Signatures topics course

Lecture 1

Quadratic forms (and signature)

K field, V K-vector space.

A quadratic form is a function

$q : V \longrightarrow K$ s.t.

- $q(av) = a^2 q(v) \quad \forall a \in K, v \in V$
- $b(v,w) := q(v+w) - q(v) - q(w)$ is bilinear

Ex For $a, b, c \in K$

$$q(x,y) = ax^2 + bxy + cy^2$$

quadratic form on K^2.

Equivalence

$$\begin{tikzcd} V' \arrow[rr, dashed, "\phi", "\cong"'] \arrow[dr, "q'"'] & & V \arrow[dl, "q"] \\ & K & \end{tikzcd}$$

$q \sim q'$ if

$\exists\ \phi$ s.t. $q' = q \circ \phi$.

eg $q = xy$ $q' = x^2 - y^2$ equivalent

(take $\phi(x, y) = (x+y, x-y)$)

but q not equivalent to $q'' = x^2 + y^2$
for $K = \mathbb{R}$

Basic problem Classify quadratic forms up to equivalence

Dictionary

Assume $\mathrm{char}(K) \neq 2$.

Quadratic forms $\longleftrightarrow$ symmetric bilinear forms

$$q \longmapsto b(v,w) := \tfrac{1}{2}\left[q(v+w) - q(v) - q(w)\right]$$

$$q(v) := b(v,v) \longleftarrow\!\shortmid\, b$$

symmetric bilinear forms on $K = \langle e_1, \dots, e_d \rangle$ $\longleftrightarrow$ symmetric matrices

$$b \longmapsto B_{ij} := b(e_i, e_j)$$

$$b(v,w) := v^t B w \longleftarrow\!\shortmid\, B$$

Sample use

- $q' = q \circ \phi \longleftrightarrow b'(v,w) = b(\phi v, \phi w) \longleftrightarrow B' = \Phi^t B \Phi$

  Φ = matrix of ϕ.

- q is "degenerate" $\longleftrightarrow$ $\exists\, u$ s.t. $b(u,-) \equiv 0$ $\longleftrightarrow$ $\det B = 0$

Sylvester's law of inertia:

B symmetric, real coefficients, $\det B \neq 0$

(i) B equivalent to $B_{n,m} = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$

(ii) no two of $B_{n,m}$ are equivalent.

Terminology

- n = positive index $\Big\}$ invariants of B
- m = negative index
- rank = $n+m$
- signature $\operatorname{sig}(B) := n-m$

Cor two nondegenerate real quad forms equivalent $\Longleftrightarrow$ same rank & signature.

Cor two nondegenerate complex quadratic forms equivalent $\Longleftrightarrow$ same rank.

$$\begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}^t \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Proof of Sylvester

$\Rightarrow B_{n,m}$ equiv to $B_{n+m,0} = I_{n+m}$ over $\mathbb{C}$.

Proof of (ii) Fix n,m. $B = B_{n,m}$

Say subspace $P \subset \mathbb{R}^{n+m}$ is *positive* if $B(v,v) > 0 \quad \forall \ v \in P \setminus \{0\}$

Suffices to prove

$$\underbrace{\max_{P \subset \mathbb{R}^{n+m}} \dim P}_{\text{obviously an invariant of } B} = n$$

$\text{span}\{e_1, \dots, e_n\}$ positive $\Rightarrow \max \geq n$.

By contradiction, suppose $\max = \dim P > n$.

Let $N = \text{span}\{e_{n+1}, \dots, e_{n+m}\}$

Dimension count $\Rightarrow$ $\underset{\substack{\uparrow \\ \text{pos.}}}{P} \cap \underset{\substack{\uparrow \\ \text{neg.}}}{N} \neq \{0\}$ ↯

Proof of (i) It suffices to diagonalize B

eg $\begin{pmatrix} \frac{1}{\sqrt{\pi}} & \\ & \frac{1}{\sqrt{2}} \end{pmatrix}^t \begin{pmatrix} \pi & \\ & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\pi}} & \\ & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Diagonalizing B:

Option 1 (Spectral Theorem)

B has $\underbrace{\text{orthonormal}}_{\text{wrt standard inner prod. on } \mathbb{R}^d}$ eigenbasis $u_1, \dots, u_d$

$Bu_j = \lambda_j u_j$ $\qquad$ $u_i^t B u_j = \lambda_j \delta_{ij}$

$\Phi = (u_1 \cdots u_d)$ $\qquad$ $\Phi^t B \Phi = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$

Option 2 (row/column operations)

Ex $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \underset{\substack{\uparrow \\ R2 \mapsto -\frac{1}{3}R1 + R2}}{\begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{pmatrix}} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \underset{\substack{\uparrow \\ C2 \to C2 - \frac{1}{3}C1}}{\begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$

Option 3 (complete the square)

$$q = 3x^2 + 2xy + y^2$$

$$= 3\left[x^2 + \frac{2}{3}xy + \left(\frac{y}{3}\right)^2 - \left(\frac{y}{3}\right)^2\right] + y^2$$

$$= 3\left(x + \frac{y}{3}\right)^2 - \frac{2}{3}y^2$$

$$\Rightarrow q \sim 3x^2 - \frac{2}{3}y^2$$

$\square$

Rmk Option 3 works over any field of char $\neq 2$

(use $\frac{1}{2}$ to complete square
$$x^2 + ax \rightsquigarrow x^2 + ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$$
)

$\boxed{\text{Course Preview}}$ "signatures everywhere"

- manifolds

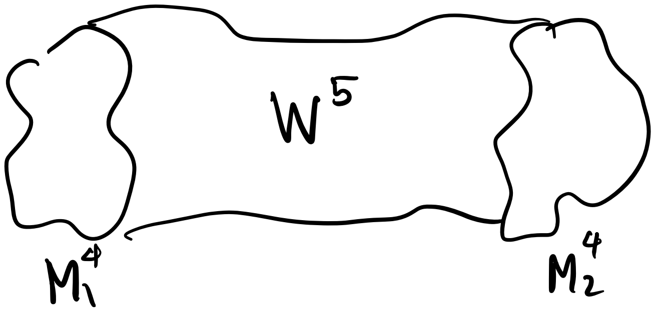
M^{4k} closed, oriented manifold, dim $= 4k$

$$B_M : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \xrightarrow[\text{product}]{\text{cup}} H^{4k}(M;\mathbb{R}) \cong \mathbb{R}$$

nondeg. symmetric bilinear form

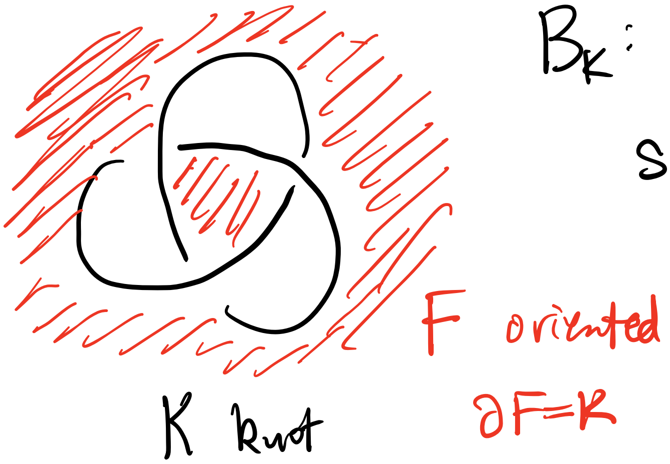
$$\text{sig}(M) := \text{sig}(B_M)$$

Thm Two 4-manifolds have same signature $\Longleftrightarrow$ they cobordant



Rmk Cobordisms generally hard to construct. Much harder than computing signature of a matrix!

- Knots

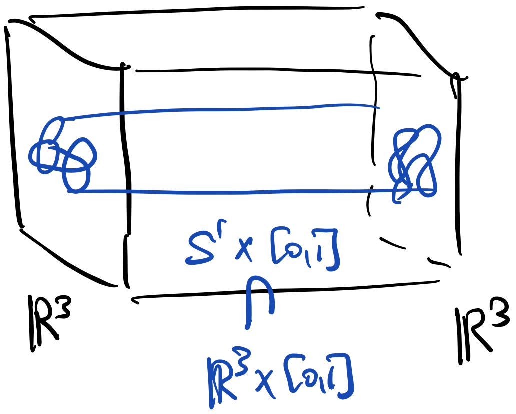


$B_K : H_1(F;\mathbb{R}) \times H_1(F;\mathbb{R}) \longrightarrow \mathbb{R}$

symmetric Seifert form.

$\mathrm{sig}(K) := \mathrm{sig}(B_K)$

use to study knots up to concordance



- Symplectic geometry

$$\underbrace{(\mathbb{R}^{2g}, \omega)}_{\text{sympl. vector space}} \qquad \underbrace{\omega(x,y) = x^t \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} y}_{\text{skew sym. form.}}$$

$$\mathrm{Lag}(\mathbb{R}^{2g}, \omega) = \left\{ L \subset \mathbb{R}^{2g} \;\middle|\; \begin{array}{l} \dim L = g \\ \omega|_L \equiv 0 \end{array} \right\}$$

Surprise: has natural charts $\cong \left\{ \begin{array}{l} \text{real sym} \\ g \times g \text{ matrices} \end{array} \right\}$

$\rightsquigarrow$

$$\mu : (\mathrm{Lag})^{\times 3} \xrightarrow[\substack{\text{defined} \\ \text{as a signature}}]{} \mathbb{Z}$$

Maslov index

(generalizes Euler class
$\mathrm{Lag}(\mathbb{R}^2) \cong S^1$)

- algebra

p real polynomial

Q: given $a<b$, how many real roots does p have in $(a,b) \subset \mathbb{R}$?

Euclidean algorithm $p_0 = p$, $p_1 = p'$

$$p_0 = q_1 p_1 - p_2$$

$$p_1 = q_2 p_2 - p_3$$

$$\vdots$$

$$p_m = q_{m-1} p_{m-1} + 0$$

Define

$$B = \begin{pmatrix} q_1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & q_m \end{pmatrix}$$

Thm For $a<b$

$$\#\text{ roots of } p \text{ in } (a,b) \text{ (w/o multiplicity)} = \frac{\operatorname{sig}(B(b)) - \operatorname{sig}(B(a))}{2}$$

First part of course:

getting familiar w/ quadratic forms,

esp. integral forms.

Lecture 2

Last time

- quadratic form $q : K^d \longrightarrow K$

  is diagonalizable

  $$q' : K^d \xrightarrow{\phi} K^d \xrightarrow{q} K$$

  $$q'(x_1, \dots, x_d) = a_1 x_1^2 + \cdots + a_d x_d^2$$

- $K = \mathbb{R} \Rightarrow$

  $$q \sim q' = x_1^2 + \cdots + x_n^2 - (x_{n+1}^2 + \cdots + x_{n+m}^2)$$

  signature $:= n - m$

Sylvester's Law

$B \in GL_d(\mathbb{R})$ symmetric

$\exists \; \Phi \in GL_d(\mathbb{R}) \qquad \Phi^t B \Phi = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$

$\mathrm{sig}(B) := n - m$

Ex $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in GL_2(\mathbb{R})$

$$\mathrm{sig}(B) = \begin{cases} 2 & \det(B) > 0,\ \mathrm{tr}(B) > 0 \\ 0 & \det(B) < 0 \\ -2 & \det(B) > 0,\ \mathrm{tr}(B) < 0. \end{cases}$$

Rational quadratic forms and p-signatures

$q : \mathbb{Q}^d \longrightarrow \mathbb{Q}$ quadratic form

Last time q can be diagonalized (over $\mathbb{Q}$)

$q \sim q' = a_1 x_1^2 + \cdots + a_d x_d^2$ $(a_i \in \mathbb{Q})$

Q: when are two diagonal forms equivalent?

Rmk Completing square doesn't give canonical diagonal form

$(3x^2 + 2xy) + y^2 \rightsquigarrow 3\left(x + \frac{y}{3}\right)^2 + \frac{2}{3} y^2$

$3x^2 + (2xy + y^2) \rightsquigarrow 2x^2 + (x+y)^2$

Q: Consider form

$\ell(x^2+y^2)$ where ℓ is prime.

when is this form equivalent over $\mathbb{Q}$ to x^2+y^2?

Rmk if $q = a_1 x_1^2 + \cdots + a_d x_d^2$ &

$q' = b_1 x_1^2 + \cdots + b_d x_d^2$ equivalent / $\mathbb{Q}$

then

- the forms have same # pos/neg sign

  ($\sim$ over $\mathbb{Q}$ $\Rightarrow$ $\sim$ over $\mathbb{R}$)

- $\prod a_i = \prod b_i$ in $\mathbb{Q}^\times / (\mathbb{Q}^\times)^2$

$$\left(\begin{aligned} B' \sim B \quad &\Longleftrightarrow \quad B' = \Phi^t B \Phi \\ &\Longrightarrow \quad \det B' = \det(\Phi)^2 \cdot \det(B) \\ &\Longrightarrow \quad \det(B') \equiv \det(B) \mod (\mathbb{Q}^\times)^2 \end{aligned} \right)$$

This doesn't help distinguish x^2+y^2 from $\ell(x^2+y^2)$.

Some observations

- $x^2+y^2 \sim (x+y)^2+(x-y)^2 = 2(x^2+y^2)$
- Similarly $x^2+y^2 \sim (ax+by)^2+(bx-ay)^2$

  $= (a^2+b^2)(x^2+y^2)$
- when $\ell \equiv 1 (4)$ can write $\ell = a^2+b^2$

so $x^2+y^2 \sim \ell(x^2+y^2)$

what about $\ell \equiv 3 (4)$?

$$3(x^2+y^2) \underset{?}{\sim} (x^2+y^2)$$

Thm (weak Hasse principle)

two quadratic forms on $\mathbb{Q}^d$ equivalent $\iff$ same det $\in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ and same p-signature for every prime p.

p-signatures (Conway)

- Fix $p \geq 3$ prime for $a \in \mathbb{Z}$

write
$$a = \underbrace{p^k}_{p\text{-part}} \cdot \underbrace{u}_{\text{rel prime to } p}.$$

say a is a p-*anti-square* if k odd and u is not a square in $(\mathbb{Z}/p\mathbb{Z})^\times$

given $a_1 x_1^2 + \cdots + a_d x_d^2 \quad a_i \in \mathbb{Z}$

the p-signature is

$$\sum_i \text{p-part}(a_i) + 4 \cdot \#\{a_i \text{ p-antisquare}\}$$

mod 8.

Example $q = 6x^2 + 20y^2 + 15z^2$

3-signature

$$\begin{aligned} &3 + 1 + 3 \\ +\ &4 + 0 + 4 \end{aligned} \equiv 7(8)$$

5-signature

$$\begin{aligned} &1 + 5 + 5 \\ +\ &0 + 0 + 4 \end{aligned} \equiv 7(8)$$

p-signature $p \geq 7$

$$\begin{aligned} &1 + 1 + 1 \\ +\ &0 + 0 + 0 \end{aligned} \equiv 3(8)$$

2-signature : weird see notes.

(-1)-signature $:= \sum \underbrace{(-1)\text{-part}(a_i)}_{\text{sign}(a_i)} \in \mathbb{Z}$

$a = (-1)^k \cdot u$ $\equiv$ signature over $\mathbb{R}$.

$u > 0$. (!)

__Exercise__ Use p-signatures to show

for l prime

$l(x^2+y^2) \sim x^2+y^2 \iff$ $l=2$ or $l \equiv 1 (4)$

__Thm__ (weak Hasse principle, restated)

Two quadratic forms over $\mathbb{Q}$ are equivalent $\iff$ equivalent over $\mathbb{R}$ & $\mathbb{Q}_p$ for each prime p.

discuss more next time

(Useful) **Prop** $f = a_1 x_1^2 + \cdots + a_d x_d^2$ quadratic form over $\mathbb{Q}$. Fix $b \in \mathbb{Q}^\times$.

① If $\exists$ $u \in \mathbb{Q}^d$ s.t. $f(u) = b$

then $f \sim b x_1^2 + g(x_2, \ldots, x_d)$

② (Witt cancellation)

Ass. $u \neq u'$ and $f(u) = b = f(u')$

write $f \sim b x_1^2 + g(x_2, \ldots, x_d)$

$f \sim b x_1^2 + g'(x_2, \ldots, x_d)$

Then $g \sim g'$.

Application (alternate proof of Sylvester)

$$[+1]^{\oplus n+k} \oplus [-1]^{\oplus m} \sim \underbrace{[+1]^{\oplus n} \oplus [-1]^{\oplus k+m}}_{x_1^2+\cdots+x_n^2-(x_{n+1}^2+\cdots)}$$

$$[+1]^{\oplus k} \sim [-1]^{\oplus k}$$

$\Rightarrow k=0.$

Proof of Prop

① Check $\mathbb{Q}^d = \mathrm{span}(u) \oplus \mathrm{span}(u)^\perp$

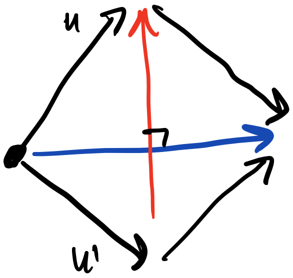
choose new basis compatible with this decomposition

② wlog assume u, u' L.I.

