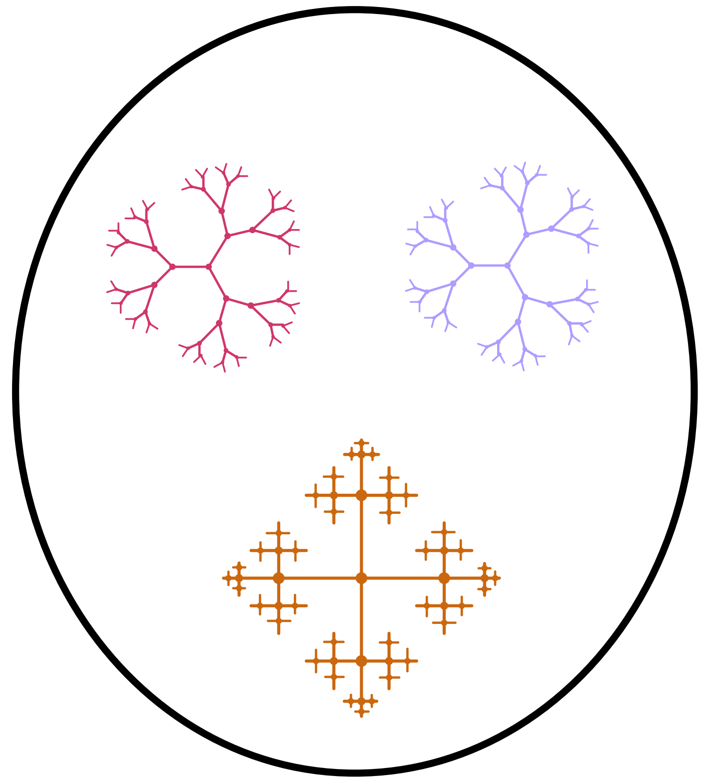
Graphs X with $\pi_1(X) \cong F_2$

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ set we are indexing over	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \cong F_n$ g metric $\updownarrow$ covering space theory T tree with $F_n \curvearrowright T$ freely and cocompactly g metric	

Example F_2

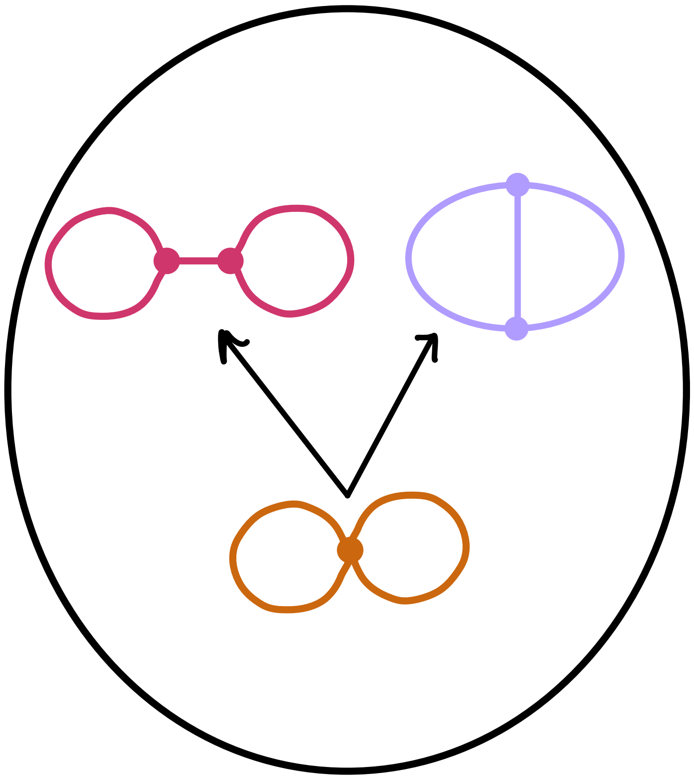


Graphs X with $\pi_1(X) \cong F_2$

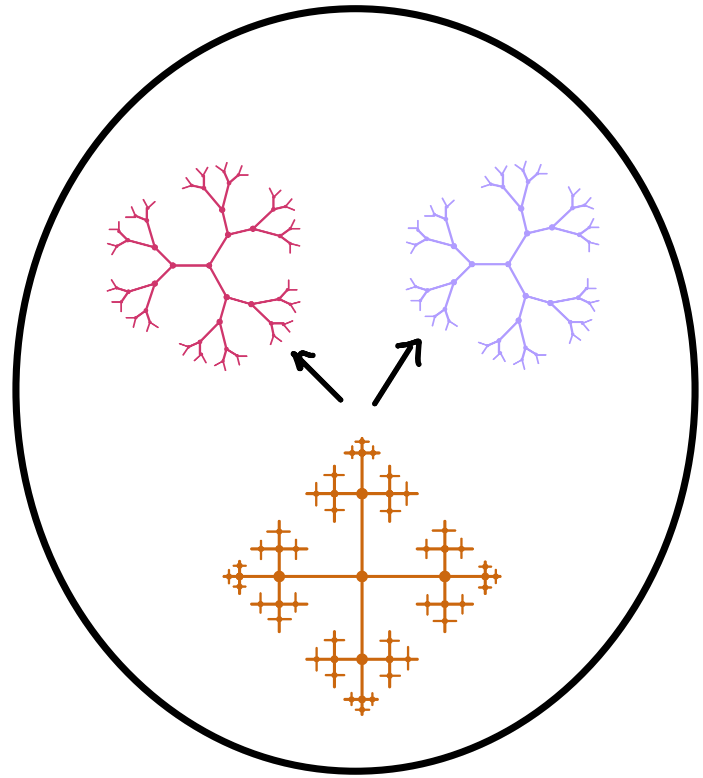


Trees T with $F_n \curvearrowright T$ freely and cocompactly

Example F_2



Graphs X with $\pi_1(X) \cong F_2$



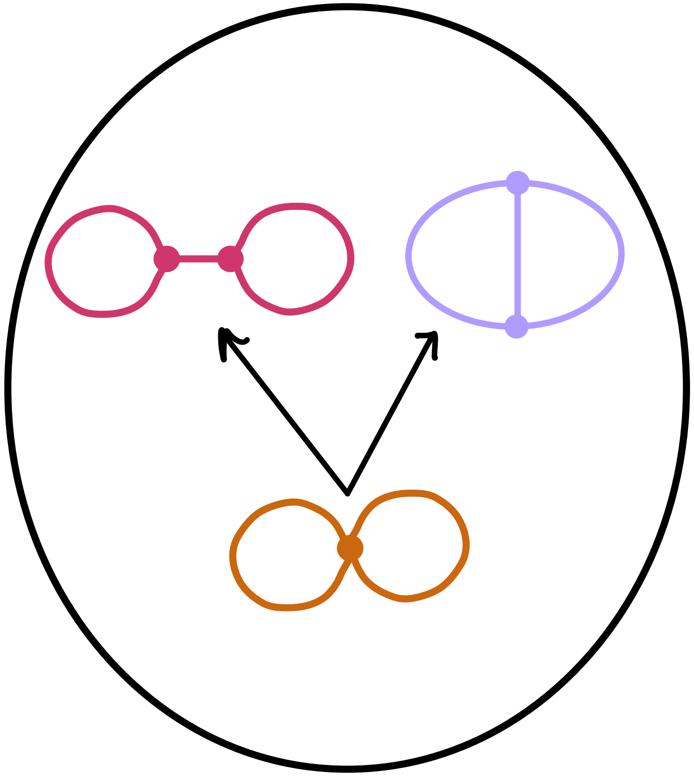
Trees T with $F_n \curvearrowright T$ freely and cocompactly

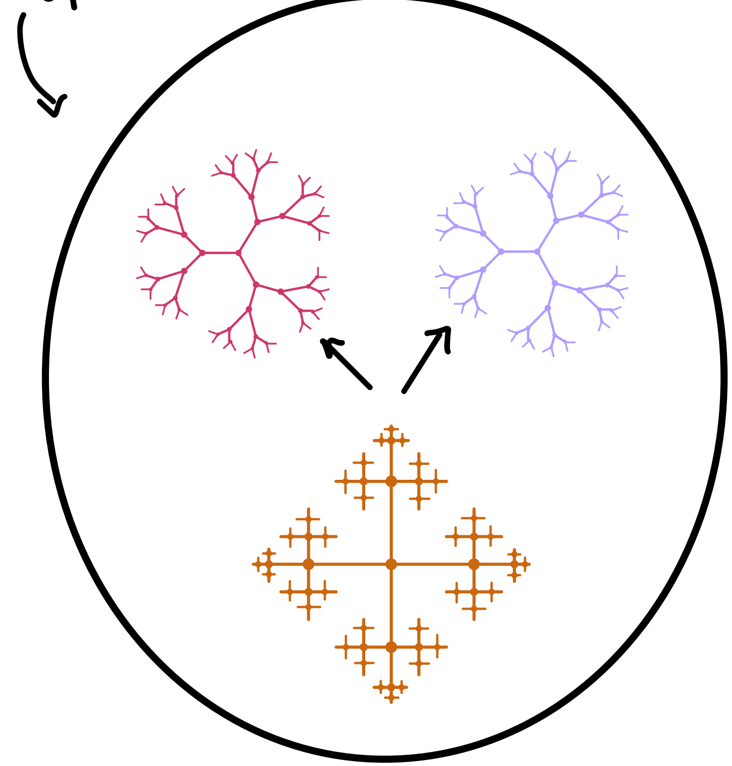
Key idea: Folding reduces minimal volume entropy.