Set $w = u - u'$ and define

$r_w : v \longmapsto v - 2\frac{B(v,w)}{B(w,w)} w$ reflection

Then $r_w(u) = u'$



So r_w maps

span$(u)^\perp$ to span$(u')^\perp$

$\square$

Possible problem: $u-u'$ is isotropic

ie $f(u-u')=0$.

Then use $u+u'$ instead.

if $f(u-u'$ $0 = f(u+u')$ then

$$\underbrace{B(u-u', u+u')}_{=0 \text{ b/c } f(u)=f(u')} = f(2u) - \underbrace{f(u-u') - f(u+u')}_{=0}$$

$\Rightarrow f(u)=0$ $\times$.

$\boxed{E_8}$ (Next: integral quad. forms. Now: one ex.)

$$D_n = \{ x \in \mathbb{Z}^n \mid \sum x_i \equiv 0 (2) \}$$

$$D_n^+ = D_n \cup \left(D_n + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)\right)$$

- $n \equiv 0(2) \Rightarrow D_n^+$ is a lattice

(when n odd D_n^+ is not closed under $+$)

$2\left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right) \notin D_n$.

- $n \equiv 0(4) \Rightarrow$

Inner product on $\mathbb{R}^n$ restricts to integral quadratic form $D_n^+ \longrightarrow \mathbb{Z}$.

- $n \equiv 0(8) \Rightarrow$ form is even $D_n^+ \longrightarrow 2\mathbb{Z}$

and $|\det| = 1$ (unimodular)

D_8^+ aka E_8 lattice.

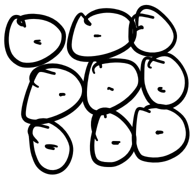
The quadratic form has matrix

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}$$

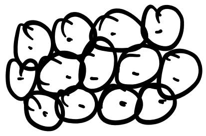
- This is the intersection form of a topological 4-manifold with no smooth structure.

- E_8 gives densest ~~lattice~~ sphere packing in dim 8 (Viazovska 2016)

(Blichfeldt 1935)



square lattice in $\mathbb{R}^2$



hex lattice (densest)

E_8 kissing number $= 240$ $\quad \mathbb{Z}^8$ kissing $\# = 16 = 2\cdot 8$

- $$\Theta_k(z) = \sum_{v \in D_{8k}^{+}} q^{\langle v,v \rangle} \qquad q = e^{2\pi i z}$$

modular form weight 4k.

$$\Theta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{4k}\,\Theta(z)$$

E_8, Θ functions, Isospectral Tori

$L \subset \mathbb{R}^n$ lattice. Assume $\langle v,v\rangle \in \mathbb{Z}$ $v \in L$.

Theta function $\Theta_L(z) = \sum_{v \in L} q^{\langle v,v\rangle}$

$q = e^{2\pi i z}$

Function on $\mathbb{H} := \{Im(z) > 0\}$

Ex $\mathbb{Z} \subset \mathbb{R}$

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i z \cdot n^2}$$

$$\Theta(z+2) = \sum_{n \in \mathbb{Z}} e^{\pi i (z+1) n^2} = \Theta(z).$$

Claim $\Theta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}}\, \Theta(z).$

$\Rightarrow$ "θ is modular form of weight $\frac{1}{2}$ for $\Gamma = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{Z})$"

$$\theta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{1/2}\,\theta(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$z \in \mathbb{H}$$

Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)$$

$$\hat{f}(y) = \int_{\infty}^{\infty} e^{2\pi i x y} f(x)\,dx$$

For $f(x) = e^{\pi i z \cdot x^2}$

$$\hat{f}(y) = \sqrt{\frac{i}{z}}\; e^{-\pi i y^2/z} \qquad \left(\begin{matrix}\text{exercise/} \\ \text{computation}\end{matrix}\right)$$

Poisson $\Rightarrow$ $$\underbrace{\sum_n e^{\pi i z n^2}}_{\theta(z)} = \sqrt{\frac{i}{z}}\; \underbrace{\sum_m e^{-\pi i m^2/z}}_{\theta(-1/z)}$$

More generally if $L \subset \mathbb{R}^n$

$\underbrace{\text{unimodular}}_{\mathrm{vol}(\mathbb{R}^n/L)=1}$, $\underbrace{\text{even}}_{\langle v,v\rangle \in 2\mathbb{Z}\ \forall v.}$ lattice $\left(\begin{matrix}\text{only exist}\\ \text{if } n \equiv 0 (8)\end{matrix}\right)$

then $\theta_L(z+1) = \theta_L(z)$ $\quad \theta_L\left(\frac{-1}{z}\right) = z^{\frac{n}{2}}\,\theta(z)$

$\Rightarrow \theta_L$ modular form for $SL_2(\mathbb{Z})$.
$\qquad\quad^{\wedge}$
$\quad$ weight $\frac{n}{2}$

Application (isospectral tori)

Given $L \subset \mathbb{R}^n$

get torus $\mathbb{R}^n/L \cong T^n$
$\quad ^{\wedge}$
Riemannian

$$\theta_L \longleftrightarrow \text{lengths of geodesics.}$$

$$\overset{\shortparallel}{\sum_{v \in L}} q^{\langle v,v\rangle} = \sum_N \#\{v \in L \mid \langle v,v\rangle = N\} \cdot q^N$$

Two tori are isospectral if they have same geodesic lengths.
($\longleftrightarrow$ eigenvalues of Laplacian)

Q: Are isospectral manifolds isometric?
(Can you hear the shape of a drum?)

Thm (Milnor) $\exists$ non-isometric isospectral tori of dim $= 16$.

About proof. Recall from last time

$$D_n = \{ x \in \mathbb{Z}^n \mid \Sigma x_i \equiv 0(2) \}$$

$$D_n^+ = D_n \cup \left(D_n + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)\right)$$

if $n \equiv 0(8)$ D_n^+ is even, unimodular lattice

$$D_8^+ \cong E_8$$

Claim $\mathbb{R}^{16}/D_8^+ \oplus D_8^+$ & $\mathbb{R}^{16}/D_{16}^+$

$\underbrace{\text{isospectral}}_{\text{harder / more interesting}}$ but not isometric.

$\hookrightarrow$ Θ functions are weight 8 modular forms.

There is a unique such form up to scaling $\Rightarrow$ $\Theta_{D_8^+ \oplus D_8^+} = \Theta_{D_{16}^+}$. $\square$

Rational Quadratic forms & Strong Hasse principle

Last time

Weak Hasse principle Two quadratic forms / $\mathbb{Q}$ are equivalent $\iff$ equivalent over $\mathbb{Q}_p$ for each prime p (including $p=-1$, $\mathbb{Q}_{-1} \equiv \mathbb{R}$)

($\iff$ same p-signature for each p.)

Today explain how to deduce from

Strong Hasse Principle (Hasse-Minkowski)

① A rational quadratic form f represents 0 (i.e $\exists$ $x \in \mathbb{Q}^d \setminus \{0\}$ s.t. $f(x)=0$)

$\iff$ f represents 0 over $\mathbb{Q}_p$ for each p.

② Same for "f represents $b \in \mathbb{Q}$"

Rmk (Hasse principles)

This is about solving (quadratic) equations:

$\exists ?\ x \in \mathbb{Q}^d$ s.t. $f(x) = a_1 x_1^2 + \cdots + a_d x_d^2 = b$

An equation satisfies Hasse principle if.....

Rmk ① $\Rightarrow$ ②

if f represents b over $\mathbb{Q}_p$ $\forall p$

then $g = f(x) - by^2$ reps 0 over $\mathbb{Q}_p$ $\forall p$

① $\Rightarrow$ g reps 0 over $\mathbb{Q}$

$\Rightarrow$ f reps b over $\mathbb{Q}$. ✓

Rmk Strong $\Rightarrow$ Weak. Proof by induction

Base case $f = ax^2$, $f' = a'x^2$

Assume $f \sim f'$ over $\mathbb{Q}_p$ $\forall p$.

WTS $f \sim f$ over $\mathbb{Q}$.

Suffices to show f' represents a over $\mathbb{Q}$

since then $f' \sim ax^2$ (last time)

$f \sim f'$ over $\mathbb{Q}_p \Rightarrow f'$ reps a over $\mathbb{Q}_p$ $\forall p$

$\xRightarrow[\text{Strong Hasse}]{}$ f' reps a over $\mathbb{Q}$ ✓

Induction Step basically the same.

Suppose f, f' equiv over $\mathbb{Q}_p$ $\forall p$.

Fix $b \in \mathbb{Q}^\times$ represented by f. (over $\mathbb{Q}$)

Suffices to show b represented by f' too.

Last time: if b rep'd by

$$f \sim b x_1^2 + g(x_2, \ldots, x_d)$$

$$f' \sim b x_1^2 + g'(x_2, \ldots, x_d)$$

and Witt cancellation $\Rightarrow$ $g \sim g'$ over $\mathbb{Q}_p$ $\forall p$.

$\Rightarrow$ $g \sim g'$ over $\mathbb{Q}$ by induction

$\Rightarrow$ $f \sim f'$

Rational Forms & Hyperbolic manifolds

(In response to Sam: why geometers care about rational forms .)

$SO(n,1;\mathbb{Z})$

$$= \left\{ A \in SL_{n+1}(\mathbb{Z}) \;\middle|\; A^t \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \right\}$$

Prop For $n \geq 3$ $SO(n,1;\mathbb{Z})$ contains a surface subgroup.

↳ $\pi_1($

$)$

Rmk $SO(3,1;\mathbb{Z}) \curvearrowright \mathbb{H}^3$

$\mathbb{H}^3 / SO(3,1;\mathbb{Z})$ finite volume hyperbolic 3-manifold (noncompact)

Kahn-Markovic:

M^3 closed hyperbolic

$\Rightarrow$ $\pi_1(M)$ contains surface subgroup.

Prop is much easier b/c $SO(n,1;\mathbb{Z})$

is arithmetic group.

Correction / Additions

Claim from last time

If $g(x_0, \dots, x_d) = bx_0^2 - f(x_1, \dots, x_d)$

represents 0 then f represents b.

(over any field of char $\neq 2$)

Proof By assumption $\exists \ (y_0, \dots, y_d) \in K^{d+1}$

s.t. $by_0^2 = f(y_1, \dots, y_d)$

Case 1 $y_0 \neq 0 \Rightarrow$

$$b = \left(\frac{1}{y_0}\right)^2 f(y_1, \dots, y_d) = f\left(\frac{y_1}{y_0}, \dots, \frac{y_d}{y_0}\right)$$

$\Rightarrow f$ represents b.

Case 2 $y_0 = 0$.

Then f represents 0 ("f isotropic")

Lemma f isotropic nondegenerate $\Rightarrow$ f reps every $b \in K^\times$.

Pf of Lem $f : K^d \longrightarrow K$

b associated bilinear form.

Fix $u \in K^n$ w/ $f(u) = 0$

f nondegen $\Rightarrow$ $\exists$ $v \in K^d$ s.t.

$b(u,v) \neq 0$. Rescale v so $b(u,v) = 1$.

$$\text{matrix of } b\big|_{\text{span}(u,v)} = \begin{array}{c|cc} & u & v \\ \hline u & 0 & 1 \\ v & 1 & t \end{array}$$

$b(su+v, su+v) = 2s + t = 0$ if $s = -\frac{t}{2}$

replace v by $-\frac{t}{2}u + v$ so then matrix of $b|_{\mathrm{span}(u,v)} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{hyperbolic form}}$

quadratic form of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is

$q(x,y) = 2xy.$

In particular $q(\frac{b}{2}, 1) = b.$ $\square$

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i z \cdot n^2}$$

(Trent): convergence?

Claim converges on $\mathbb{H} = \{ \mathrm{Im}(z) > 0 \} \subset \mathbb{C}$.

write $z = x + iy$

$$\left| e^{\pi i z \cdot n^2} \right| = \left| e^{\pi i (x+iy) n^2} \right| = \Big| e$$

$$= \underbrace{\left|e^{\pi i x n^2}\right|}_{=1} \cdot \left|e^{-\pi y n^2}\right|$$

decays very fast as $n \to \infty$.

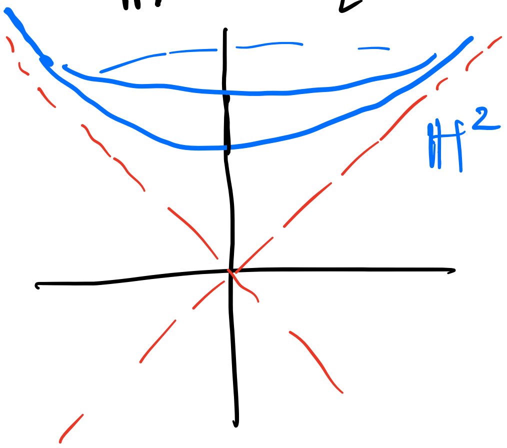
as long as $y > 0$.

If $y \leq 0$ there's trouble...

Rational forms & hyperbolic mflds.

$$f_n := -x_0^2 + x_1^2 + \dots + x_n^2$$

$$\mathbb{H}^n := \{ x \in \mathbb{R}^{n+1} \mid f_n(x) = -1, \ x_0 > 0 \}$$



hyperboloid model of hyperbolic space

For $x \in \mathbb{H}^n$

$$T_x \mathbb{H}^n \cong x^{\perp} \qquad f_n|_{x^{\perp}} \text{ pos. def.}$$

⇝ Riem. metric on $\mathbb{H}^n$

(hyperbolic metric)

$$O(f_n;\mathbb{R}) = \left\{ A \in GL_{n+1}(\mathbb{R}) \;\middle|\; f_n(Av) = f_n(v) \ \forall v \in \mathbb{R}^{n+1} \right\}$$

$\cup$

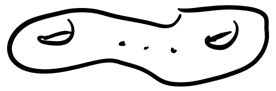
$O^+(f_n;\mathbb{R})$ index 2 subgp preserving $\mathbb{H}^n$

$\cup$

$SO^+(f_n;\mathbb{R}) = O^+ \cap SL_n(\mathbb{R})$

$\cup$

$SO^+(f_n;\mathbb{Z}) = SO^+ \cap SL_n(\mathbb{Z})$.

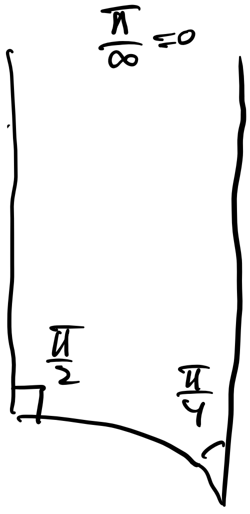
Prop $SO^+(f_n;\mathbb{Z})$ contains a surface subgroup. $\pi_1($  $)$

genus ≥ 2.

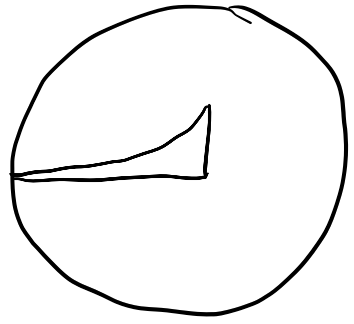
Ex $SO^+(f_n;\mathbb{Z})$ contains

$$SO^+(f_2;\mathbb{Z}) = \pi_1\Big(\underbrace{\mathbb{H}^2/SO^+(f_2;\mathbb{Z})}_{\substack{\text{noncompact surface}\\ \text{(orbifold)}}}\Big)$$

Fact $SO^+(f_2;\mathbb{Z}) \cong \Delta(2,4,\infty)$ triangle group.



generated by reflections in sides of triangle w/ angles $(\frac{\pi}{2}, \frac{\pi}{4}, 0)$



$\Delta(2,4,\infty)$ virtually free

$\Rightarrow$ doesn't contain π_1(closed surface).

Thm (Mahler Compactness application)

$q : \mathbb{Q}^d \longrightarrow \mathbb{Q}$ nondegen. quadratic form

$SO(q;\mathbb{Z}) < SO(q;\mathbb{R})$ as above.

$SO(q;\mathbb{R}) / SO(q;\mathbb{Z})$ compact

$\iff$ q is anisotropic ie $q(v) \neq 0$
$\forall v \in \mathbb{Q}^d \setminus \{0\}$

Ex. f_n isotropic $\forall n$.

so $\mathbb{H}^n / SO(f_n;\mathbb{Z})$ always noncompact.

Ex $q = -7x_0^2 + x_1^2 + x_2^2$. anisotropic

$-7a_0^2 + a_1^2 + a_2^2 = 0$ $\qquad a \neq 0 \Rightarrow a_0 \neq 0$

clear denominators $\Rightarrow a_0, a_1, a_2 \in \mathbb{Z}$.

$$a_1^2 + a_2^2 = 7a_0^2$$

Number theory: $n \in \mathbb{Z}_{>0}$ is sum of 2 squares

$\iff$ prime factorization contains no
p^k where $p \equiv 3(4)$ & k odd.

$\implies$ $\mathbb{H}^2 / SO(q;\mathbb{Z})$ compact hyperbolic 2-orbifold.

finitely covered by a compact hyp surface

$\implies$ $SO(q;\mathbb{Z})$ has surface subgp.

Proof of Prop $\quad q = -7x_0^2 + x_1^2 + x_2^2$

$f_n = -x_0^2 + x_1^2 + \dots + x_n^2 \qquad n \geq 3.$

WTS $\exists$ $SO(q;\mathbb{Z}) \hookrightarrow SO(f_n;\mathbb{R})$

Trick: $f_n' := -7x_0^2 + 7x_1^2 + x_2^2 + \cdots + x_n^2$

Observe

- $SO(q;\mathbb{Z}) \hookrightarrow SO(f_n';\mathbb{Z})$ for $n \geq 3$

- $f_n' \sim f_n$ over $\mathbb{Q}$ since

$$x^2 - y^2 \sim (4x+3y)^2 - (3x+4y)^2$$
$$= 7(x^2-y^2).$$

This implies $SO(f_n;\mathbb{Z})$ and $SO(f_n';\mathbb{Z})$ have common finite index subgroup (commensurable)

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}^t \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}^{-1} SO(f_2;\mathbb{Q}) \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} = SO(f_2';\mathbb{Q})$$

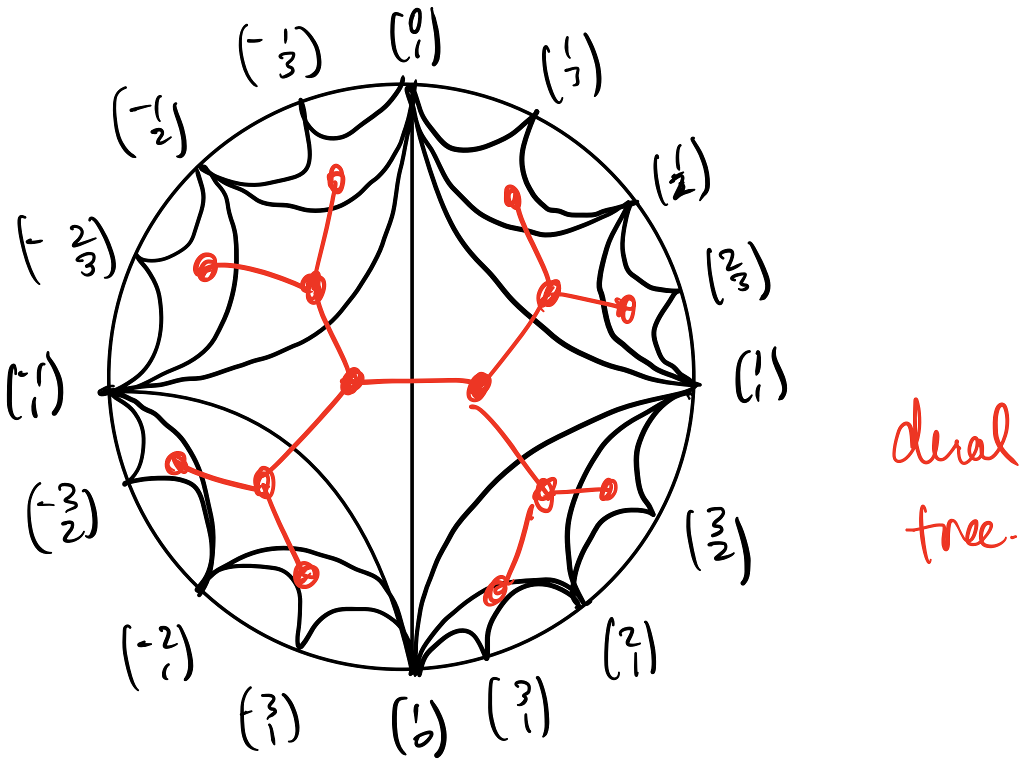
$SO(f_n'; \mathbb{Z})$ contains surface group $\Rightarrow$ $SO(f_n; \mathbb{Z})$ does too $\square$

Final algebraic chapter

Integral Quadratic Forms

- classification of unimodular, indefinite forms (useful for study manifolds)
- positive definite forms & mass formula
- forms on $\mathbb{Z}^2$

  very concrete classification using the Farey graph.



vertices: primitive vectors in $\mathbb{Z}^2$ (up to ±)

edges: pairs of vectors that that form basis for $\mathbb{Z}^2$

triangles: triples, each pair a basis.

Quadratic Forms on $\mathbb{Z}^2$

$q(x,y) = ax^2 + hxy + by^2 \qquad a,b,h \in \mathbb{Z}$

Goal given q, q' determine if $q \sim q'$ in finite time based on their values

Assume q is nondegenerate.

Dichotomy

- q definite
  - positive ($q > 0$)
  - negative ($q < 0$)
- q indefinite
  - isotropic ($\exists\ v \neq 0 \quad q(v) = 0$)
  - anisotropic ($q(v) \neq 0 \quad \forall v \neq 0$)

Observe q determined by values

$q(e)$, $q(f)$, $q(e+f)$ whenever e, f basis for $\mathbb{Z}^2$

Check $q(xe+yf) = q(e)x^2 + [q(e+f) - q(e) - q(f)]xy$
$+ q(f)y^2$

equivalently $B(u,v) := q(u+v) - q(u) - q(v)$

det. by $B(e,e)$ $B(e,f)$ $B(f,f)$

(but there are infinitely many choices of e,f)
...

Case If q, q' positive definite

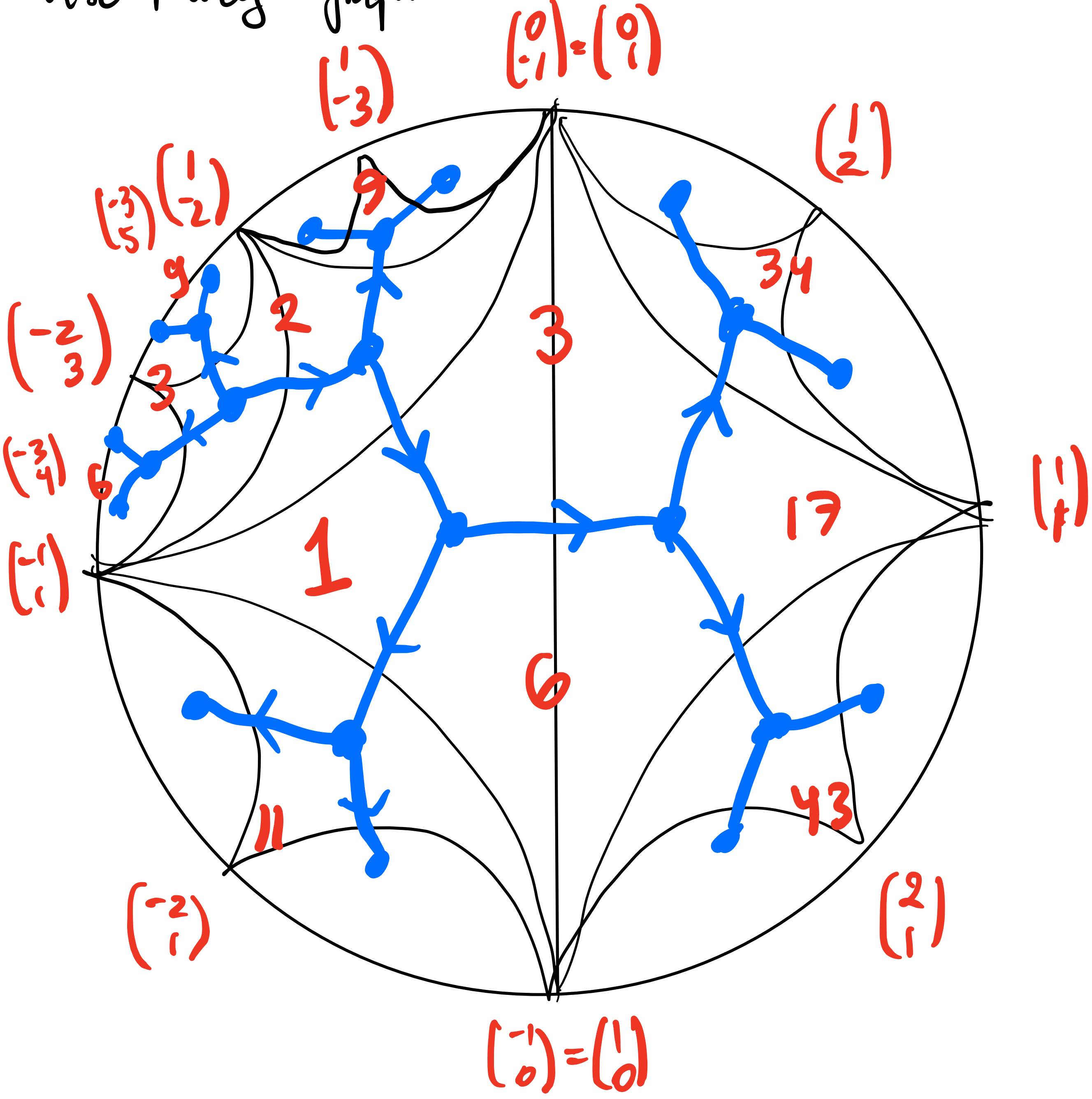
distinguish them by ... shortest vectors.

(But how to find these?)

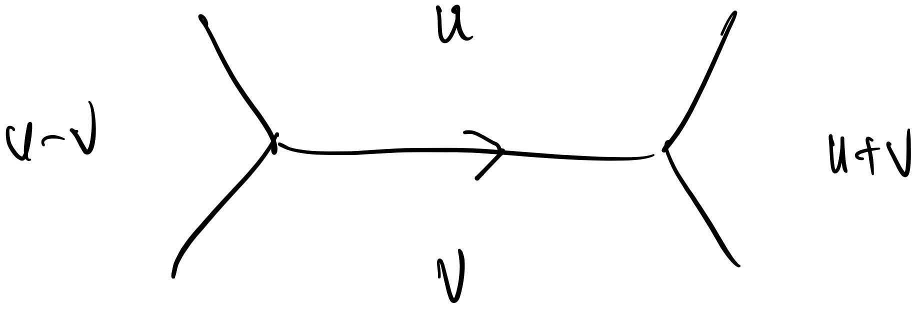
Eg what's smallest value of

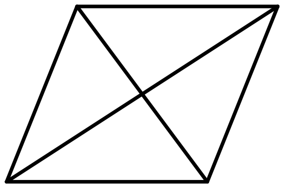
$$q = 6x^2 + 8xy + 3y^2$$

Use Farey graph



Generaley





Parallelogram law

$$q(u+v) + q(u-v) = 2\left[q(u) + q(v)\right]$$

equiv

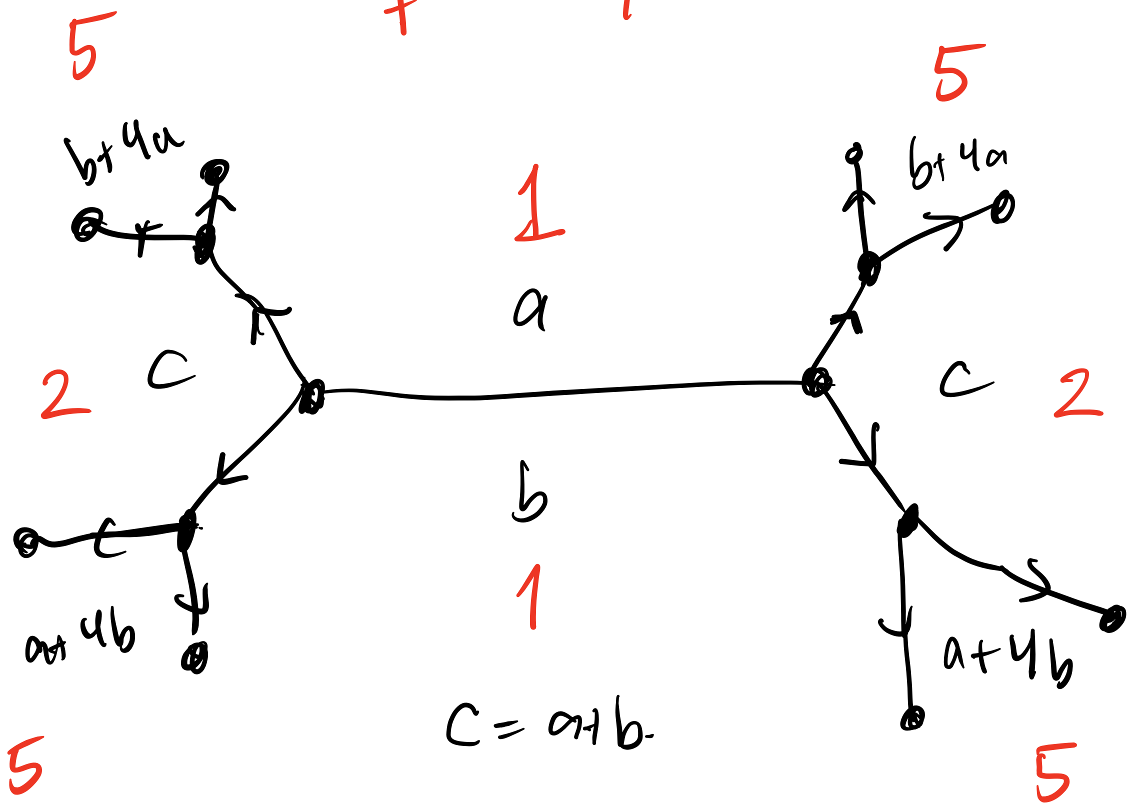
$$\left[q(u)+q(v)-q(u+v)\right] + \left[q(u)+q(v)-q(u-v)\right] = 0$$

- exactly one positive denote it δ

$q(u+v) = q(u-v) + 2\delta$ OR

- each is zero

$q = x^2 + y^2$



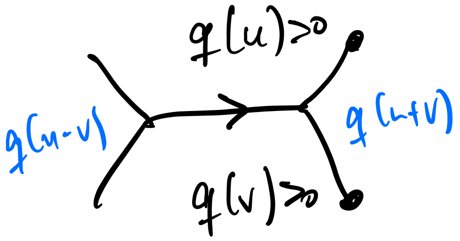
$c = a+b.$

$a+c-b = a+(a+b)-b = 2a > 0$

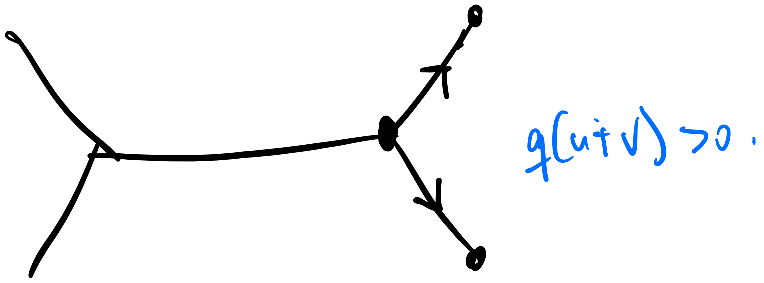
$b+c-a = b+(a+b)-a = 2b > 0$

Climbing Lemma

q any form on $\mathbb{Z}^2$
$u, v \in \mathbb{Z}^2$ any basis

If 

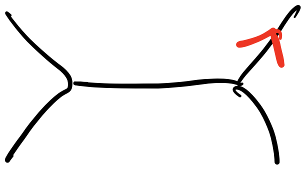
then



Proof

① $q(u+v) = 2\left[q(u)+q(v)\right] - q(u-v)$

$$= \underbrace{q(u)}_{>0} + \underbrace{q(v)}_{>0} + \underbrace{\left[q(u)+q(v)-q(u-v)\right]}_{=\delta>0} > 0$$

② To prove 

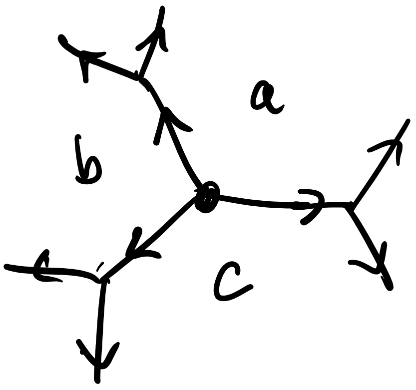
<u>WTS</u> $q(u)+q(u+v)-q(v) > 0.$

$$q(u)+q(u+v)-q(v) = q(u)+[q(u)+q(v)+\delta]-q(v)$$

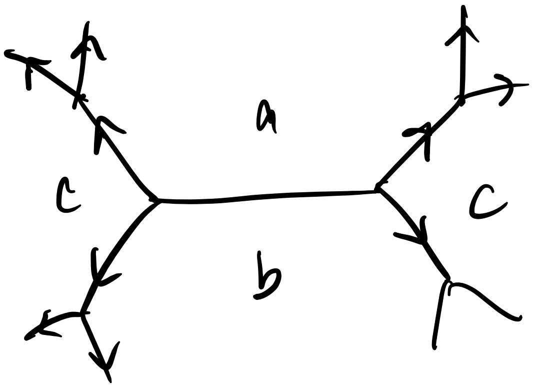
$$= 2q(u)+\delta > 0. \quad ✓$$

Summary

Given pos. def. form. q can use climbing lemma to find smallest values of q.



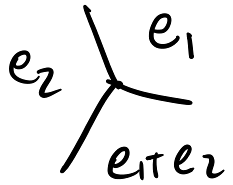
or



This gives algorithm to determine if $q\sim q'$ in definite case.

Cor For $q:\mathbb{Z}^2\to\mathbb{Z}$ pos. def.

$SO(q)$ is a subgroup of $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.



permuted $\begin{pmatrix}0&1\\-1&1\end{pmatrix}$ order 6.

Positive integral quadratic forms + mass formula

Say $q,q':\mathbb{Z}^d\to\mathbb{Z}$ have

same *genus* if equivalent over $\mathbb{Z}_p$

(p-adic integers) for each prime p.