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \text{\_\_\_\_}\}$ (set we are indexing over)	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \simeq F_n$ g metric ↕ covering space theory T tree with $F_n \curvearrowright T$ freely and cocompactly g metric	$ent(F_n) = 3(n-1)\log 2$

Example F_2

For all n, T_3 is in the corresponding space of trees





Graphs X with $\pi_1(X) \cong F_2$

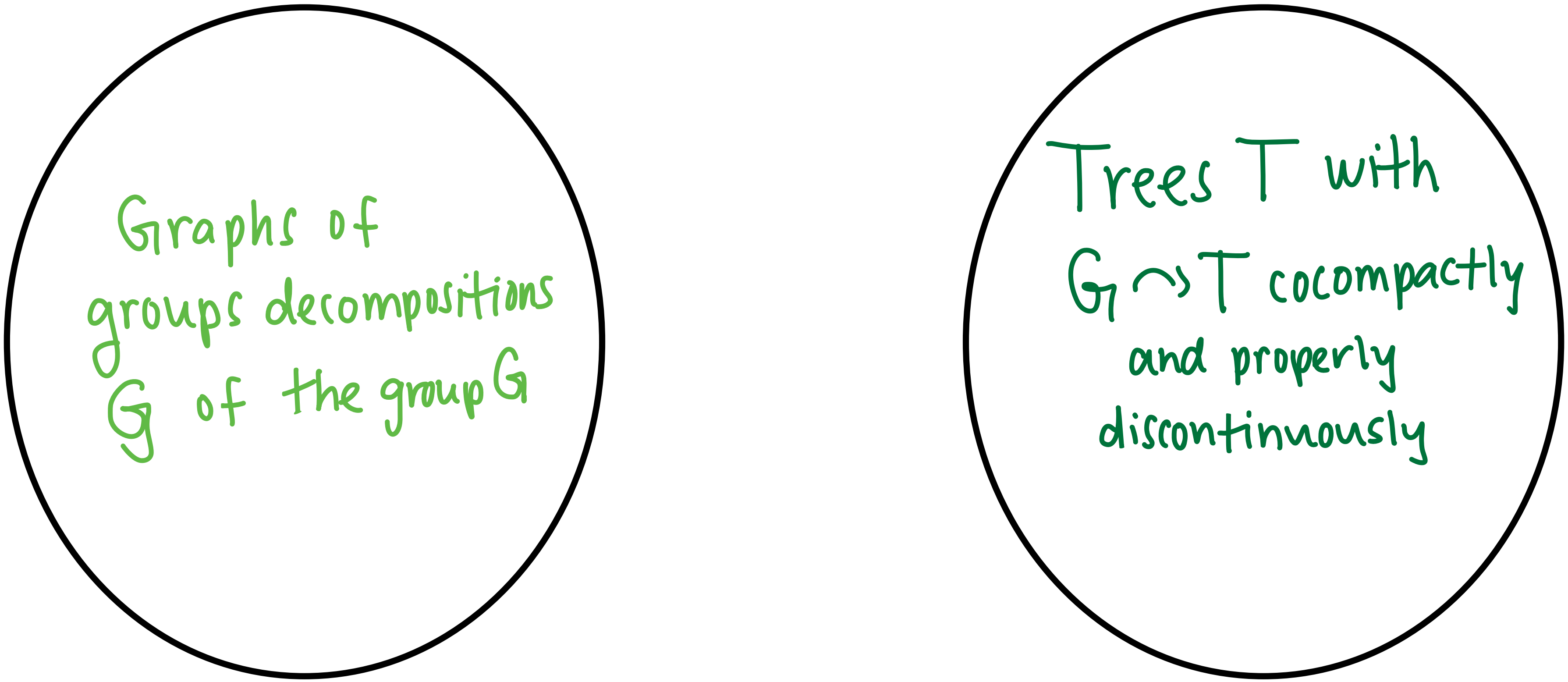
Trees T with $F_n \curvearrowright T$ freely and cocompactly

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_ \}$ (set we are indexing over)	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \simeq F_n$, g metric ↕ covering space theory T tree with $F_n \curvearrowright T$ freely and cocompactly, g metric	$ent(F_n) = 3(n-1)\log 2$
G Virtually free group $F_n \underset{k}{\leq} G$		

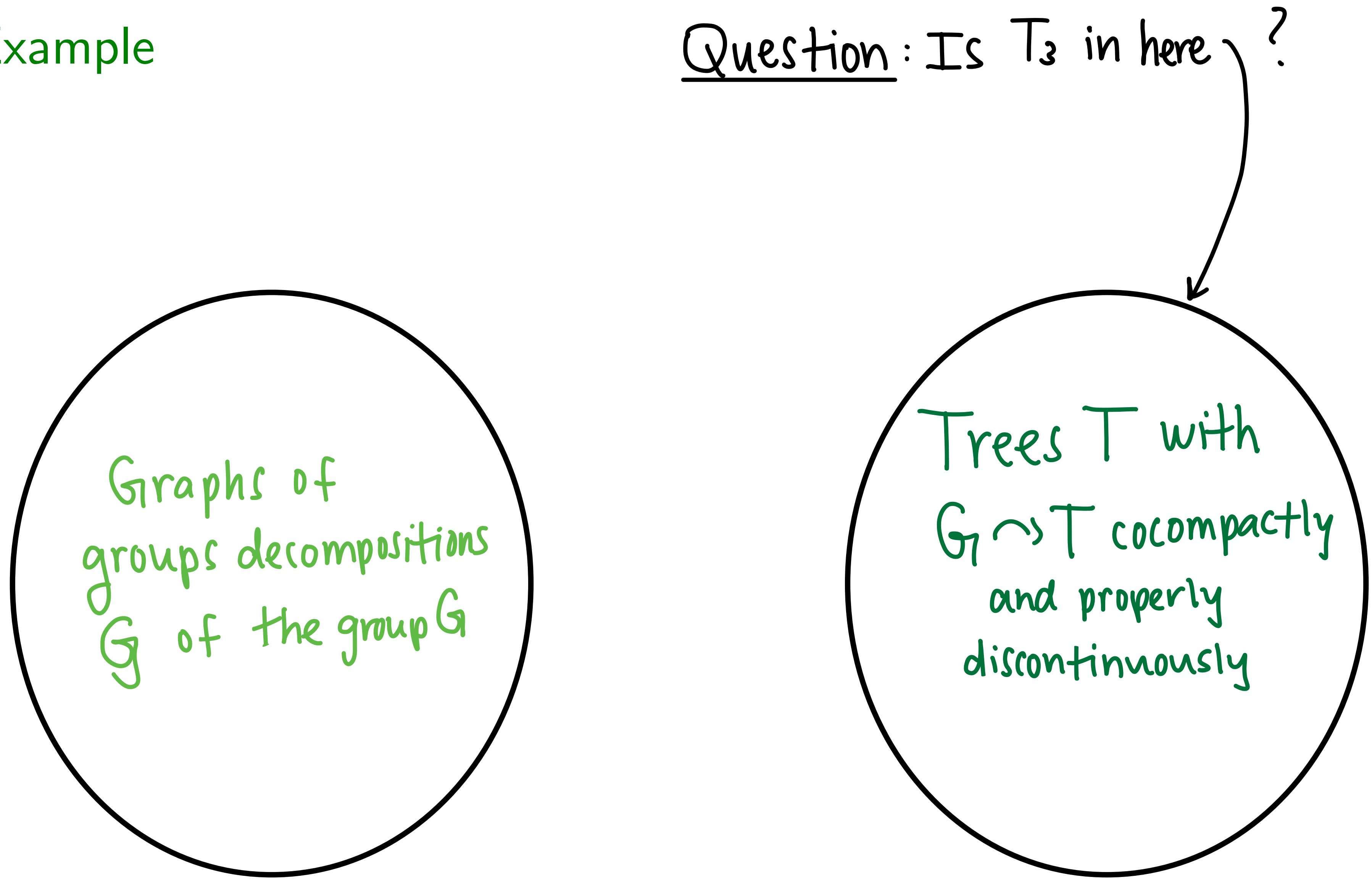
The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ set we are indexing over	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \simeq F_n$ g metric ↕ covering space theory T tree with $F_n \curvearrowright T$ freely and cocompactly g metric	$ent(F_n) = 3(n-1)\log 2$
G Virtually free group $F_n \leq_k G$	T tree with $G \curvearrowright T$ properly discontinuously and cocompactly g metric	

The minimal volume entropy of ______	$\inf\{ent(X,g)Vol(X,g) \mid \_\_\_\_\}$ (set we are indexing over)	Results
X Finite graph	g metric on X	Lim calculates the minimal volume entropy for every finite graph.
F_n Free group	X finite graph with $\pi_1(X) \cong F_n$, g metric ↕ covering space theory T tree with $F_n \curvearrowright T$ freely and cocompactly, g metric	$ent(F_n) = 3(n-1)\log 2$
G Virtually free group $F_n \leq_K G$	$\mathcal{G}$ graph of groups decomposition of G, g metric ↕ Bass-Serre theory T tree with $G \curvearrowright T$ properly discontinuously and cocompactly, g metric	(on future slide)

Example


Graphs of
groups decompositions
G of the group G
Trees T with
G ↷ T cocompactly
and properly
discontinuously