$(\mathbb{Z}_{-1}=\mathbb{R})$

Unfortunately genus doesn't determine the form (no Hasse principle over $\mathbb{Z}$...)

For a genus $\mathcal{G}$ define the

<u>mass</u> $$m(\mathcal{G}) = \sum_{q \in \mathcal{G}} \frac{1}{|O(q)|}$$

$O(q) \equiv$ orthogonal group $\left(\begin{matrix} \text{finite b/c} \\ q \text{ pos. def} \end{matrix} \right)$

<u>Mass formula</u> for $\overset{\text{pos. def.}}{\text{unimodular}}$ forms of rank $8k$.

$$m(\mathcal{G}) = 2^{1-8k} \frac{1}{(4k)!} B_{2k} \prod_{j=1}^{4k-1} B_j$$

B_n = Bernoulli numbers.

rank	mass	# forms
8	$\sim 10^{-9}$	1 $\rightsquigarrow E_8 \cong D_8^+$
16	$\sim 10^{-18}$	2 $\rightsquigarrow D_8^+ \oplus D_8^+ , D_{16}^+$
24	$\sim 10^{-15}$	24
32	$\sim 10^{7}$	$> 10^{7}$

each summand in $m(G)$ contributes at most $\frac{1}{2}$ …

Indefinite Forms on $\mathbb{Z}^2$ (after Conway)

Last time

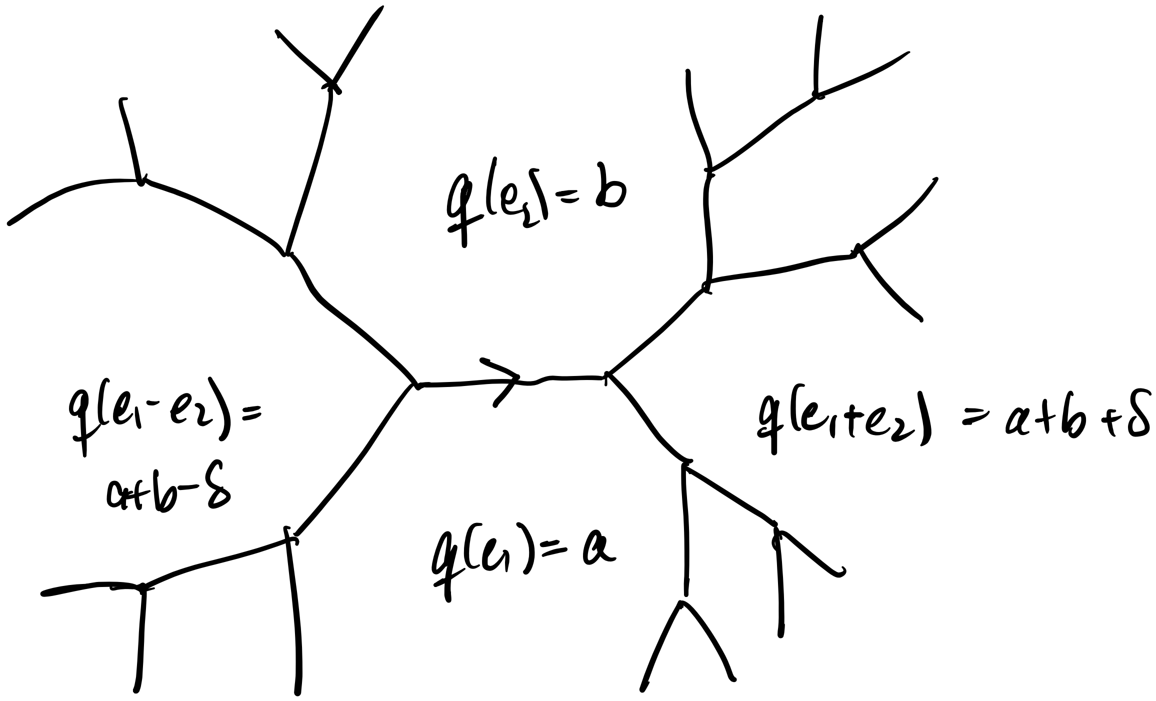
- For any quadratic form on $\mathbb{Z}^2$

  get labeling of vertices of Farey graph.

  $\hookrightarrow$ labeling of regions of dual Farey tree. + direction on edges

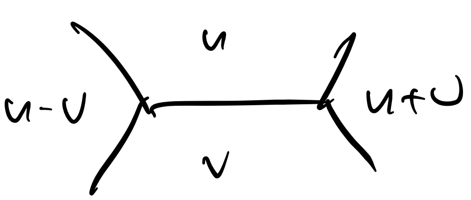
---

$q = ax^2 + \delta xy + by^2$ wlog $\delta > 0$



Observe $\underset{q(e_1-e_2)}{\underset{\shortparallel}{a+b-\delta}}$, $\underset{q(e_1)+q(e_2)}{\underset{\shortparallel}{a+b}}$, $\underset{q(e_1+e_2)}{\underset{\shortparallel}{a+b+\delta}}$ arithmetic progression

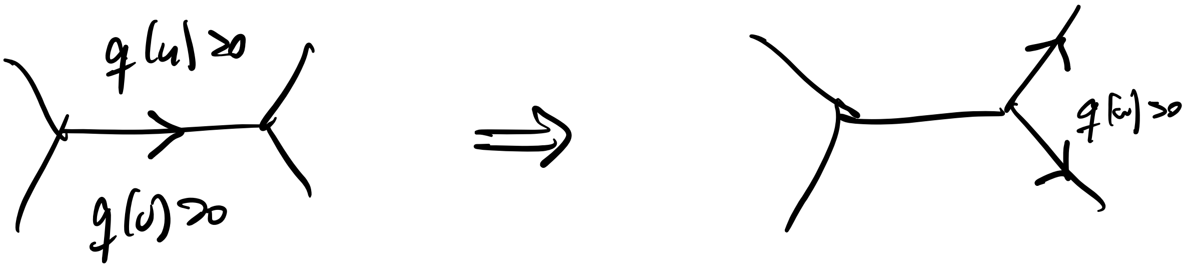
- see this pattern around every edge



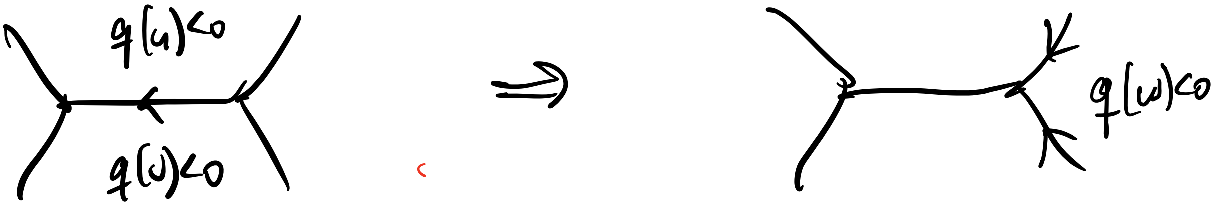
(parallelogram law)

u, v basis for $\mathbb{Z}^2$

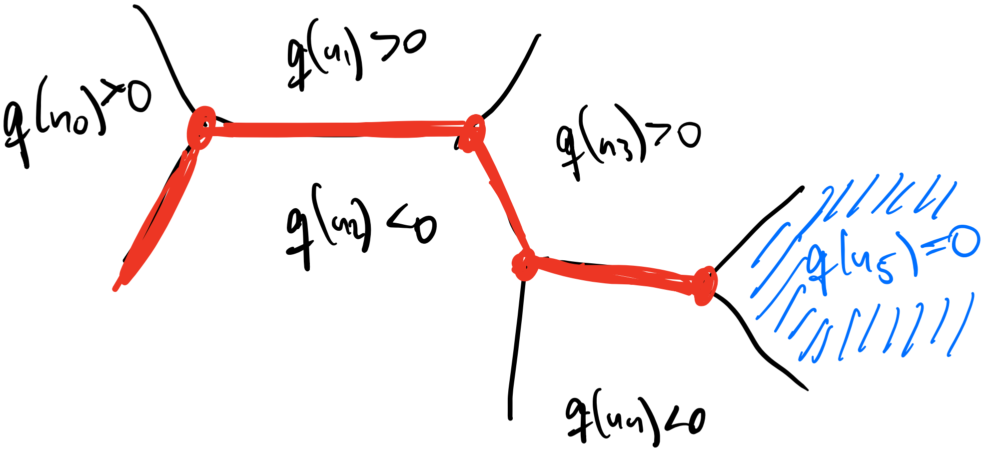
- Climbing lemma



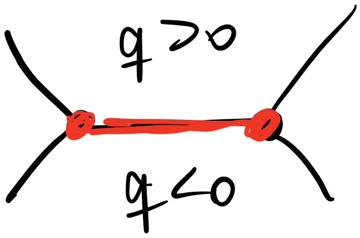
Similarly



Situation not covered by Climbing lemma:



river edges = edges where

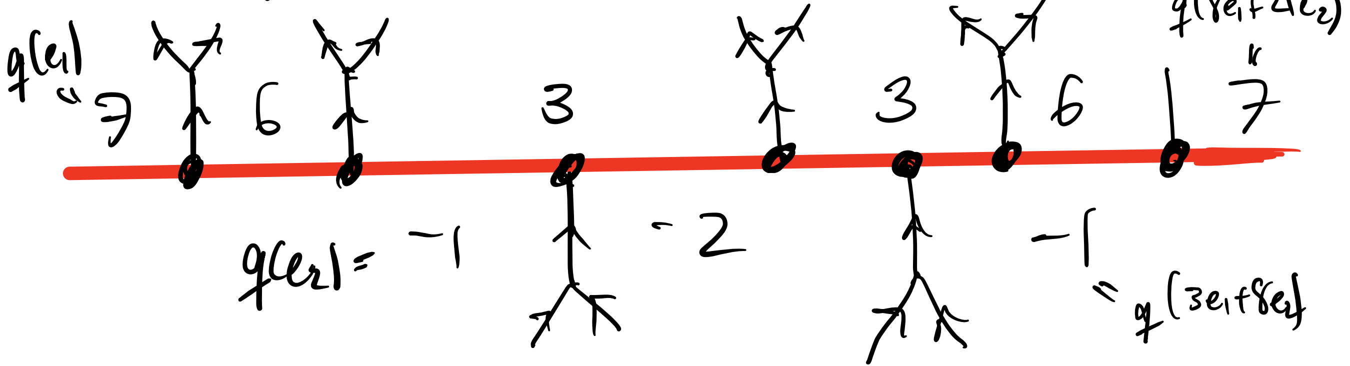


- Rivers keep flowing
- until they reach a lake = region where $q=0$.

So q anisotropic $\Rightarrow$ $\exists$ river is bi-infinite.

q isotropic $\Rightarrow$ $\exists$ a lake. (maybe more?)

Ex $q = 7x^2 - y^2$ (anisotropic)



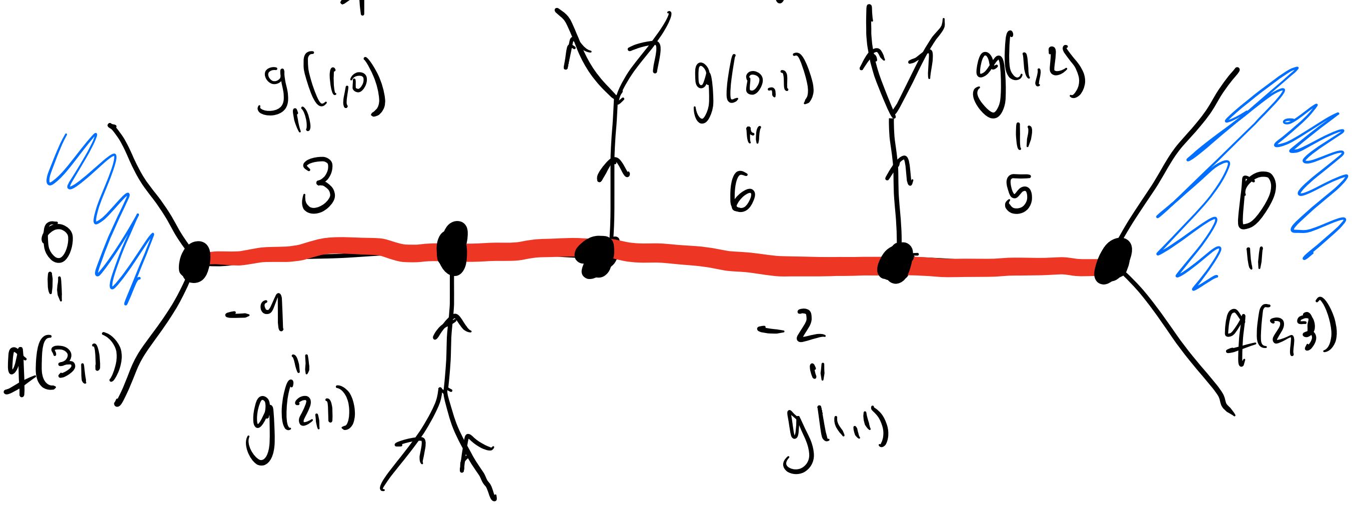
Labels along the river are periodic!

⇒ river is periodic.

This reflects the fact that

$SO(q) \ni \begin{pmatrix} 8 & 3 \\ 21 & 8 \end{pmatrix}$ has infinite order.

Ex $q = 3x^2 - 11xy + 6y^2$



Thm (rivers & lakes)

$q : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ indefinite

(1) q anisotropic

$\Rightarrow \exists$ unique river,

it's bi-infinite, its labels are periodic

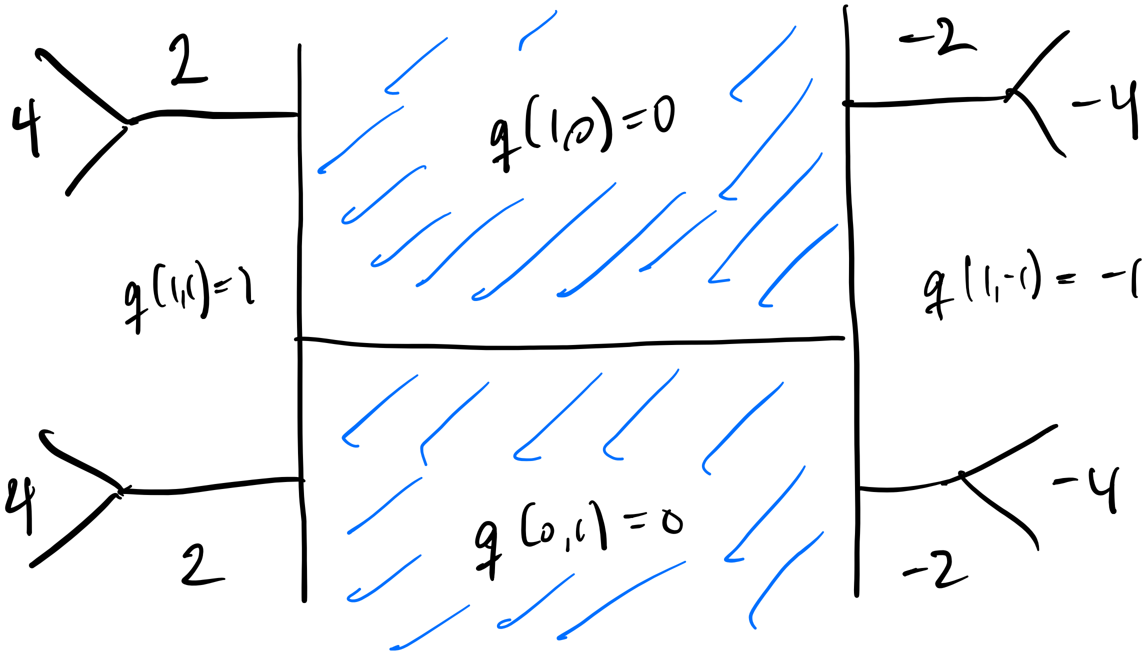
(2) q isotropic

$\Rightarrow \exists$ exactly two lakes,

either adjacent or connected by a river

Rmk Two adjacent lakes is a weir

eg $q(x,y) = xy$.



Application (isomorphism prob for indefinite forms on $\mathbb{Z}^2$)

Given q, q' use climbing to find river (or weir) Either periodic or finite.

Compute values along river to determine

Proof (Anisotropic ⇒ River periodic)

- Fix basis $e, f \in \mathbb{Z}^2$.

$$q(xe+yf) = q(e)x^2 + \Big[\underbrace{q(e+f) - q(e) - q(f)}_{\delta > 0}\Big] xy + q(f)y^2$$

Bilinear form B has matrix $B = \begin{pmatrix} 2q(e) & \delta \\ \delta & 2q(f) \end{pmatrix}$

Key $|\det(B)| = |4q(e)q(f) - \delta^2|$ is invariant of q. ($\Phi \in GL_2(\mathbb{Z})$ has $\det \in \mathbb{Z}^\times = \{\pm 1\}$)

- if $q(e) > 0$, $q(f) < 0$ then $|4q(e)q(f) - \delta^2| = 4|q(e)q(f)| + \delta^2$

$\exists$ finitely many integer solutions

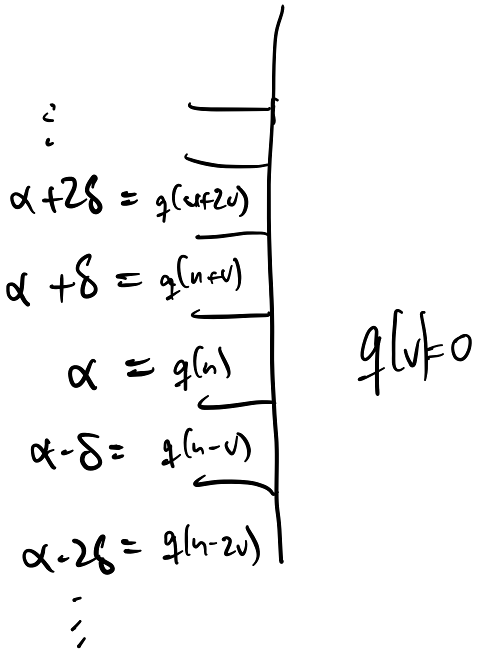
to $4 \cdot \alpha\beta + \delta = \det$

$\Rightarrow$ periodic.

<u>Proof</u> (isotropic $\Rightarrow$ two lakes)

if $\exists \geq 1$ lake

must be a river somewhere... or a weir



river is ∞ $\Rightarrow$ periodic

$\Rightarrow$ bi-infinite $\times$.

$\Rightarrow \exists$ 2 lakes. $\square$

Aside Topography $\overset{?}{=}$ Topology

Cor $q : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$

- q anisotropic $\Rightarrow$ $SO(q)$ virtually $\mathbb{Z}$
- q isotropic $\Rightarrow$ $SO(q)$ finite

$\square$

(see examples)

Recall (Mahler Compactness)

For $q : \mathbb{Z}^d \longrightarrow \mathbb{Z}$

$SO(q;\mathbb{R}) / SO(q;\mathbb{Z})$ compact

$\Longleftrightarrow$ q anisotropic.

In special case above $(d=2)$

$SO^+(q;\mathbb{R}) \cong \mathbb{R}$

Indefinite unimodular forms on $\mathbb{Z}^d$

- B symmetric integer matrix
- $\det(B) = \pm 1$ (unimodular)

- eg D_n^+ when $n \equiv 0 \, (8)$

  $\uparrow$ pos. definite

$\exists > 10^7$ inequivalent pos. definite B in dim $= 32$ (mass formula)

Thm (Serre) B as above and indefinite

- B odd ($\exists\, v$ s.t. $v^t B v$ odd)

$$\Rightarrow B \text{ equiv to } \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

$$= [+1]^{\oplus n} \oplus [-1]^{\oplus m}$$

- B even ($v^t B v \in 2\mathbb{Z} \quad \forall v$)

$$\Rightarrow B \text{ equiv to } (E_8)^{\oplus n} \oplus H^{\oplus m}$$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

---

odd case is exercise modulo

Thm (Meyer) $q : \mathbb{Z}^d \to \mathbb{Z}$

indefinite, unimodular $\Rightarrow$ isotropic.

(not true w/o unimodular
eg $x^2+y^2-7z^2$ anisotropic)

See Notes

Intersection Forms of (4-)manifolds

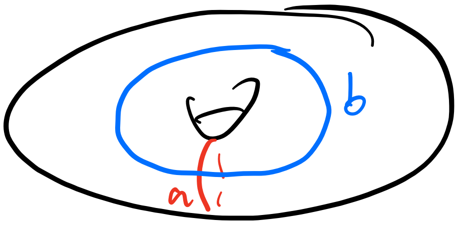
Source: Scorpan's Wild World of 4-manifolds

- M^4 closed oriented 4-manifold
- First examples S^4, $S^2 \times S^2$, $\mathbb{CP}^2$, $T^4 = S^1 \times \cdots \times S^1$
- Basic principle: a lot of the topology of M is captured by how surfaces intersect in M. especially when $\pi_1(M) = 0$.

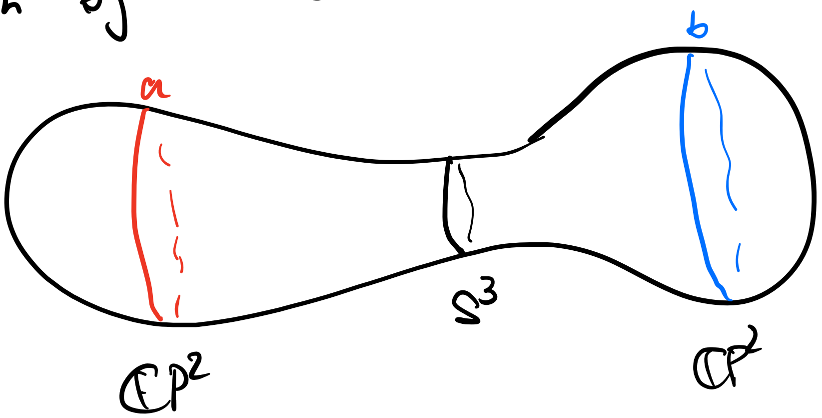
Eg $S^2 \times S^2$ vs $\mathbb{CP}^2 \# \mathbb{CP}^2$

$$H_i(S^2 \times S^2) \cong \left\{ \begin{array}{ll} \mathbb{Z} & i = 0,4 \\ \mathbb{Z}^2 & i = 2 \\ \text{else} & \end{array} \right\} \cong H_i(\mathbb{CP}^2 \# \mathbb{CP}^2)$$

$H_2(S^2 \times S^2)$ gen by $a = [S^2 \times pt]$ and $b = [pt \times S^2]$



$H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ gen by $a = [\mathbb{CP}^1]$ and $b = [\mathbb{CP}^1]$



" intersection matrices "

$$\begin{array}{c|cc} & a & b \\ \hline a & 0 & 1 \\ b & 1 & 0 \end{array} \not\cong \begin{array}{c|cc} & a & b \\ \hline a & 1 & 0 \\ b & 0 & 1 \end{array}$$

$S^2 \times S^2$ $\qquad$ $\mathbb{CP}^2 \# \mathbb{CP}^2$

So $S^2 \times S^2 \not\cong \mathbb{CP}^2 \# \mathbb{CP}^2 \ldots$

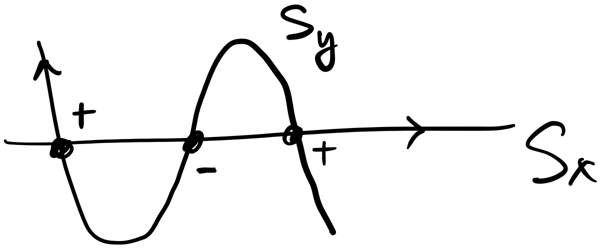
General Fact: Every $x \in H_2(M^4)$ is represented by an embedded surface $S_x \subset M$

intersection form $H_2(M) \times H_2(M) \longrightarrow \mathbb{Z}$

$$\langle x,y\rangle := S_x \cdot S_y$$

Properties:

symmetric, bilinear, unimodular on $H_2(M)/\text{torsion}$



These props best seen using equivalent formulation:

$$\langle -,- \rangle : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\langle \alpha,\beta\rangle := \underset{\substack{\uparrow\\ \text{cupprod.}}}{(\alpha\cup\beta)}\overset{\swarrow\ H^4,\, H_4 \text{ pairing}}{\ }\underset{\substack{\nwarrow\\ \text{fundamental class } \in H_4(M)}}{[M]}$$

Scorpan: "Think w/ intersections, prove w/ cup products"

(warm up w/ looking for examples ...)

Geography Question: Which integral sym. bilinear forms arise as intersection forms?

Recall (last time) $\overset{\text{indefinite}}{\text{unimodular}}$ $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}$

is one of $[+1]^p \oplus [-1]^q$ or $E_8^{\oplus n} \oplus H^{\oplus m}$

(Serre)

which are intersection forms?

$B_{\mathbb{C}P^2} = [+1]$ $\quad B_{\overline{\mathbb{C}P^2}} = [-1]$ $\quad B_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\equiv H.$

$B_{M_1 \# M_2} = B_{M_1} \oplus B_{M_2}$

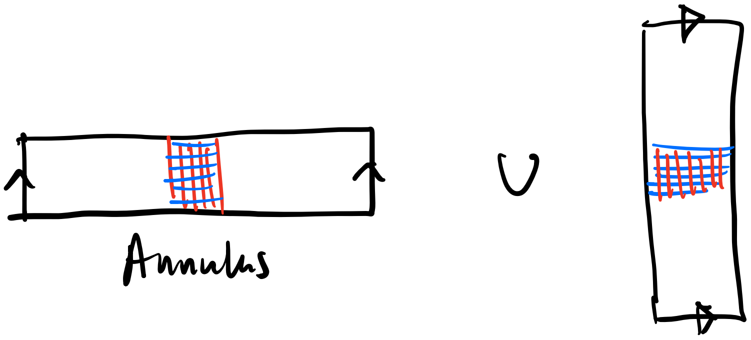
Thm $\exists$ closed simply connected topological 4-manifold M with $B_M = E_8$.

Warning M has not smoothable.

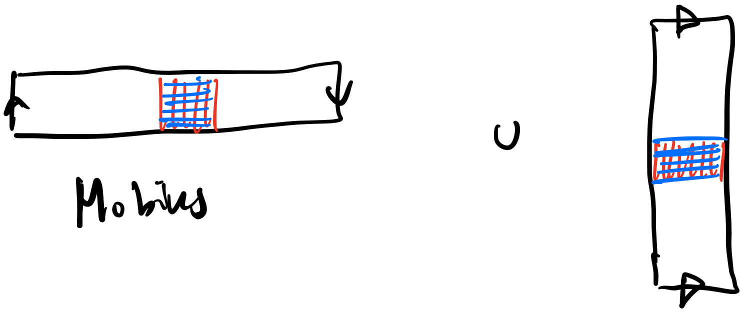
Thm (Donaldson) M smooth B_M definite

$\Rightarrow$ $B_M = [+1]^{\oplus p}$ or $B_M = [+1]^{\oplus q}$.

Key to (core) construction of E_8-manifold: plumbing



$= T^2 \setminus D^2$



$= K \setminus D^2$

(Klein bottle)

Annulus, Mobius $\longleftrightarrow$ D^1-bundles over S^1.

Build 4-manifolds by plumbing D^2-bundles over S^2
(for example)

Ex Plumb two copies of $S^2 \times D^2$.

$\rightsquigarrow$ N 4-manifold with boundary.

What is N? Some clues:

① $N \sim S^2 \vee S^2$. intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

② $\partial N = D^2 \times S^1 \cup S^1 \times D^2 = S^3$

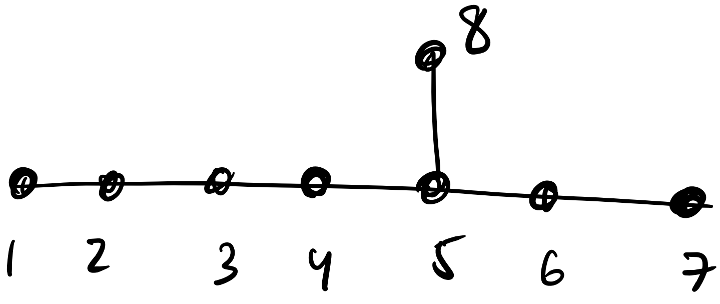
$\uparrow$ closed 3-manifold

Cap to get $S^2 \times S^2$. (usual cell structure)

so $N = S^2 \times S^2 \setminus D^4$.

(analogous to surface case: got $S^1 \times S^1 \setminus D^2$)

E_8 intersection form



vertices: self intersection 2

edges: intersect once.

$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

Plumb 8 copies of $T^{\leq 1}S^2$ unit disk bundle.

(next time).

Representing Homology by Submanifolds

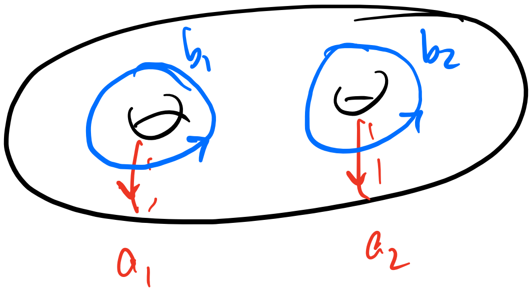
Thm (Thom) M^m sm. or. mfld. $x \in H_i(M;\mathbb{Z})$.

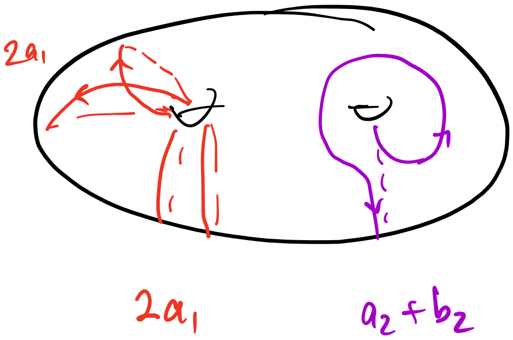
$\exists\ k \geq 1$ s.t. $kx = [N]$ where $N \hookrightarrow M$ embedded submanifold.

If $i \leq 8$ or $i = m-1, m-2$ then $x = [N]$ for some N...

Ex. $m=2$ $M=$ surface.

Any $x \in H_1(M)$ represented by a multicurve

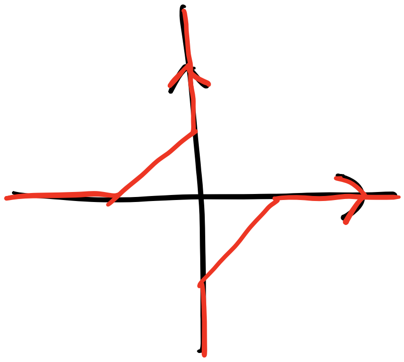




$\pi_1(M) \twoheadrightarrow H_1(M) \cong \pi_1(M)^{ab}$ (Hurewicz)

$\Rightarrow$ x rep'd by smooth immersed curve

surger intersections to get multicurve.

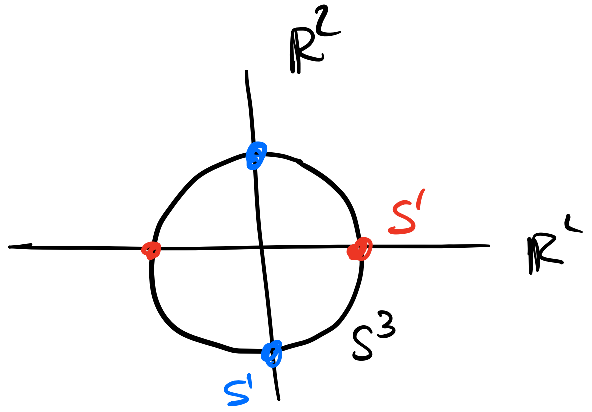


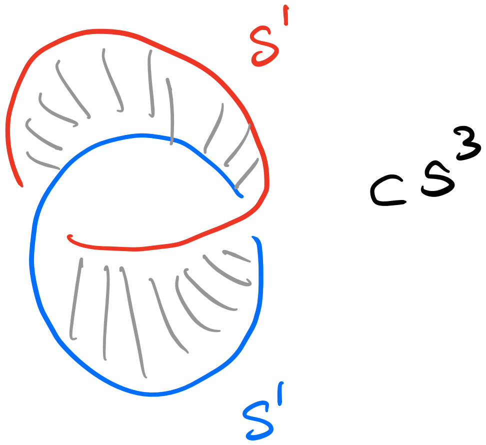
Same argument works for M^4 w/ $\pi_1(M)=0$.

$\pi_2(M) \xrightarrow{\cong} H_2(M)$ (Hurewicz)

Given immersion $S^2 \longrightarrow M^4$ self transverse

intersections look like $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{R}^4 \ldots$





Alternative: intersection locally is

$\{(z,w)\in \mathbb{C}^2 \mid zw=0\}$.

Replace w/ $\{(z,w)\in \mathbb{C}^2 \mid zw=\varepsilon\}$ ...

<u>Note</u> in 4d case resulting submfld is

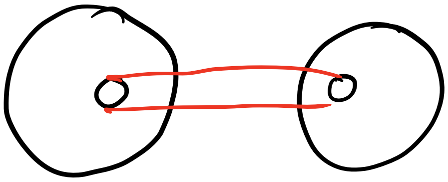
connected, but not nec. S^2.

Eg $M = \mathbb{CP}^2$ $\quad H_2(\mathbb{CP}^2) = \langle a \rangle \cong \mathbb{Z}$.

$a = [\mathbb{CP}^1]$. $\quad a \cdot a = 1$

$2a$

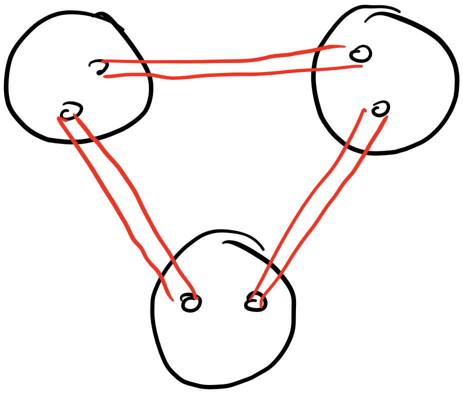
rep'd by.