Example


Question: Is T_3 in here?
Graphs of groups decompositions $\mathcal{G}$ of the group G
Trees T with $G \curvearrowright T$ cocompactly and properly discontinuously

Lemma

Let G be a virtually free group with index$-k$ free subgroup F_n. Then

geometrically = cocompactly and properly discontinuously

$$ent(G) = \frac{ent(F_n)}{k} \quad \Longleftrightarrow \quad G \overset{\text{geo}}{\curvearrowright} T_3$$

Lemma

Let G be a virtually free group with index$-k$ free subgroup F_n. Then

$$ent(G) = \frac{ent(F_n)}{k} \qquad \Longleftrightarrow \qquad G \overset{\text{geo}}{\curvearrowright} T_3$$

Theorem (Z)

If G is a virtually free group

$$G \overset{\text{geo}}{\curvearrowright} T_3 \Longleftrightarrow \begin{array}{c} G \text{ has a graph of groups decomposition} \\ \text{which satisfies the} \\ \text{link subgroup series condition} \end{array}$$

Lemma

Let G be a virtually free group with index$-k$ free subgroup F_n. Then

$$ent(G) = \frac{ent(F_n)}{k} \qquad \Longleftrightarrow \qquad G \overset{\text{geo}}{\curvearrowright} T_3$$

Theorem (Z)

If G is a virtually free group

$$G \overset{\text{geo}}{\curvearrowright} T_3 \Longleftrightarrow \begin{array}{c} G \text{ has a graph of groups decomposition} \\ \text{which satisfies the} \\ \text{link subgroup series condition} \end{array}$$

Theorem

Every virtually free right angled Coxeter group acts geometrically on T_3.

Thank you!

Connectivity in the space of pointed hyperbolic 3-manifolds

Matthew Zevenbergen

Boston College

2024

Pointed hyperbolic 3-manifolds

$$\mathcal{H} = \left\{ (M,p) : \begin{array}{l} M \quad \text{complete} \quad \text{oriented} \\ \text{hyperbolic} \quad \text{3-manifold,} \\ p \in M \end{array} \right\} \Big/ \text{pointed isometry}$$

Definition (The geometric topology on $\mathcal{H}$, informally)

Pointed manifolds are close in the **geometric topology** *on $\mathcal{H}$ if they are almost isometric on large neighborhoods of their basepoints.*

Connected components

Def: For a fixed hyperbolic 3-manifold M, the **leaf** of $\mathcal{H}$ corresponding to M is

$$\ell(M) := \{(M, p) \in \mathcal{H} \mid p \in M\}.$$

Connected components

Def: For a fixed hyperbolic 3-manifold M, the **leaf** of $\mathcal{H}$ corresponding to M is

$$\ell(M) := \{(M, p) \in \mathcal{H} \mid p \in M\}.$$

Theorem (Z.)

The connected components of $\mathcal{H}$ are

1. $\ell(M)$ *for each* M *with* $vol(M) < \infty$
2. $\mathcal{H}_\infty := \{(N, p) \in \mathcal{H} \mid vol(N) = \infty\}$.

Idea of proof: Use the *density theorem* of Namazi-Souto and Ohshika to construct a dense path connected subset of $\mathcal{H}_\infty$. □

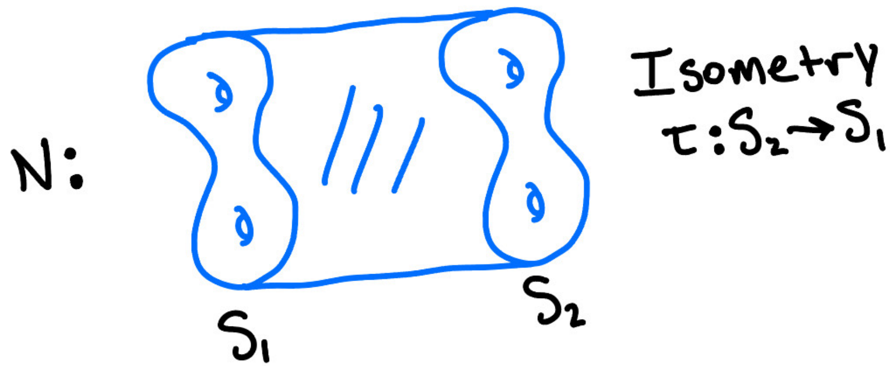
Path connectivity

Theorem (Z.)

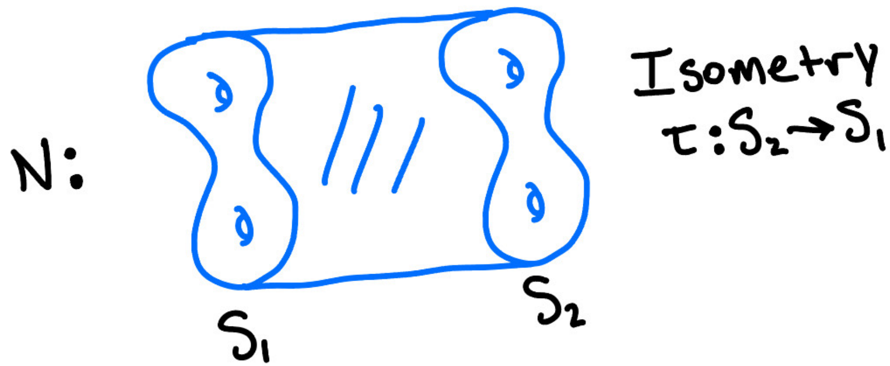
$\mathcal{H}_\infty$ is not path connected. In particular, there exists a hyperbolic 3-manifold M such that $\ell(M)$ is a path component of $\mathcal{H}_\infty$.

Here, $\mathcal{H}_\infty = \{(N, p) \in \mathcal{H} \mid \mathrm{vol}(N) = \infty\}$.

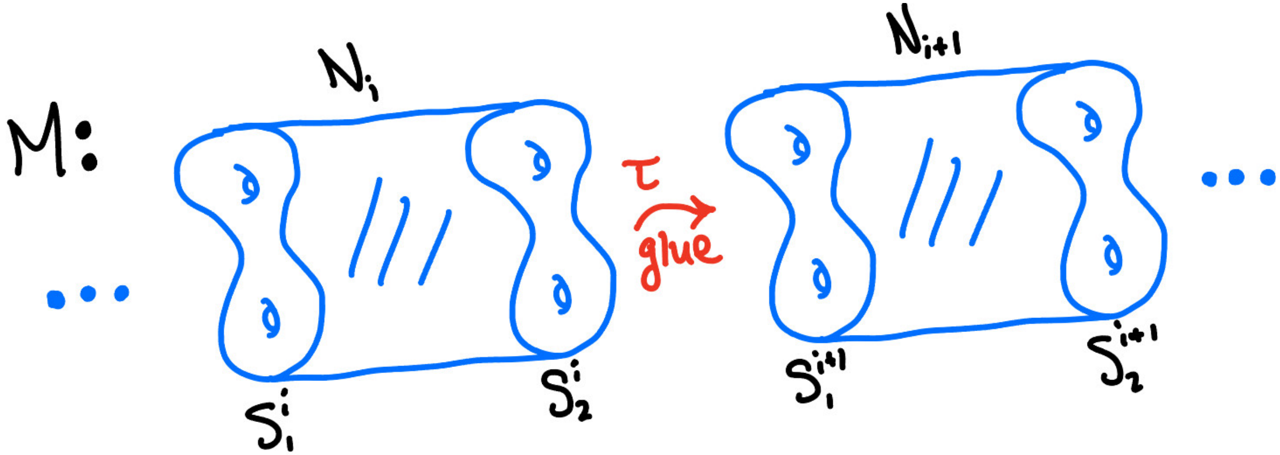
Construction of M with $\ell(M)$ a path component of $\mathcal{H}_\infty$:
Building block: Let N be a connected, compact, oriented hyperbolic 3-manifold with two totally geodesic isometric boundary components S_1, S_2 with an isometry $\tau : S_2 \to S_1$.



Construction of M with $\ell(M)$ a path component of $\mathcal{H}_\infty$:
Building block: Let N be a connected, compact, oriented hyperbolic 3-manifold with two totally geodesic isometric boundary components S_1, S_2 with an isometry $\tau : S_2 \to S_1$.



Gluing: For $i \in \mathbb{Z}$ enumerate copies N_i of N with $\partial N_i = S_1^i \sqcup S_2^i$. For all $i \in \mathbb{Z}$, glue S_2^i to S_1^{i+1} via τ. The result is M.



Homeomorphisms, Isotopy, and Group Actions

Trent Lucas

Brown University

The Basic Goal

Suppose a finite group G acts on a (closed, oriented) manifold M.

The Basic Goal

Suppose a finite group G acts on a (closed, oriented) manifold M.

Broad goal: Understand the group $\underbrace{Homeo_G(M)}_{G\text{-equivariant homeomorphisms}}$.