$\cong S^2$

$3a$

rep'd by

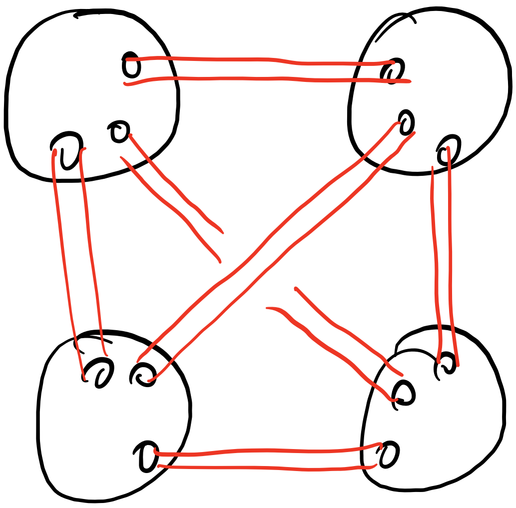


$\cong T^2$

generally $d \cdot a$ rep'd by surface of genus

$$g = \frac{(d-1)(d-2)}{2}$$ (degree-genus formula)

Can choose rep'l to be algebraic "plane curves in $\mathbb{CP}^2$"



$d=4 \Rightarrow g=3.$

Min genus rep.

tools: Gauge theory

& Seiberg-Witten theory

Representing Homology by submanifolds

M^{2n} closed oriented manifold

intersection form

$$H^n(M;\mathbb{Z}) \times H^n(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\langle \alpha_1, \alpha_2 \rangle = (\alpha_1 \cup \alpha_2)[M].$$

Exercise (see notes)

If $\alpha_i = \text{Poincare dual}(x_i)$ $\quad x_i \in H_n(M)$

and $x_i = [N_i]$ where $N_i \hookrightarrow M$ submanifold

then $\langle \alpha_1, \alpha_2 \rangle = N_1 \cdot N_2$

This allows us to think/reason geometrically.

Given $x \in H_k(M^n)$

Want: $N^k \xhookrightarrow{f} M^n$ embedded submfld

s.t. $f_*([N]) = x$.

Thm (Thom, 1950s) (stated incorrectly last time)

x is rep'd by a submanifold if

$\boxed{k \leq 6}^{\star}$ or $k = n-1, n-2$.

Ex For M^9 every $x \in H_k(M)$

rep'd by a submanifold $0 \leq k \leq 8$.

Rmk This is sharp:

E.g. $Sp(2) = \{ A \in GL_2(\mathbb{H}) \mid A^* A = 1 \}$

compact symplectic group, $\mathbb{H}$ = quaternions.

10 dimensional compact Lie group.

$$S^3 \cong Sp(1) \longrightarrow Sp(2) \longrightarrow S^7$$

analogous to $\quad S^1 \cong U(1) \longrightarrow U(2) \longrightarrow S^3$

and $\quad S^0 \cong O(1) \longrightarrow O(2) \longrightarrow S^1$

Compute $\quad H_k(Sp(2)) \cong \begin{cases} \mathbb{Z} & k=0,3,7,10 \\ 0 & \text{else} \end{cases}$

(exercise in Serre spectral sequence)

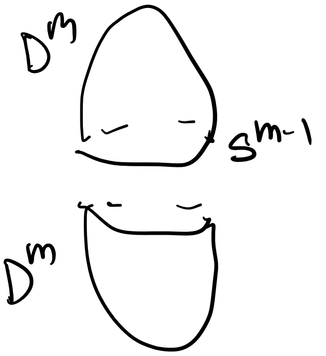
Thm (Bohr - Hanke - Kotschick 2001)

Generator $x \in H_7(Sp(2))$ is not represented by a submanifold.

Rmk This won't work for $S^3 \times S^7 \ldots$

so $S^3 \longrightarrow Sp(2) \longrightarrow S^7$ must be nontrivial.

A principle G-bundle over $S^m = D^m \cup D^m$



det. by htpy class of map $S^{m-1} \longrightarrow G$.

Here $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z} \neq 0$.

(Clutching)

Representing $x \in H_{n-1}(M^n)$

$$
\begin{aligned}
H_{n-1}(M) &\cong H^1(M;\mathbb{Z}) && \text{(Poincare duality)} \\
&\cong [M, K(\mathbb{Z},1)] && \text{(Brown representability)} \\
&= [M, S^1].
\end{aligned}
$$

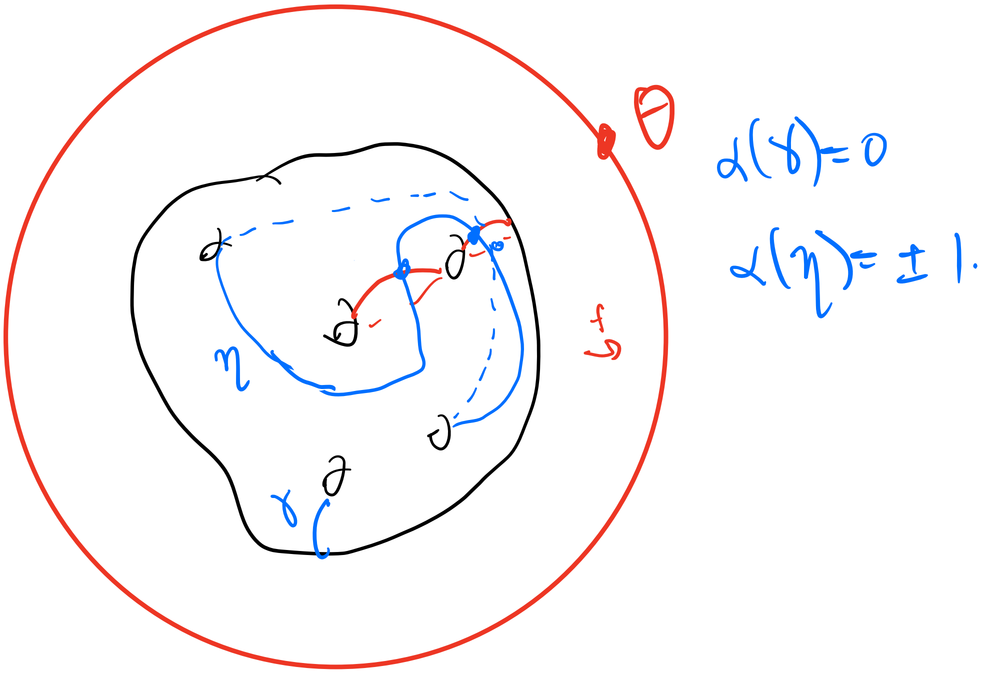
$$\begin{tikzcd}
f\colon M \to S^1 \arrow[r, squiggly] & \pi_1(M) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, squiggly] & \alpha \in H^1(M;\mathbb{Z}) \\
& H_1(M) \arrow[ur, dashed, "\alpha"'] & &
\end{tikzcd}$$

wlog f smooth. Fix reg. value $\theta \in S^1$

$N^{n-1} := f^{-1}(\theta) \hookrightarrow M^n$ submanifold.

For $\alpha : H_1(M) \to \mathbb{Z}$ $\alpha([\gamma]) = \gamma \cdot N$

$\Rightarrow$ $\alpha = PD([N])$.



Representing $x \in H_{n-2}(M^n)$

$$H_{n-2}(M) \cong H^2(M;\mathbb{Z}) \cong [M, K(\mathbb{Z},2)]$$
$$= [M, \mathbb{C}P^\infty]$$

$$\begin{tikzcd} f : M \arrow[r] & \mathbb{C}P^\infty & \rightsquigarrow & \mathbb{C} \arrow[r] & E \arrow[d] & \rightsquigarrow & c_1(E) \in H^2(M;\mathbb{Z}) \\ & & & & M & & \end{tikzcd}$$

For $\sigma : M \to E$ section transverse to 0-section

$N^{n-2} := \sigma(M) \cap M \hookrightarrow M$ emb. submfld.

$c_1(E) = PD([N])$.

($c_1(E)$ on a surface $\overset{\Sigma \subset M}{}$ evaluates obstruction to section. This is given by $\Sigma \cdot N$.)

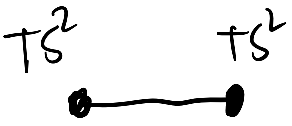
E_8 manifold.

<u>Thm</u> $\exists$ simply connected closed 4 mfld

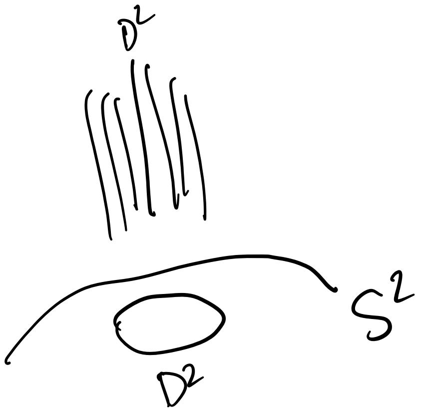
with intersection form $B_M = E_8$

<u>Key</u> Plumbing

Eg.



$$\begin{tikzcd} D^2 \arrow[r] & T^{\leq 1}S^2 \arrow[d] & \supset D^2\times D^2 \arrow[r, leftrightarrow] & D^2\times D^2 \subset & T^{\leq 1}S^2 \arrow[d] & D^2 \arrow[l] \\ & S^2 & (x,y) \arrow[r, leftrightarrow] & (y,x) & S^2 & \end{tikzcd}$$

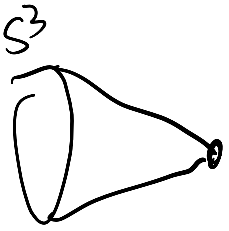


result is noncompact.

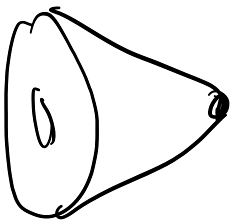
N with $B_N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\left(2 = \chi(S^2) \right)$

To obtain closed mfld want $\partial N = S^3$



cone $\cong D^4$



T^3 cone not a manifold!

want $H^i(M, M-x) \cong \begin{cases} \mathbb{Z} & i=4 \\ 0 & \text{else} \end{cases}$

Lemma N^4 simply conn. $\partial N \neq \phi$.
connected.

∂N homology sphere $\iff$ B_N unimodular

Proof bread + butter alg top.

LES of $(N, \partial N)$

$$\begin{tikzcd}
\cancelto{0}{H_3(N,\partial N)} \arrow[r] & H_2(\partial N) \arrow[r] & H_2(N) \arrow[r, "\phi"] \arrow[d, "{B_N(\cdot,-)}"'] & H_2(N,\partial N) \arrow[r] \arrow[d, "\cong"', "PD"] & H_1(\partial N) \arrow[r] & \cancelto{0}{H_1(N)} \\
 & & H_2(N)^* \arrow[r, "\cong", "UCT"'] & H^2(N) & &
\end{tikzcd}$$

$x \mapsto B_N(x, -)$

B unimodular $\iff$ $B_N(\cdot, -)$ iso

$\iff$ ϕ iso

$\iff$ $H_1(\partial N) \cong H_2(\partial N) = 0.$ $\square$

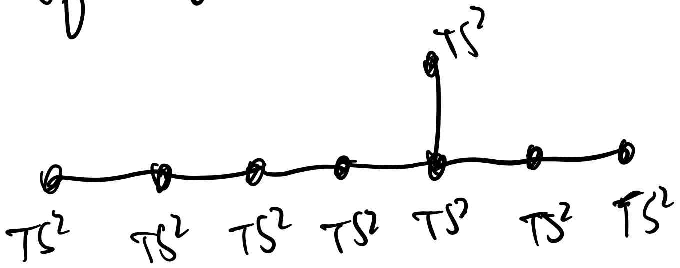
Thm (Freedman on fake 4-balls)

See Scorpan.

X homology 3-sphere. ∃ contractible topological 4-mfld Y with $\partial Y = X$.

Construction of E_8 manifold.

① Plumb



to get N w/

$$B_N = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \equiv E_8$$

② E_8 unimodular $\Rightarrow$ ∂N is homology 3-sphere

by lemma

③ Freedman $\Rightarrow$ $\exists$ contractible Y w/ $\partial Y = \partial N$.

$M := N \underset{\partial}{\cup} Y$

closed top. mfld $B_M = E_8$. $\square$

<u>Rmk</u> In fact ∂N is <u>NOT</u> S^3.

It's the Poincare Homology sphere!

(Poincare's counterex. to original Poincaré conj)

Alternate model: quotient of dodecahedron...

Intersection form & cobordism

M (smooth) closed oriented 4-manifold.

intersection form $B_M : H_2(M) \times H_2(M) \to \mathbb{Z}$.

The <u>signature</u> of M is defined as

$$\mathrm{sig}(M) := \mathrm{sig}(B_M). \quad \left(\begin{array}{c}\text{homotopy} \\ \text{invariant}\end{array}\right)$$

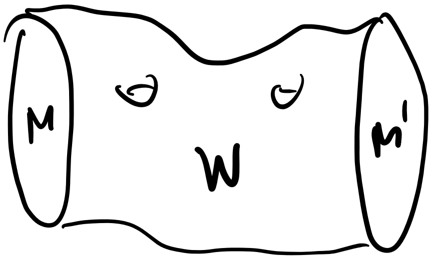
eg $\mathrm{sig}\left(\#_g S^2 \times S^2\right) \equiv \mathrm{sig}\left(H^{\oplus g}\right) = 0$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Sig}\left(\#_n \mathbb{C}P^2 \# \#_m \overline{\mathbb{C}P^2}\right) = \mathrm{sig}\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix} = n-m.$$

<u>Thm</u> (geometric significance of $\mathrm{sig}(M)$)

$\mathrm{Sig}(M) = \mathrm{sig}(M') \iff M$ & M' are <u>cobordant</u>



W^5 oriented manifold

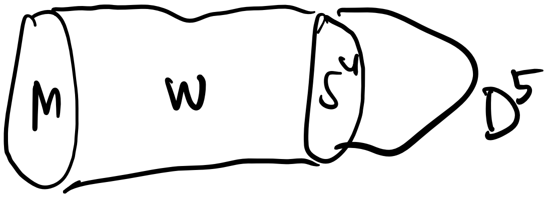
and

$\partial W \cong M' \sqcup \bar{M}$

(as oriented manifolds.
$\bar{M} = M$ w/ opposite orientation)

Ex $\text{sig}(M)=0 \iff M$ cobordant to S^4
(since $\text{sig}(S^4)=0$)

$\iff M$ bounds (an oriented 5-manifold)
$M=\partial W'$



$W'=W\cup D^5$

$W=W'\setminus D^5$

Cor $\mathbb{CP}^2$ doesn't bound. (impossible (!?) w/o algebra ...)

but $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ does bound.

Elementary argument:

$$\begin{tikzcd}
 & D^2 \arrow[d] & \\
\mathbb{CP}^2 \setminus D^4 \cong & N(\mathbb{CP}^1) \arrow[d] & \text{tubular nbhd} \\
 & \mathbb{CP}^1 \cong S^2 &
\end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}
\underbrace{D^2 \cup D^2}_{S^2} \arrow[r] & \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \arrow[d] \\
 & \mathbb{CP}^1 \cong S^2
\end{tikzcd}$$

Any
$$\begin{tikzcd}
S^2 \arrow[r] & M^4 \arrow[d] \\
 & S^2
\end{tikzcd}$$
bounds
$$\begin{tikzcd}
D^3 \arrow[r] & W^5 \arrow[d] \\
 & S^2
\end{tikzcd}$$

Rmk $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \not\cong S^2 \times S^2$ (Why?)

since $B_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \not\cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B_{S^2 \times S^2}.$

By clutching an oriented (linear) bundle $S^2 \to M \to S^2$

is determined by (homotopy class of)

map $S^1 \longrightarrow SO(3)$

$\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$ so $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

is the unique nontrivial S^2-bundle over S^2

Sometimes written $S^2 \tilde{\times} S^2$

<u>Quick argument</u> $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$

$$SO(3) \cong \mathbb{RP}^3 = D^3 / \pm 1 \text{ on } \partial D^3 = S^2$$

$\overset{\rotatebox{90}{$\in$}}{A}$ is CW rotation along axis $v \in S^2$ by angle $\theta \in [0, \pi]$

$$\Rightarrow SO(3) \cong S^2 \times [-\pi, \pi] / \sim$$

rotation by θ along $v = 0$ $\equiv$ id.

rotation by π at v = rotation by π at $-v$

Strategy for Thm

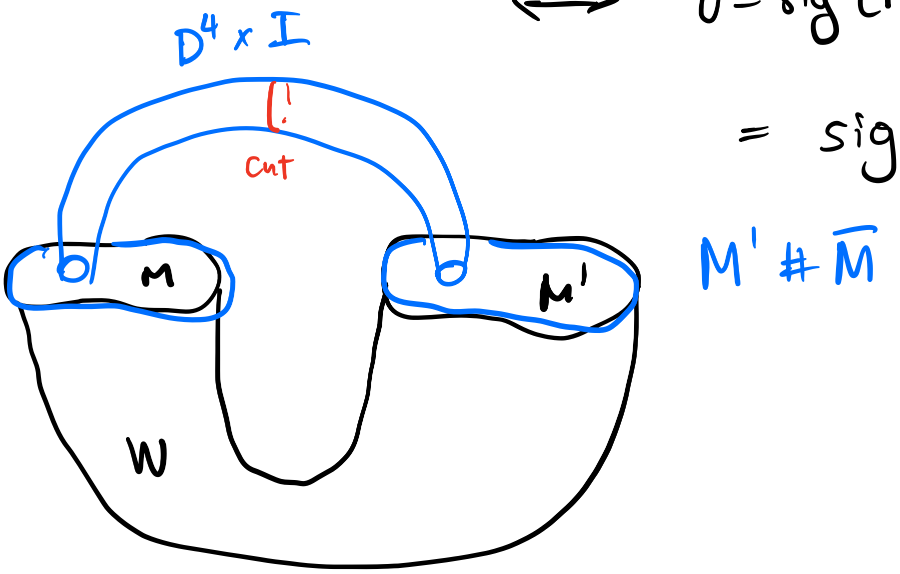
① Suffices to show $\operatorname{sig}(M) = 0 \iff M$ bounds.