G-equivariant homeomorphisms

Path Components

$$Homeo_G(M) \hookrightarrow Homeo(M)$$

⬇

$$\mathcal{P}: \pi_0\left(Homeo_G(M)\right) \to \pi_0(Homeo(M))$$

Path Components

$$Homeo_G(M) \hookrightarrow Homeo(M)$$

⬇

$$\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \underbrace{\pi_0(Homeo(M))}$$

Isotopy classes
(*mapping class group*)

Path Components

$$Homeo_G(M) \hookrightarrow Homeo(M)$$

⬇

$$\mathcal{P}: \underbrace{\pi_0\big(Homeo_G(M)\big)}_{\text{G-equivariant isotopy classes}} \to \underbrace{\pi_0(Homeo(M))}_{\text{Isotopy classes (mapping class group)}}$$

G-equivariant
isotopy classes

Isotopy classes
(*mapping class group*)

Path Components

$$Homeo_G(M) \hookrightarrow Homeo(M)$$

$$\Downarrow$$

$$\mathcal{P}: \underbrace{\pi_0\big(Homeo_G(M)\big)}_{\text{G-equivariant isotopy classes}} \to \underbrace{\pi_0\big(Homeo(M)\big)}_{\text{Isotopy classes (mapping class group)}}$$

Today's Question: Is $\mathcal{P}$ injective?

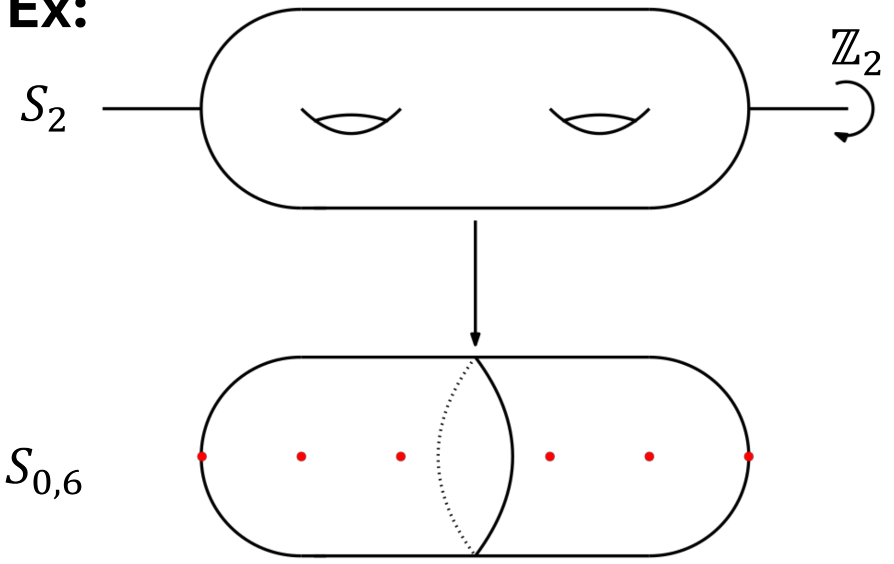
For surfaces: yes!

Birman-Hilden, MacLachlan-Harvey (70s): If M is a hyperbolic surface, then $\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is injective.

For surfaces: yes!

Birman-Hilden, MacLachlan-Harvey (70s): If M is a hyperbolic surface, then $\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is injective.

Ex:

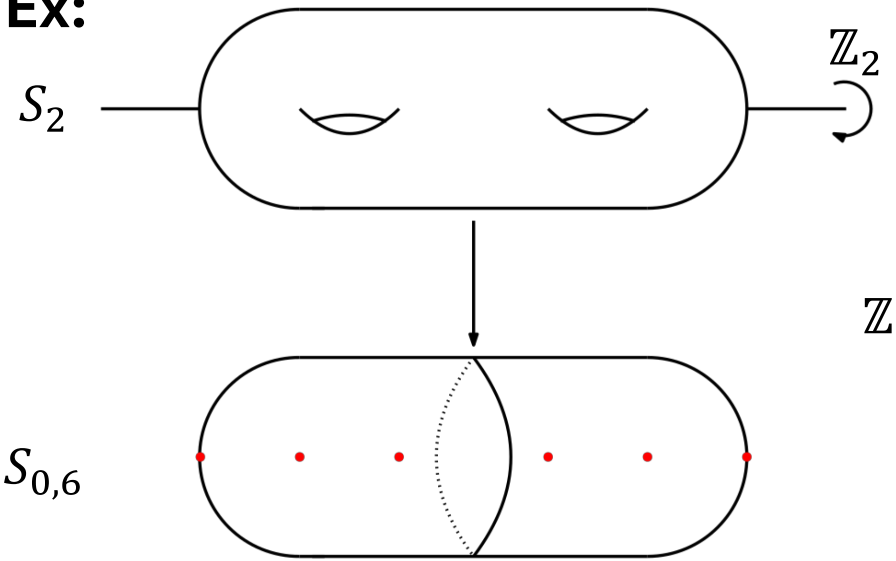


$\hookrightarrow$

For surfaces: yes!

Birman-Hilden, MacLachlan-Harvey (70s): If M is a hyperbolic surface, then $\mathcal{P}: \pi_0(Homeo_G(M)) \to \pi_0(Homeo(M))$ is injective.

Ex:



Birman-Hilden: There's a SES

$$\mathbb{Z}_2 \hookrightarrow \pi_0(Homeo(S_2)) \twoheadrightarrow \pi_0\left(Homeo(S_{0,6})\right)$$

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds, $\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is not injective.

For 3-manifolds: no!

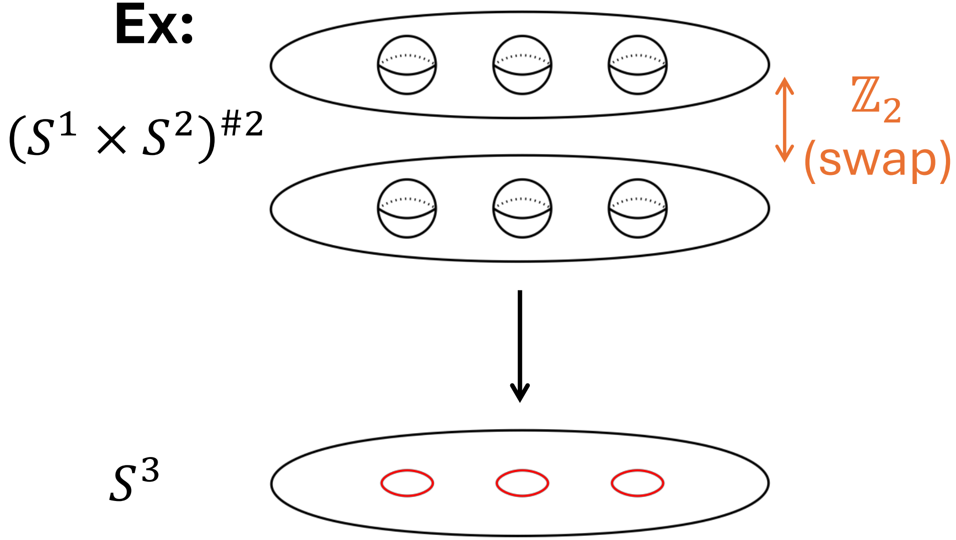
Theorem (L.): For most group actions on 3-manifolds, $\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is not injective.

Need:
- G does not act freely
- M/G has at least 3 prime factors

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds,
$\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is not injective.

Ex:



Need:

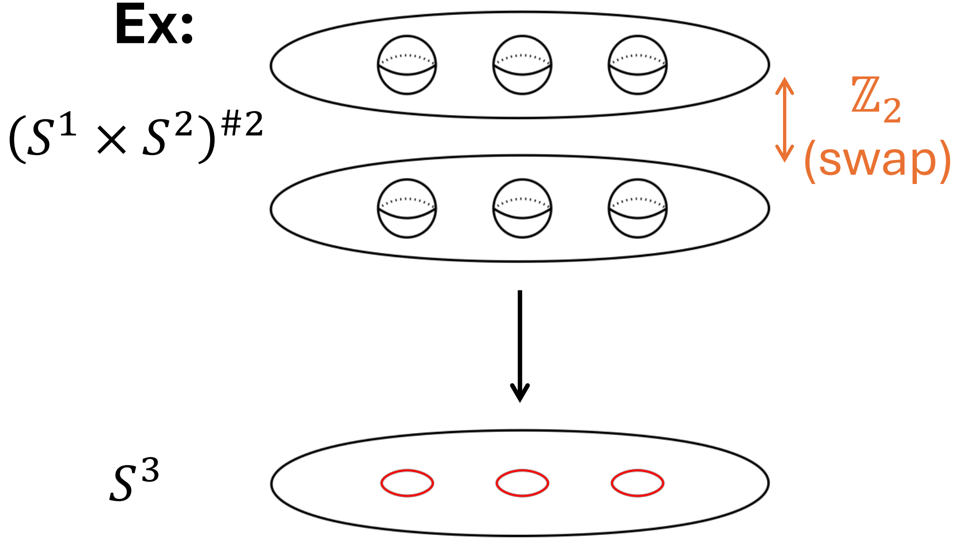
- G does not act freely
- M/G has at least 3 prime factors

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds,
$\mathcal{P}: \pi_0(Homeo_G(M)) \to \pi_0(Homeo(M))$ is not injective.

Need:
- G does not act freely
- M/G has at least 3 prime factors

Ex:



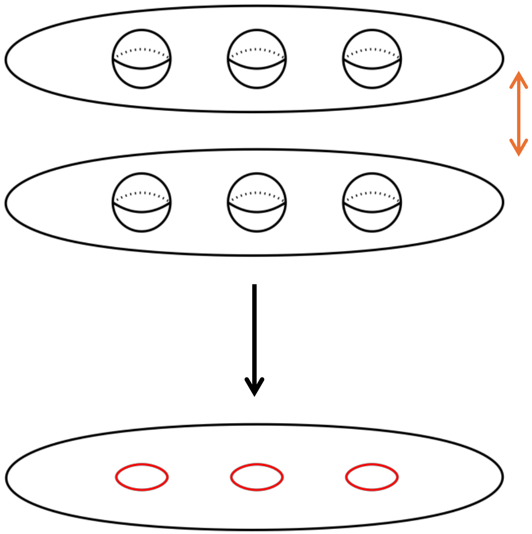
Next step: What is $Ker(\mathcal{P})$?