Proof.

M cobordant to $M' \iff M' \# \overline{M}$ bounds

$\iff 0 = \operatorname{sig}(M' \# \overline{M})$

$= \operatorname{sig}(M') - \operatorname{sig}(M)$



② $M = \partial W \Rightarrow \mathrm{sig}(M) = 0$ (elementary)

③ $\mathrm{sig}(M) = 0 \Rightarrow M = \partial W$ (Rokhlin)

Proof of ② (③ next time)

Lemma 1 (half-lives, half-dies)

$M^{2k} = \partial W^{2k+1}$ oriented manifolds

- $\dim_{\mathbb{Q}} \ker \left[H_k(M) \longrightarrow H_k(W) \right] = \frac{1}{2} \dim_{\mathbb{Q}} H_k(M)$

($\mathbb{Q}$-coefficients)

- ker is isotropic wrt B_M.

Lemma 2 if $B : \mathbb{Q}^{2d} \times \mathbb{Q}^{2d} \to \mathbb{Q}$ nondeg. has d-dim'l isotropic subspace then $\mathrm{sig}(B) = 0$

Pf of lem 2 By assumption

$\exists\ u$ st. $B(u,u)=0$

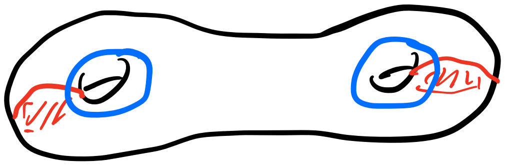
$\exists\ v$ st. $B(u,v)=1$.

$$B = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \oplus B' \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus B'$$

induct □

About lem 1:

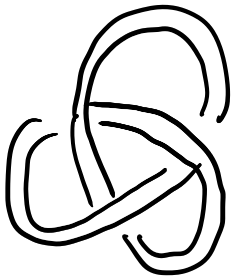
Ex.



M = genus g surface

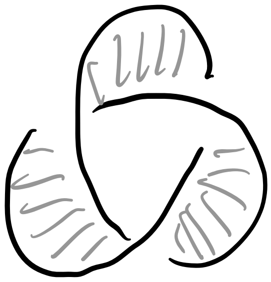
W = handlebody.

Ex



$W = S^3 \setminus N(K)$

$M \cong T^2$



$\exists !$ primitive $v \in H_1(T^2) \cong \mathbb{Z}^2$

that bounds in W.

Proof of Lem 1   w/ $\mathbb{Q}$-coeff

$$\begin{tikzcd}
{} \arrow[d, phantom, "\text{PD}"] & H_{k+1}(W,M) \arrow[r, "\partial"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H_k(M) \arrow[r, "i"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H_k(W) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
{} \arrow[d, phantom, "\text{UCT}"] & H^k(W) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H^k(M) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H^{k+1}(W,M) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
{} & H_k(W)^* \arrow[r, "i^*"] & H_k(M)^* \arrow[r, "\partial^*"] & H_{k+1}(W,M)^*
\end{tikzcd}$$

$$\begin{aligned}
\dim \ker(i) &= \dim \ker(\partial^*) \\
&= \dim \operatorname{im}(i^*) \\
&= \dim H_k(M) - \dim \ker(i)
\end{aligned}$$

(linear alg $T : U \to V$ $T^* : V^* \to U^*$

$\dim \ker T + \dim \operatorname{im} T^* = \dim U$)

ker(i) is isotropic: Fix $x_1, x_2 \in \ker(i) = \operatorname{im}(\partial)$

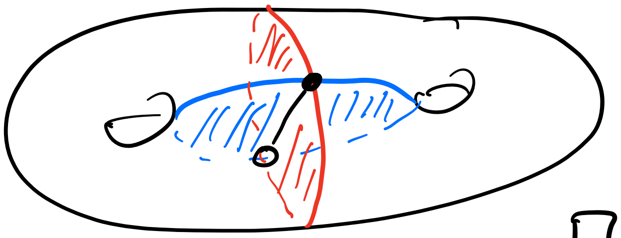
$x_i = \partial y_i$

$y_i = [N_i] \in H_{k+1}(W, M)$, $N_i \subset W$ submfld

$[\partial N_i] = x_i$ <u>WTS</u> $(\partial N_1) \cdot (\partial N_2) = 0.$

$N_1^{k+1} \cap N_2^{k+1} \subset W^{2k+1}$ 1-manifold (with ∂)

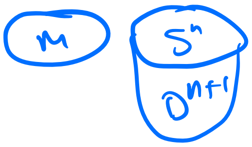
$\Rightarrow$ intersections of ∂N_1 & ∂N_2 occur in pairs w/ opposite signs.



$\square$

Cobordism groups

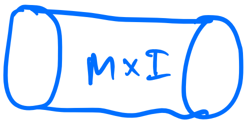
$$\Omega_n = \left\{ \begin{array}{l} n\text{-dim'l closed} \\ \text{oriented mflds} \end{array} \right\} \Big/ \begin{array}{l} \text{(oriented)} \\ \text{cobordism.} \end{array}$$



abelian group under $\sqcup$.

Identity: $[S^n]$ = mflds that bound. $[M \sqcup S^n] = [M]$

Inverses: $-[M] = [\bar{M}]$



Eg $\Omega_1 = 0$, $\Omega_2 = 0$

By Thm $\Omega_4 \cong \mathbb{Z}$ given by signature.
generated by $\mathbb{C}P^2$

Utility : every cobordism invariant
determined by value on $\mathbb{C}P^2$

Ex. $p_1 : \Omega_4 \longrightarrow \mathbb{Q}$

M^n closed or. mfld $\hookrightarrow \mathbb{R}^N$

$$\rightsquigarrow \quad M \xrightarrow{\varphi_M} Gr_n \mathbb{R}^N \subset Gr_n \mathbb{R}^\infty \sim BO(n)$$
$$x \longmapsto T_x M \subset \mathbb{R}^\infty$$

$$H^\bullet (Gr_n \mathbb{R}^\infty ; \mathbb{Q}) \cong \mathbb{Q}[p_1, \dots, p_{\lfloor n/2 \rfloor}]$$

$$p_1 : \Omega_4 \longrightarrow \mathbb{Q} \qquad M \longmapsto \varphi_M^*(p_1)[M]$$

well-defined :

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "i"'] & Gr_n \mathbb{R}^\infty \arrow[d] \\ W \arrow[r, "\varphi_W"'] & Gr_{n+1} \mathbb{R}^\infty \end{tikzcd}$$

if $M = \partial W$ then

$$\varphi_M^*(p_1)[M] = i^* \varphi_W^*(p_1)[M].$$

$$= \varphi_W^*(p_1)\big(\underbrace{i_*[M]}_{=0.}\big) = 0.$$

$\mathrm{sig}(\mathbb{C}P^2) = 1$, $p_1(\mathbb{C}P^2) = 3$

$\Omega_4 = \langle \mathbb{C}P^2 \rangle \Rightarrow$

$\mathrm{sig}(M) = \frac{1}{3} p_1(M)$ $\forall$ 4-manifolds M

special case of
(Hirzebruch signature theorem)

Next time: Finish proof of Thm

More on cobordism.

Signature & cobordism

Thm M, N closed oriented 4-manifolds

M, N cobordant $\iff$ $\text{sig}(M) = \text{sig}(N)$.

Last time

- equivalent: M bounds $\iff$ $\text{sig}(M) = 0$.
- M bounds $\Rightarrow$ $\text{sig}(M) = 0$.

Example K3 manifold.

(so far haven't seen very complicated 4-manifolds)

Definition 1 Smooth quartic in $\mathbb{C}P^3$

eg $\{ x^4 + y^4 + z^4 + w^4 = 0 \} \subset \mathbb{C}P^3$

K3 "surface" (quick but opaque)

Defn 2 $T^4 = S^1 \times \cdots \times S^1$ $\quad S^1 \subset \mathbb{C}$

$\circlearrowright_{\sigma}$ $\quad \sigma(x_1, \dots, x_4) = (\bar{x}_1, \dots, \bar{x}_4)$

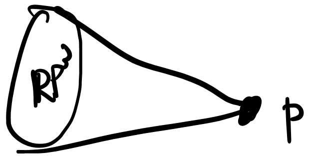
σ involution, 16 fixed points $(\pm 1, \dots, \pm 1)$

$X = T^4/\sigma$ orbifold

$p \in \mathrm{Fix}(\sigma)$ $\quad \sigma$ acts on $T_p(T^4)$ by $\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Nbhd of singular points $\cong$ Cone$(\mathbb{R}P^3)$

Last time: $\mathbb{R}P^3 \cong SO(3)$



Also $SO(3) \curvearrowright T^1 S^2$ simple transitive

$\Rightarrow SO(3) \cong T^1 S^2$

Remove Cone$(\mathbb{R}P^3)$ replace w/ $T^{\leq 1} S^2$

unit disk $_{\wedge}$ bundle.
cotangent

Get closed 4-manifold $\quad K = K3$ manifold.

Facts • K simply connected.

• $H_2(K;\mathbb{Q}) \cong \mathbb{Q}^{22}$ generated by

16 S^2's (zero section of disk bundles)

and $6 = \binom{4}{2}$ T^2's (coming from $H_2(T^4)$)

Intersection form on $H_2(K;\mathbb{Q})$ equivalent

to $[-2]^{\oplus 16} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$

$\Rightarrow$ $\operatorname{sig}(K) = 16$

By Thm K cobordant to $\#_{16} \mathbb{CP}^2$

(not obvious at all!)

Q: is $K \cong \#_{16} \mathbb{CP}^2$ diffeo?

A: No wrong intersection form.

Actually not obvious... $B_K \neq [-2]^{\oplus 16} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$

b/c not unimodular

(our basic is not a basis for $H_2(K;\mathbb{Z})$)

in fact

$$B_K \cong (-E_8)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}.$$

Thm (Rokhlin) $\mathrm{sig}(M)=0 \Rightarrow M$ bounds.

Idea of Proof

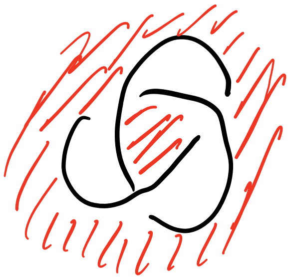
① (Immersion theory) Every M^4 (closed oriented) immerses in $\mathbb{R}^6$. $M^4 \looparrowright \mathbb{R}^6$

(strong Whitney immersion $\Rightarrow M^4 \looparrowright \mathbb{R}^7$ ...)

② (Thom) If oriented N^k embeds in $\mathbb{R}^{k+2}$ then N bounds oriented W^{k+1} in $\mathbb{R}^{k+2}$

(eg $k=1$: says Seifert surfaces exist)

Discuss general proof later



By ①+② suffices to show can improve

$M^4 \looparrowright \mathbb{R}^6$ to $M^4 \hookrightarrow \mathbb{R}^6$

There is an obstruction to doing this!

If $M \hookrightarrow \mathbb{R}^6$ then

$TM \oplus \nu M \cong T\mathbb{R}^6\big|_M \cong M \times \mathbb{R}^6$ as vector bundles

Furthermore νM is also trivial (See Kirby... Top 4mflds)

Recall Real vector bundles have Pontryagin characteristic classes.

$$H^\bullet(BO(6);\mathbb{Q}) \cong \mathbb{Q}[p_1,p_2,p_3] \quad p_i \in H^{4i}$$

characteristic class computation:

$$0 = p_1(M \times \mathbb{R}^6) = p_1(TM \oplus \nu_M)$$

$$= p_1(TM) + \cancel{p_1(\nu M)} = p_1(TM)$$

Conclude:

If $M^4 \hookrightarrow \mathbb{R}^6$, then $p_1(TM)=0$. $\Leftrightarrow$ $\mathrm{sig}(M)=0$

last time

$3\,\mathrm{sig}(M) = p_1[TM][M]$

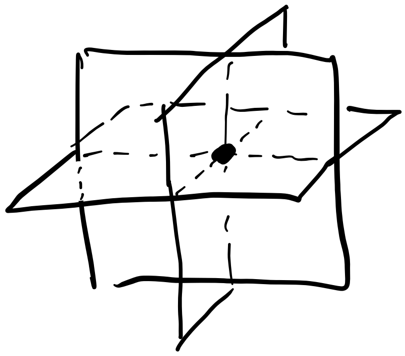
So if $\mathrm{sig}(M)=0$

have hope to have $M \hookrightarrow \mathbb{R}^6$ (won't quite realize)

What's the difficulty geometrically?

Given $M^4 \looparrowright \mathbb{R}^6$ generically $(M \pitchfork M)$

has finitely many triple points



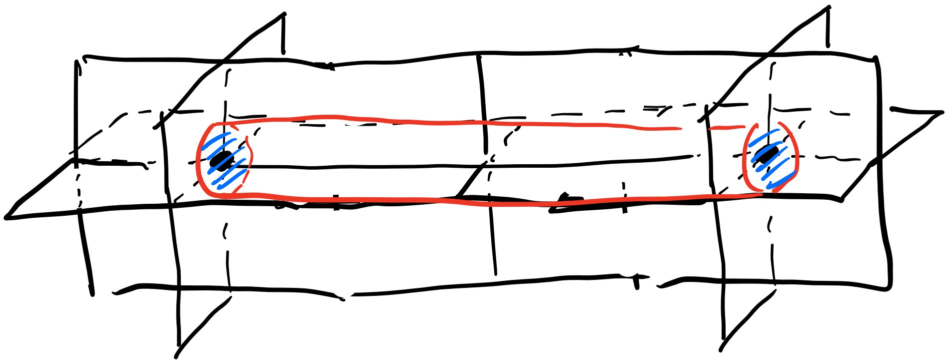
Amazing fact: for $M^4 \looparrowright \mathbb{R}^6$

sig(M) is signed count of triple points (!)

(Key connection algebra to geometry)

So sig(M) = 0 ⇒ triple points occur in ±1 pairs.

Plan: try to "cancel" ±1 pairs

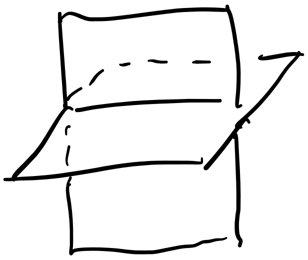


half-dim'l picture

Change M to cobordant manifold (obtained by surgery / handle attachment)

to get $M' \looparrowright \mathbb{R}^6$ w/ no triple points.

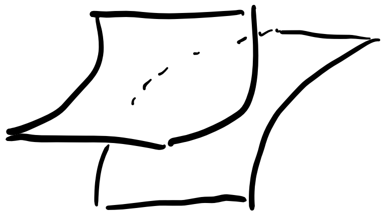
only see



locally at self-intersections

looks like $\mathbb{R}^4 \cap \mathbb{R}^4$ in $\mathbb{R}^6$

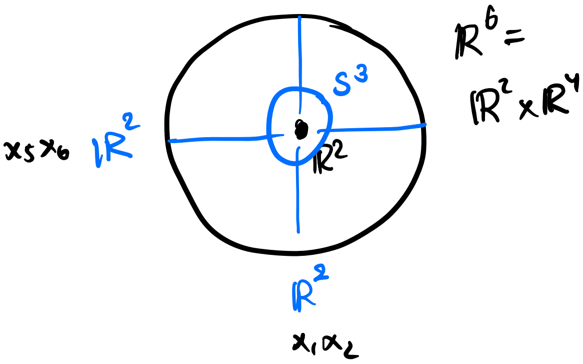
$\rightsquigarrow$



$x_1\, x_2\, x_3\, x_4$

$x_3\, x_4\, x_5\, x_6$

We've encountered this before!



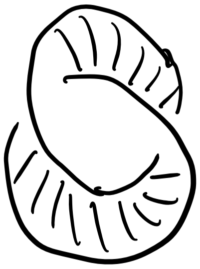
Conclude:

after replacing M by cobordant* manifold

can assume $M \hookrightarrow \mathbb{R}^6$ embedded. $\square$

$\circledast$ requires argument for the last step.

$D^2 \times S^0 \rightsquigarrow S^1 \times D^1$



More cobordism groups

$\Omega_n = \left\{ \begin{matrix} \text{closed oriented} \\ n\text{-manifolds} \end{matrix} \right\} / \text{cobordism}$ is a group.

$\left(\text{Actually can view } \Omega := \bigoplus_{n \geq 0} \Omega_n \text{ as a ring with } [M]\cdot[N] = [M \times N] \ldots \right)$

Thm (Thom) $\Omega_n = \pi_n(\text{something})$

Key is Pontryagin - Thom construction

Given $M^n \xhookrightarrow[\text{Whitney}]{} \mathbb{R}^{n+k} \subset \mathbb{R}^\infty$

$$\begin{tikzcd}
\mathbb{R}^k \arrow[d] & \mathbb{R}^k \arrow[d] \\
S^{n+k} \supset \nu M \arrow[r] \arrow[d] & \xi_k = \{(P, v) : P \in Gr_k \mathbb{R}^\infty, v \in P\} \arrow[d] \\
M \arrow[r] & Gr_k \mathbb{R}^\infty
\end{tikzcd}$$

universal/tautological bundle

Collapse map $f : S^{n+k} \longrightarrow (\nu M)^+ \longrightarrow (\xi_k)^+$

$(-)^+$ = 1 pt compactification (Thom space)

get element of $\pi_{n+k}(\xi_k^+)$

if instead $M \hookrightarrow \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$

get element of $\pi_{n+k+1}(\xi_{k+1}^+)$ ...