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds, $\mathcal{P}: \pi_0\big(Homeo_G(M)\big) \to \pi_0\big(Homeo(M)\big)$ is not injective.

Ex:

$(S^1 \times S^2)^{\#2}$



$\mathbb{Z}_2$ (swap)

S^3

Need:

- G does not act freely
- M/G has at least 3 prime factors

Next step: What is $Ker(\mathcal{P})$?

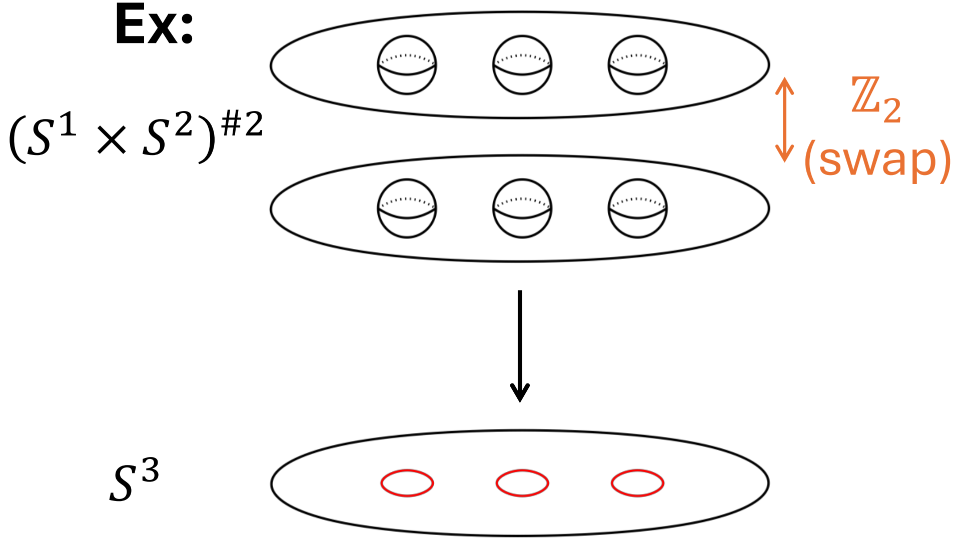
- For $(S^1 \times S^2)^{\#2} \to S^3$, $Ker(\mathcal{P}) \cong F_\infty \rtimes \mathbb{Z}_2$.

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds,
$\mathcal{P}: \pi_0(Homeo_G(M)) \to \pi_0(Homeo(M))$ is not injective.

Need:
- G does not act freely
- M/G has at least 3 prime factors

Ex:



Next step: What is $Ker(\mathcal{P})$?

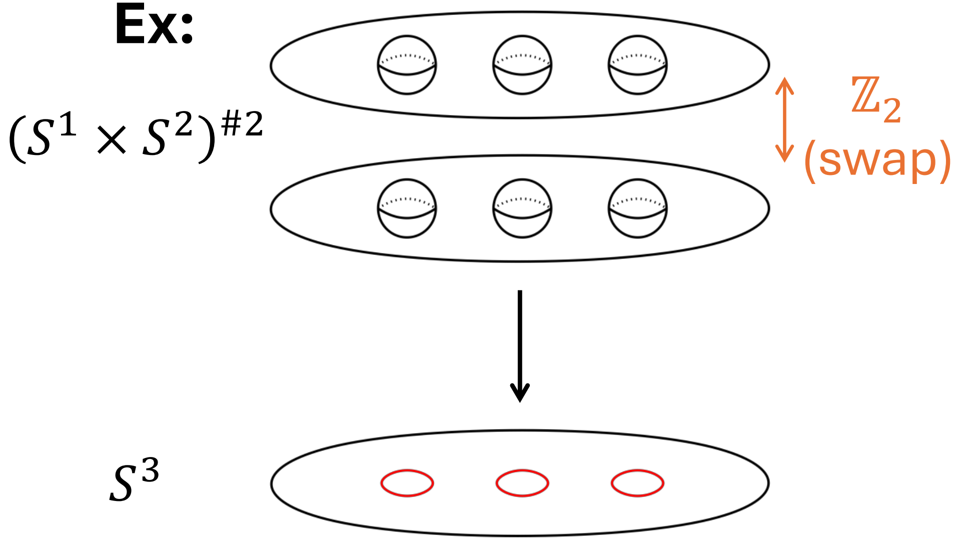
- For $(S^1 \times S^2)^{\#2} \to S^3$, $Ker(\mathcal{P}) \cong F_\infty \rtimes \mathbb{Z}_2$.
- **Theorem (L.):** For $(S^1 \times S^2)^{\#n} \to S^3$, $Ker(\mathcal{P})$ is normal closure of a single element.

For 3-manifolds: no!

Theorem (L.): For most group actions on 3-manifolds,
$\mathcal{P}: \pi_0(Homeo_G(M)) \to \pi_0(Homeo(M))$ is not injective.

Need:
- G does not act freely
- M/G has at least 3 prime factors

Ex:



Next step: What is $Ker(\mathcal{P})$?

- For $(S^1 \times S^2)^{\#2} \to S^3$, $Ker(\mathcal{P}) \cong F_\infty \rtimes \mathbb{Z}_2$.
- **Theorem (L.):** For $(S^1 \times S^2)^{\#n} \to S^3$, $Ker(\mathcal{P})$ is normal closure of a single element.
- We study $Ker(\mathcal{P})$ using tools from geometric group theory ("McCullough-Miller space").

“Epstein surfaces” in Higher Teichmuller theory

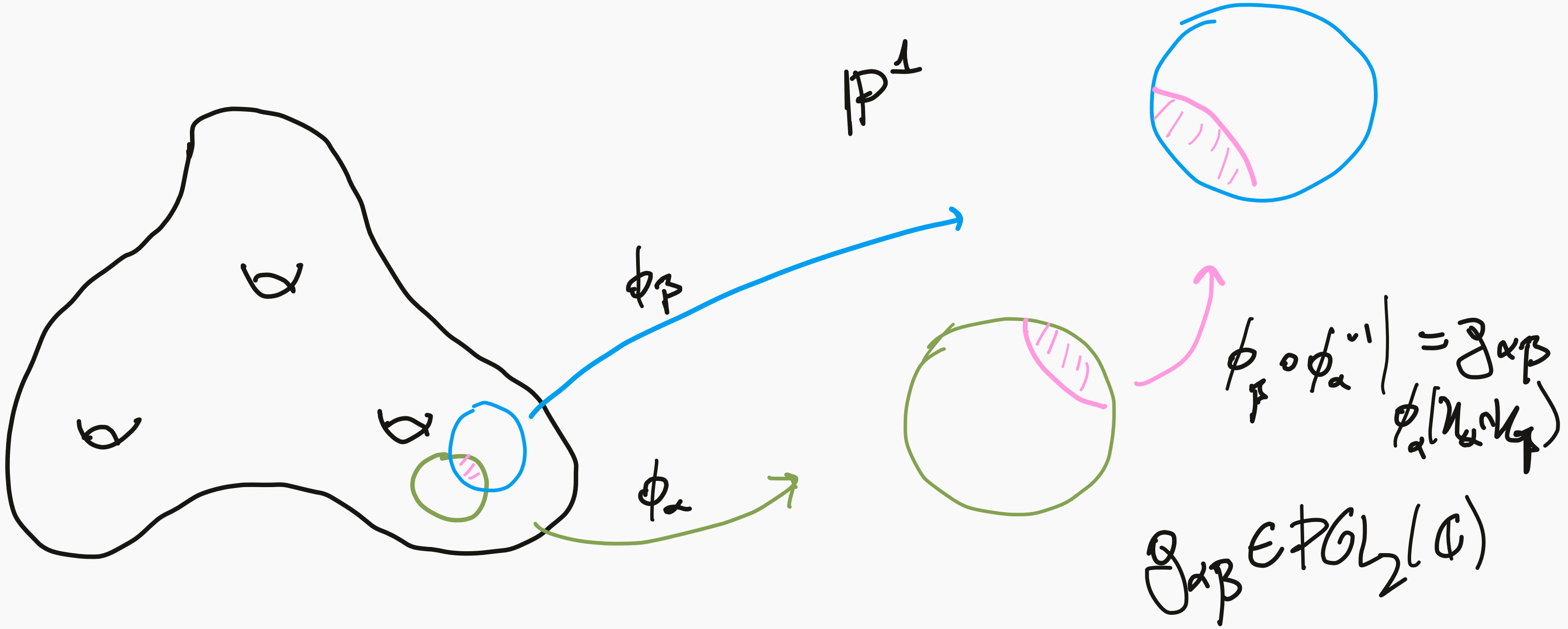
Joaquín Lema

Nov 16, 2024

Boston College

Motivation:

- Opers generalize the notion of complex projective structures on surfaces:



- A projective structure induces a Riemann surface structure on S. A monodromy construction implies that this data is equivalent to a pair (f, ρ), for $f : \tilde{S} \to \mathbb{P}^1$ locally biholomorphic for the induced complex structure, and equivariant for $\rho : \pi_1(S) \to PGL_2(\mathbb{C})$ some representation.

Motivation

- Denote by $\mathbb{CP}^1(X)$ the space of complex projective structures inducing a complex structure X on S. Fixing $[(f_0, \rho_0)] \in \mathbb{CP}^1(X)$, we can write any other $[(f, \rho)]$ as $f(z) = Osc(z)(f_0(z))$, for $Osc : \tilde{X} \to PGL_2(\mathbb{C})$ holomorphic the **osculating Mobius map**.
- This map Osc satisfies that:

$$(Osc(z))^{-1}(Osc(z))' = \frac{-1}{2}\{f, f_0\} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}.$$

where $\{f, f_0\}$ is the Schwarzian derivative of f w.r.t. f_0. An object that can be naturally identified with $H^0(K^2)$ (the space of quadratic differentials on X).

Ahlfors-Weil

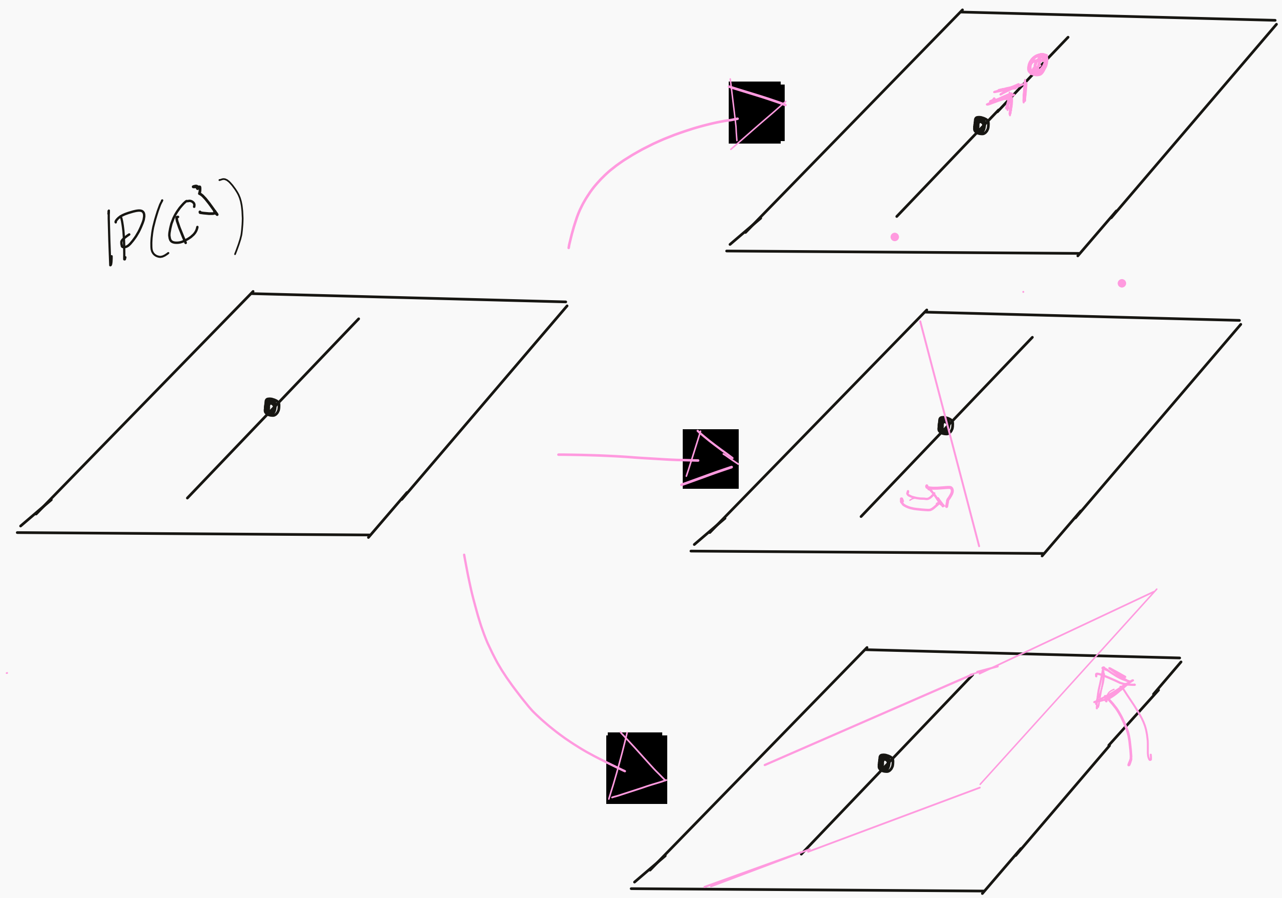
- Fixing a marking on X, we can always identify $\tilde{X} \to \mathbb{D} \subset \mathbb{P}^1$, and $\rho_0 : \pi_1(X) \to PGL_2(\mathbb{R})$ the Fuchsian representation. This lets us identify $\mathbb{CP}^1(X)$ with $H^0(K^2)$.

Baby Ahlfors-Weil
Let $q \in H^0(K^2)$ such that $||q||_2 < \frac{1}{2}$, then the associated complex projective structure $[(f_q, \rho_q)]$ satisfies that ρ_q is **convex cocompact**.

Sketch:
Identifying $PGL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$, we can think that $PGL_2(\mathbb{R})$ preserves a totally geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. We can embed $Ep_0 : \tilde{X} \to \mathbb{H}^2 \subset \mathbb{H}^3$ equivariantly for our Fuchsian representation. One can define $Ep : \tilde{X} \to \mathbb{H}^3$ as $Ep(z) = Osc(z)(E_0(z))$, for $Osc(z)$ the osculating map for the projective structure $[(f_q, \rho_q)]$. The bound gives us sufficient control over S to prove that it is quasi-isometrically embedded.

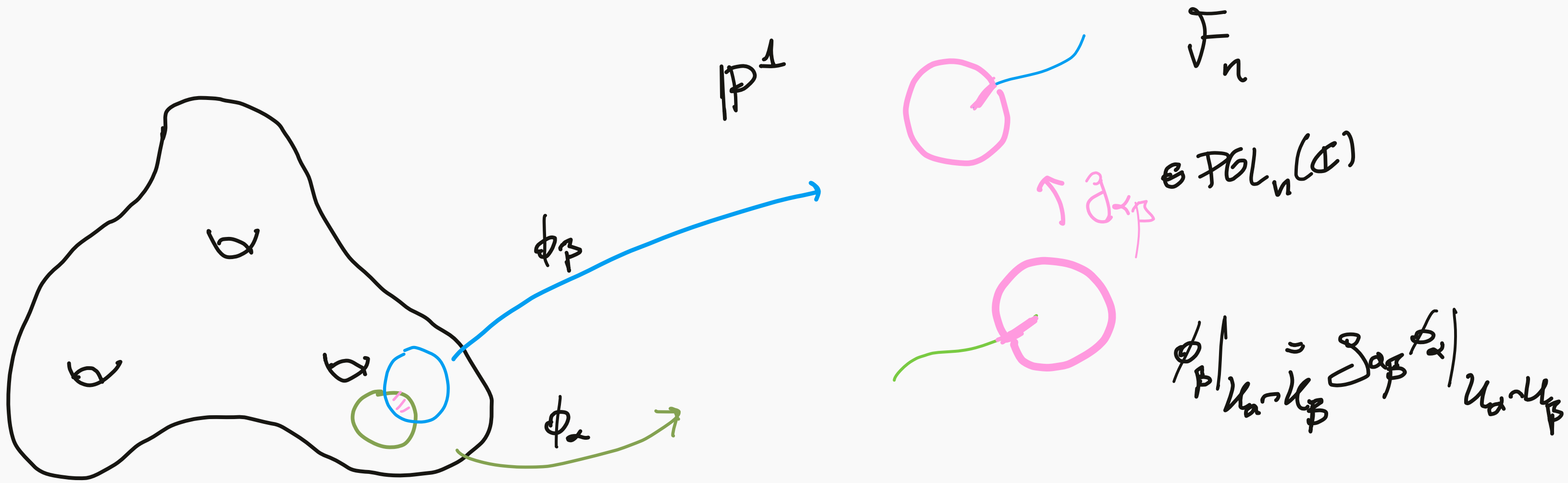
Opers

- Given V an $n-$dimensional vector space, define the full flag manifold $\mathcal{F}_n$ as the space of sequences $0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = V$, where each E_i is a subspace, $E_i \subset E_{i+1}$, and $\dim E_i = i$.
- There is a distribution $\mathcal{D} \subset T\mathcal{F}$ with some interesting geometry:



Opers

- Given a Riemann surface X, a $PGL_n(\mathbb{C})$−oper is:



  Each (complex) curve ϕ_α has to be tangent to the distribution $\mathcal{D}$ and needs to satisfy a regularity condition.
- A monodromy construction associates to every such structure a pair (f, ρ), where $f : \tilde{X} \to \mathcal{F}_n$ is a (locally injective) holomorphic map that is equivariant for $\rho : \pi_1(X) \to PGL_n(\mathbb{C})$.
- Example: if we embed $PGL_2(\mathbb{C}) \to PGL_n(\mathbb{C})$ irreducibly, this induces a map from $\iota : \mathbb{P}^1 \to \mathcal{F}_n$. One can compose the developing map of a $\mathbb{P}^1$−structure with ι to get a $PGL_n(\mathbb{C})$−oper.

The question:

Theorem (Beilinson-Drinfeld)
The space of $PGL_n(\mathbb{C})-$opers over a Riemann surface X is an affine space with underlying vector space:

$$H^0(K^2) \oplus H^0(K^3) \oplus \ldots \oplus H^0(K^n).$$

Comparing this with the Ahlfors-Weil theorem, it is natural to ask:

Question:
Are there constants $A_2, \ldots, A_n$ such that if $(q_2, \ldots, q_n)$ is a tuple of differentials with $||q_i|| < A_k$, then the monodromy of the oper is **(complex) Borel Anosov**.

- Our approach involves generalizing the osculating Möbius maps to this setting, and using those to construct an equivariant surface to the symmetric space $PGL_n(\mathbb{C})/SU_n$ similarly to the Ahlfors-Weil case.

More comments:

- The punchline is a bit different. It requires a result that allows us to promote from a surface in the symmetric space with control geometry to Anosovness of the representation (a la Kapovich-Leeb-Porti, Riestemberg).
- The strategy proves fruitful for $SL_3(\mathbb{C})$ by brutal computation of the Epstein surface for triangle groups ($A_3 = \frac{1}{3}$ works). But a general approach is in development for any complex semisimple Lie group G.

Thanks!

Disclaimer: The speaker chooses not to follow the following wise words from John Baez: "Practice your talks! ... Watch yourself struggling to turn on the laser pointer, tripping over the microphone wire, fumbling around, desperately struggling against Microsoft to get your Powerpoint presentation to work, engaging in all sorts of pointless antics that distract from the subject matter, wasting precious time, boring people to death. And resolve to do better!"

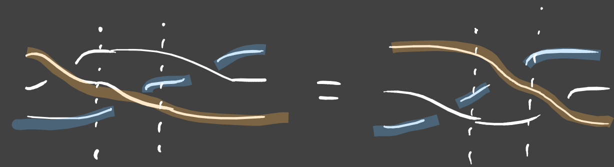
A *Crazy* theorem of Coxeter

as told by Ethan Dlugie

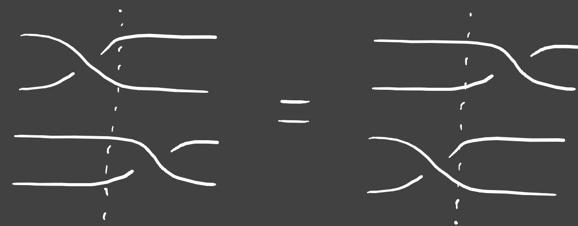
at GATSBY Fall '24

A crazy theorem of Coxeter:

- Consider the n-strand braid group

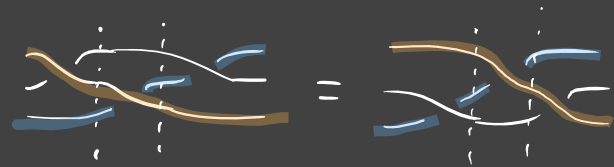


$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ else} \rangle$$

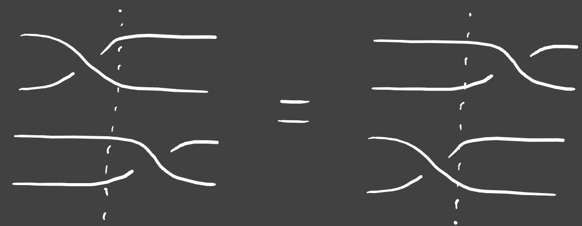


A crazy theorem of Coxeter:

- Consider the n-strand braid group



$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ else} \rangle$$



- Define the quotient

$$B_n(d) := B_n / \langle\langle \sigma^d \rangle\rangle$$

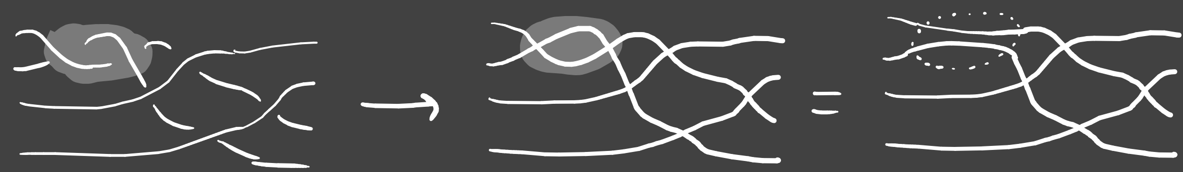
A crazy theorem of Coxeter:

- Define the quotient

$$B_n(d) = B_n / \langle\langle \sigma^d \rangle\rangle$$

- These are sometimes finite.

Ex. $B_n(2) = Sym_n$



- $B_2(d) \cong \mathbb{Z}/d\mathbb{Z}$

A crazy theorem of Coxeter:

- Define the quotient

$$B_n(d) = B_n / \langle\langle \sigma^d \rangle\rangle$$

Thm (Coxeter '59)

$B_n(d)$ is finite $\Leftrightarrow$ $(n,d) \in \{ (n,2), (2,d), (3,3), (3,4), (3,5), (4,3), (5,3) \}$

A crazy theorem of Coxeter:

- Define the quotient

$$B_n(d) = B_n / \langle\langle \sigma^d \rangle\rangle$$

Thm (Coxeter '59)

$B_n(d)$ is finite $\Leftrightarrow$ $(n,d) \in \left\{ \begin{array}{l} (n,2),\ (2,d), \\ (3,3),\ (3,4),\ (3,5), \\ (4,3),\ (5,3) \end{array} \right\}$

In this case,

$$\# B_n(d) = \left(\frac{f(n,d)}{2} \right)^{n-1} n!$$

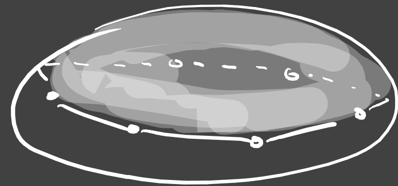
where $f(n,d)$ = # faces in Platonic solid of n-gons, d at every vertex.

A crazy theorem of Coxeter:

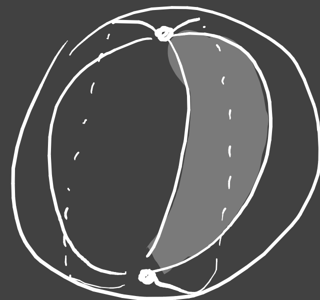
Thm (Coxeter '59) $\#B_n(d) = \left(\frac{f(n,d)}{2}\right)^{n-1} n!$

where $f(n,d)$ = # faces in Platonic solid of n-gons, d at every vertex.

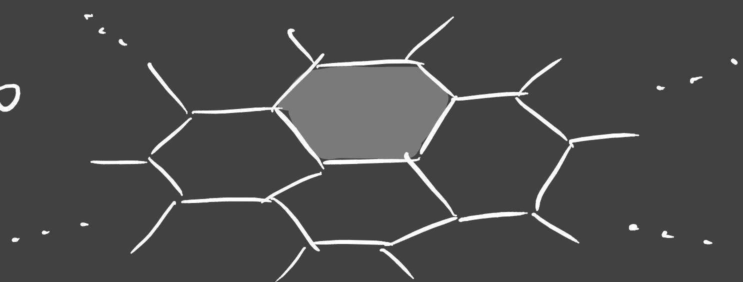
Ex • $B_n(2) = Sym_n$, $f(n,2) = 2$



• $B_2(d) \simeq \mathbb{Z}/d\mathbb{Z}$, $f(2,d) = d$



• $B_6(3)$ infinite, $f(6,3) = \infty$

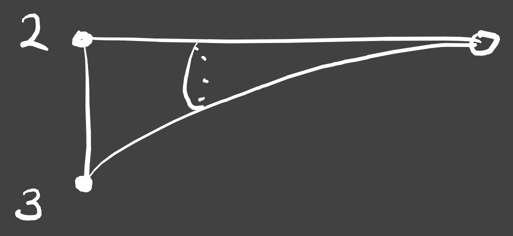


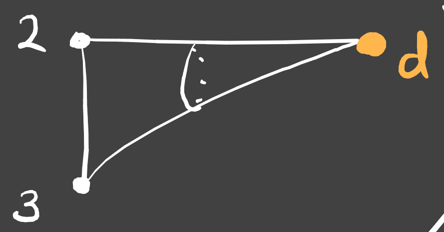
A crazy theorem of Coxeter:

Thm (Coxeter '59) $\#B_3(d) = \left(\frac{f(3,d)}{2}\right)^{n-1} n!$

where $f(3,d)$ = # faces in Platonic solid of triangles, d at every vertex.

Hint of a connection:

$B_3/ZB_3 = \mathbb{Z}/2 * \mathbb{Z}/3 = \pi_1^{orb}\Big($  $\Big)$

$B_3 / \langle ZB_3, \sigma^d \rangle = \pi_1^{orb}\Big($  $\Big)$

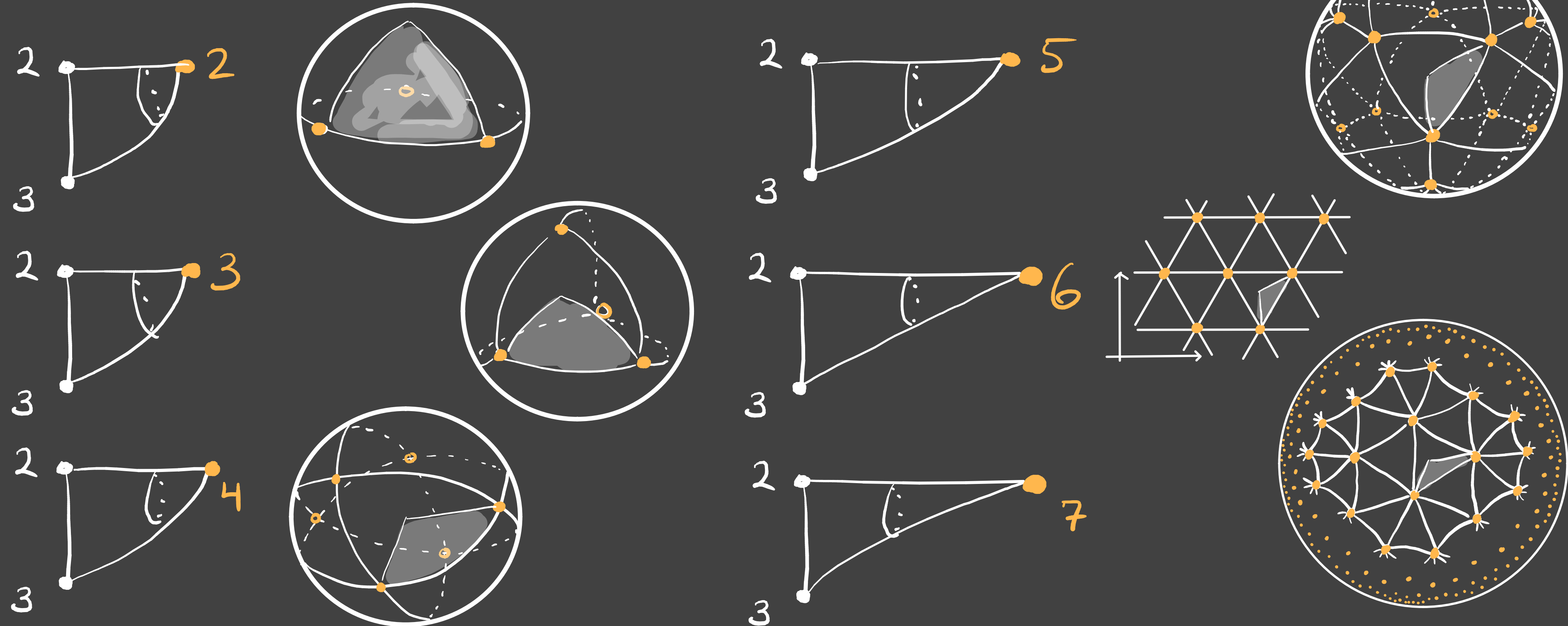
A crazy theorem of Coxeter:

Thm (Coxeter '59)

$$\#B_3(d) = \left(\frac{f(3,d)}{2}\right)^{n-1} n!$$

where $f(3,d)$ = # faces in Platonic solid of triangles, d at every vertex.

Hint of a connection:



A crazy theorem of Coxeter:

Thm (Coxeter '59) $\#B_3(d) = \left(\frac{f(3,d)}{2}\right)^{n-1} n!$

where $f(3,d)$ = # faces in Platonic solid of triangles, d at every vertex.

Hint of a connection:

- This shows $\# B_3 / \langle Z B_3, \sigma^d \rangle = 3 \cdot f(3,d)$
- Compute order of ZB_3 in $B_3(d)$?
- For larger n, $B_n / \langle Z B_n, \sigma^d \rangle = \pi_1^{orb}(\mathbb{C}\ (n-2)\text{-dim orbifold})$

  geometric structures for those?

Rigidity of Kleinian groups via higher-rank dynamics

Dongryul M. Kim

Yale University

GATSBY 2024 Fall

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: Fin. gen. Kleinian group (Z-dense)

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: Fin. gen. Kleinian group (Z-dense)

$\Lambda_\Gamma \subset \mathbb{S}^2$: Limit set of Γ

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: Fin. gen. Kleinian group (Z-dense)

$\Lambda_\Gamma \subset \mathbb{S}^2$: Limit set of Γ

$\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$: disc. faith. rep. (Z-dense)

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: Fin. gen. Kleinian group (Z-dense)

$\Lambda_\Gamma \subset \mathbb{S}^2$: Limit set of Γ

$\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$: disc. faith. rep. (Z-dense)

Theorem (Sullivan)

Suppose that ρ is a quasi-conformal deform.

If the bdry map $\partial\rho$ is conformal on $\mathbb{S}^2 - \Lambda_\Gamma$ (Beltrami diff.=0),

then ρ is trivial (conj. by Möbius transf.).

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: Fin. gen. Kleinian group (Z-dense)

$\Lambda_\Gamma \subset \mathbb{S}^2$: Limit set of Γ

$\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$: disc. faith. rep. (Z-dense)

Theorem (Sullivan)

Suppose that ρ is a quasi-conformal deform.

If the bdry map $\partial\rho$ is conformal on $\mathbb{S}^2 - \Lambda_\Gamma$ (Beltrami diff.=0),

then ρ is trivial (conj. by Möbius transf.).

- Generalization of Mostow's Rigidity
- Evidence for Ahlfors' measure conjecture

Ahlfors' meas. conj. (Proved by Canary, Agol, Calegari-Gabai)

Γ: *fin. gen. Kleinian group. Either*

$$\Lambda_\Gamma = \mathbb{S}^2 \quad \textit{or} \quad \mathrm{Leb}(\Lambda_\Gamma) = 0.$$

Canary: Tameness conj. $\Rightarrow$ Ahlfors' meas. conj.

Agol, Calegari-Gabai: Tameness

Ahlfors' meas. conj. (Proved by Canary, Agol, Calegari-Gabai)

Γ*: fin. gen. Kleinian group. Either*

$$\Lambda_\Gamma = \mathbb{S}^2 \quad \textit{or} \quad \mathrm{Leb}(\Lambda_\Gamma) = 0.$$

Canary: Tameness conj. $\Rightarrow$ Ahlfors' meas. conj.

Agol, Calegari-Gabai: Tameness

Theorem (Sullivan)

Suppose that ρ *is a quasi-conformal deform.*

If $\Lambda_\Gamma = \mathbb{S}^2$, ~~*the bdry map* $\partial\rho$ *is conformal on* $\mathbb{S}^2 - \Lambda_\Gamma$,~~

then ρ *is trivial (conj. by Möbius transf.).*

Ahlfors' meas. conj. (Proved by Canary, Agol, Calegari-Gabai)

Γ*: fin. gen. Kleinian group. Either*

$$\Lambda_\Gamma = \mathbb{S}^2 \quad \textit{or} \quad \mathrm{Leb}(\Lambda_\Gamma) = 0.$$

Canary: Tameness conj. $\Rightarrow$ Ahlfors' meas. conj.

Agol, Calegari-Gabai: Tameness

Theorem (Sullivan)

Suppose that ρ *is a quasi-conformal deform.*

If $\Lambda_\Gamma = \mathbb{S}^2$, ~~*the bdry map* $\partial\rho$ *is conformal on* $\mathbb{S}^2 - \Lambda_\Gamma$,~~

then ρ *is trivial (conj. by Möbius transf.).*

Question

What if $\mathrm{Leb}(\Lambda_\Gamma) = 0$?

In general,

$$\partial\rho : \Lambda_\Gamma \to \mathbb{S}^2$$

What is ‘conformality’ on a Leb-null set?

Circular slice: $\Lambda_\Gamma \cap C$ for circle $C \subset \mathbb{S}^2$

In general,

$$\partial\rho : \Lambda_\Gamma \to \mathbb{S}^2$$

What is 'conformality' on a Leb-null set?

Circular slice: $\Lambda_\Gamma \cap C$ for circle $C \subset \mathbb{S}^2$

Theorem (K.-Oh)

Suppose that $\mathbb{S}^2 - \Lambda_\Gamma$ has at least two components.

If $\partial\rho$ is conformal 'on Λ_Γ', i.e.,

if $\partial\rho$ maps every circular slice into a circle,

then ρ is trivial.

Indeed, setting $\Lambda_\rho =$ union of all such circular slices,

$$\text{Int}(\Lambda_\rho) \neq \emptyset \Rightarrow \rho \text{ is trivial.}$$

$\Lambda_\rho \subset \Lambda_\Gamma$: union of all circular slices mapped into circles

Theorem (K.-Oh)

Suppose further: Γ *and* $\rho(\Gamma)$ *are convex cocompact. Either*

$$\Lambda_\rho = \Lambda_\Gamma \quad \textit{or} \quad \textit{Hausdorff meas.}(\Lambda_\rho) = 0$$

and the former implies that ρ *is trivial.*

$\Lambda_\rho \subset \Lambda_\Gamma$: union of all circular slices mapped into circles

Theorem (K.-Oh)

Suppose further: Γ *and* $\rho(\Gamma)$ *are convex cocompact. Either*

$$\Lambda_\rho = \Lambda_\Gamma \quad \textit{or} \quad \textit{Hausdorff meas.}(\Lambda_\rho) = 0$$

and the former implies that ρ *is trivial.*

Proof Key Idea (for both thms).

Dynamics on **higher-rank** homogeneous spaces

(e.g. Transitivity/Ergodicity of a higher-rank flow,

higher-rank Patterson-Sullivan measures,)

and relate them to fractal geometry of limit sets

□