But $\xi_{k+1}^+ \simeq \Sigma \xi_k^+$ (suspension) and $S^{n+k+1} \longrightarrow \xi_{k+1}^{+}$ is the

$\quad\;\; \shortparallel \qquad\qquad\qquad \shortparallel$

$\Sigma(S^{n+k}) \qquad \Sigma(\xi_k^{+})$ suspension Σf

Get well-defined element $\operatorname{colim}_{k} \pi_{n+k}(\xi_k^{+})$

(could describe as homotopy group of of spectrum)

This process can be reversed (hint transversality)

Signatures of Knots

Previously

$$\Omega_4 = \left\{ \begin{matrix} \text{closed or.} \\ \text{4-mflds} \end{matrix} \right\} / \text{cobordism} \xrightarrow[\cong]{\text{sig}} \mathbb{Z}$$

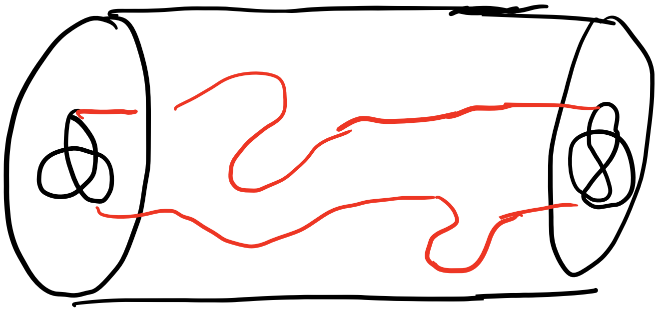
abelian group under $\sqcup$ or $\#$

Next

$$\mathcal{C} = \left\{ \begin{matrix} \text{or.} \\ \text{knots} \end{matrix} \right\} / \text{concordance} \overset{\text{signature(s)}}{\twoheadrightarrow} \mathbb{Z}^{\infty}$$

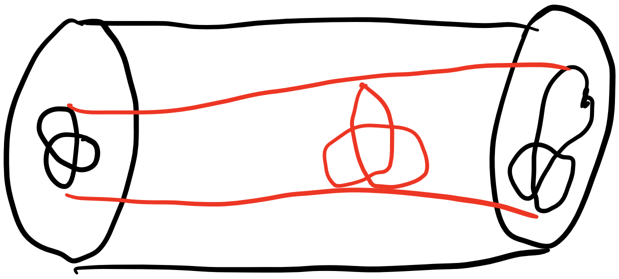
abelian group under $\#$

K, K' **concordant**



$[0,1] \times S^1 \hookrightarrow [0,1] \times S^3$

K, K' **isotopic**

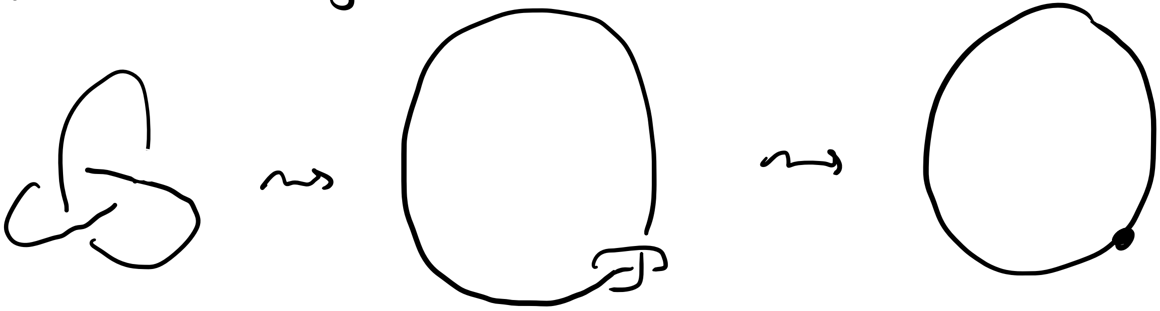


$[0,1] \times S^1 \hookrightarrow [0,1] \times S^3$

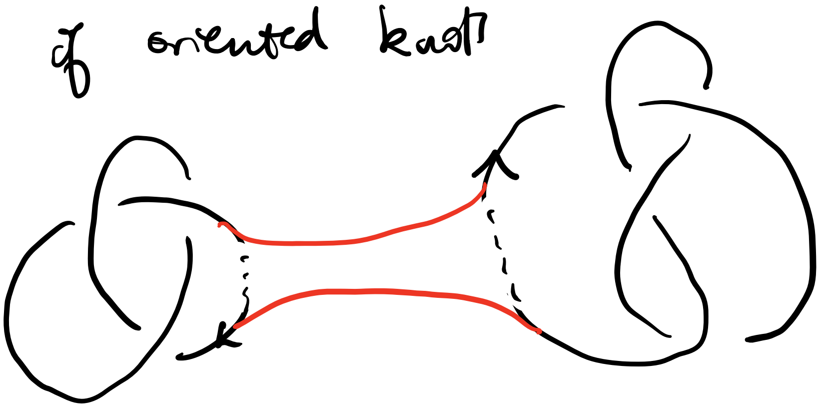
level-preserving

Here embedding is smooth (or at least locally flat).

Otherwise any two knots are isotopic



Connected sum of oriented knots



<u>Prop</u> $\mathcal{C} := \left(\left\{ \begin{matrix} \text{oriented} \\ \text{knots} \end{matrix} \right\} \Big/ \text{concordance} \, , \, \# \right)$

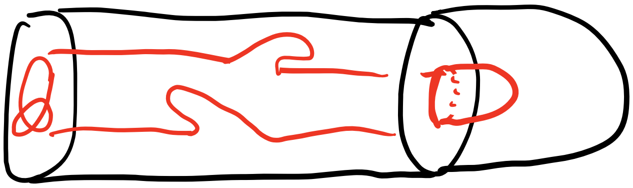
is an abelian group

"Knot concordance group"

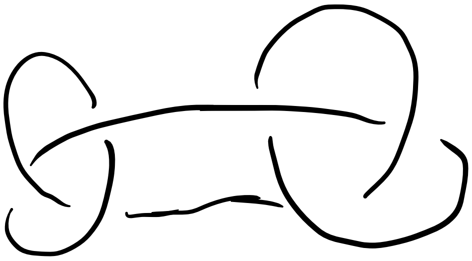
- identity = $[\text{unknot} = \bigcirc] =$

Knots that bound disk in D^4

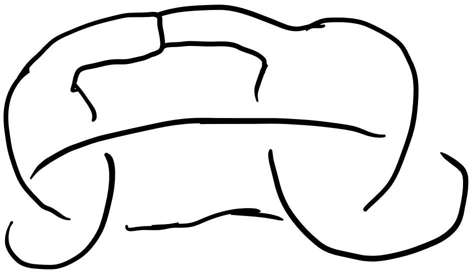
"slice knots"

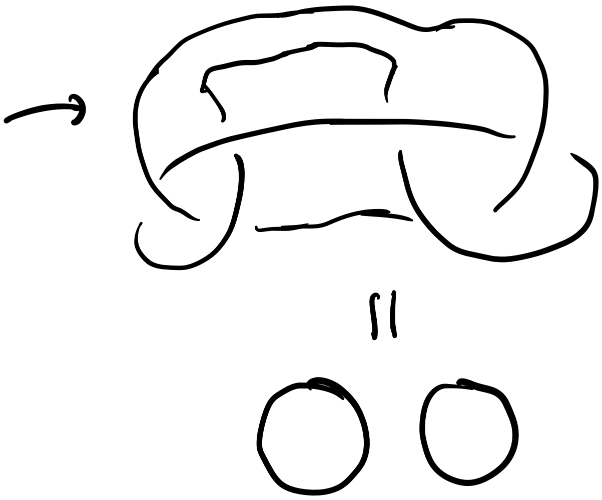


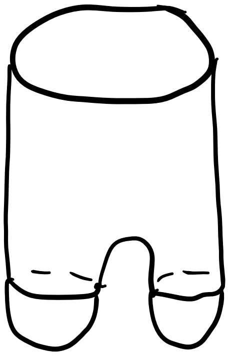
e.g.



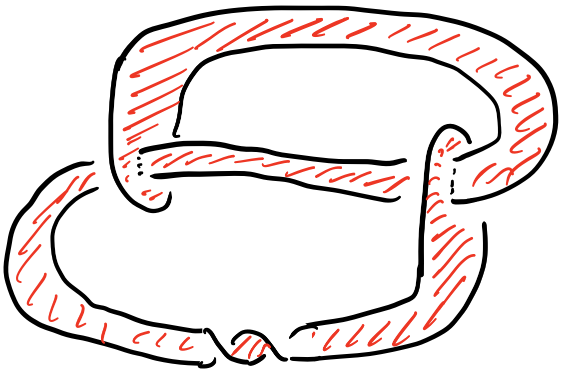
granny knot is slice





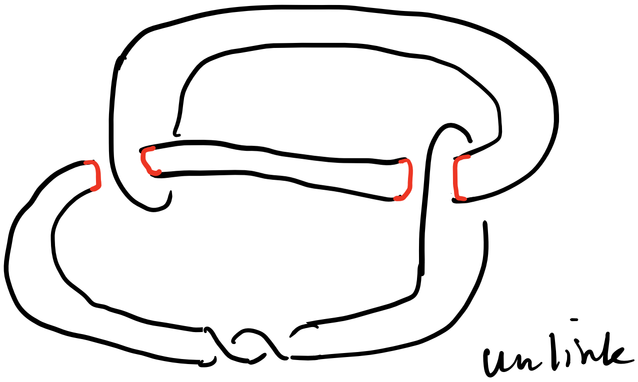


- eg ribbon knots are slice

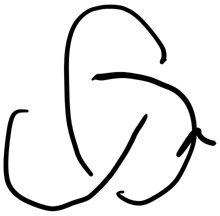


ribbon knot = knot that bounds immersed disk w/ "ribbon singularities"

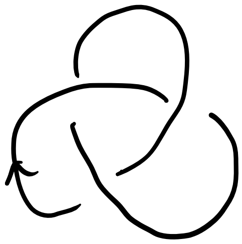
(perturb interior intersections into D^4 direction) or



- inverses

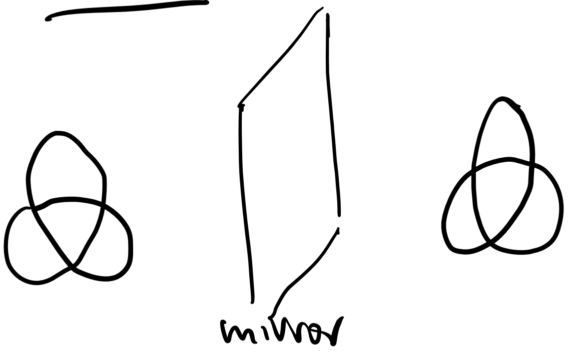


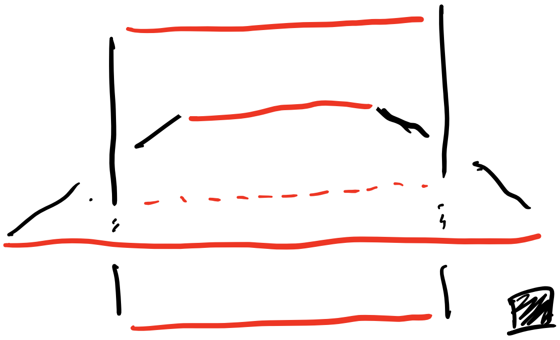
K



−K

exercise: K # −K always ribbon knot





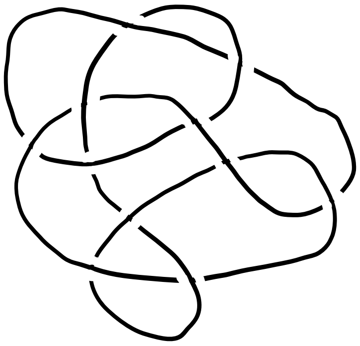
∎

Rmk $(\{\text{Knots}\}/\text{isotopy}, \#)$ doesn't have inverses.

Rmk sometimes $K, -K$ isotopic / concordant
⇝ 2-torsion in $\mathcal{C}$. amphichiral
open Q: is there 3-torsion? 5-torsion? …

Rmk Ribbon-Slice conjecture (Fox 1960s)
Every (smoothly) slice knot is ribbon.

Rmk Smooth vs topologically slice is different!



Piccirillo: Conway knot is not smoothly slice
(but it is topologically slice)

Knot signature

$K \subset \mathbb{R}^3$

① Seifert surface: $\exists$ oriented surface

$$F \subset \mathbb{R}^3 \qquad \partial F = K.$$

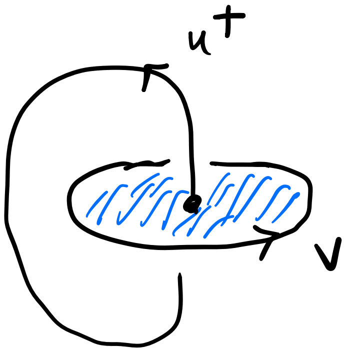
② Seifert bilinear form

$$\Sigma : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$\boxed{\text{prop}}$

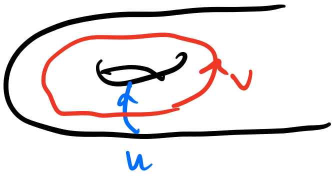
$$\Sigma(u,v) = \mathrm{Link}(u^+, v)$$

u^+ = push of u in pos. normal direction to F.



N.B. Σ is not symmetric!

$\mathrm{Lk}(u^+,v) = 1$



$$\Sigma(u,v) \neq \Sigma(v,u)$$

$B(u,v) = \Sigma(u,v) + \Sigma(v,u)$ symmetrization

$\mathrm{sig}(K) :=$ signature of B.

Thm ① $\mathrm{sig}(K)$ well-defined indep of choice of Seifert surface F.

② if K, K' concordant, then $\mathrm{sig}(K) = \mathrm{sig}(K')$.

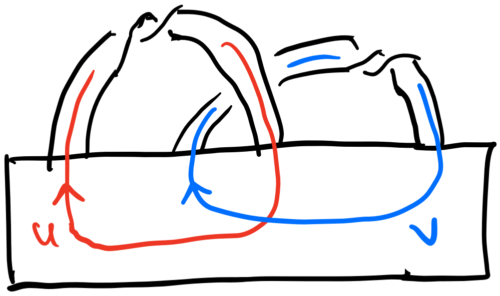
③ sig defines a homomorphism

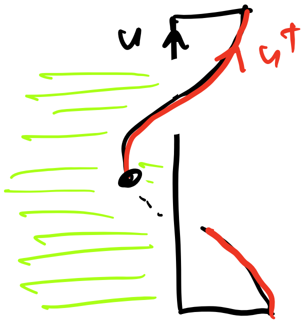
$$\mathrm{sig} : \mathcal{C} \longrightarrow \mathbb{Z}$$

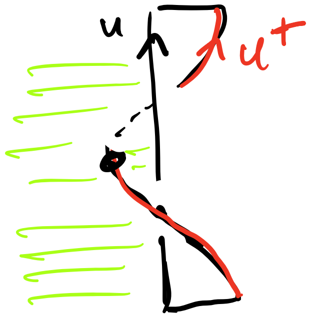
proofs next time

Example

$F \cong$







$Lk(u^+, u) = 1$

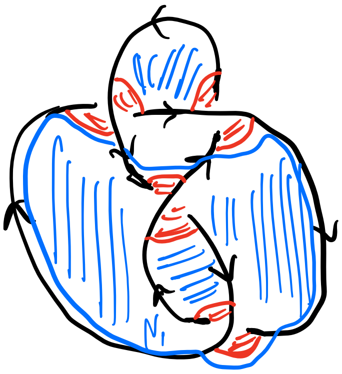
$Lk(u^+, u) = -1$

$$\Sigma = \begin{array}{c|cc} & u & v \\ \hline u & -1 & 1 \\ v & 0 & -1 \end{array} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} \lambda_1 \cdot \lambda_2 = \det > 0 \\ \lambda_1 + \lambda_2 = \operatorname{tr} < 0 \end{array}\right. \Rightarrow \operatorname{sig} = -2$$

(trefoil not slice)

Example



$F \cong$

$$\Sigma = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{matrix} \det < 0 \\ \operatorname{tr} = 0 \end{matrix} \Rightarrow \text{sig} = 0$$

Q: Is Fig8 slice?

A: No... unlike for 4mflds sig is not complete concordance invariant.

other signatures $\omega \in S^1$

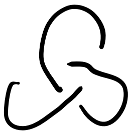
$$H_\omega := (1-\omega)\Sigma + (1-\bar{\omega})\Sigma^t$$

Hermitian form. $H_\omega^* = H\omega$.

$$\operatorname{sig}(K,\omega) := \operatorname{sig}(H_\omega)$$

unfortunately these also vanish for fig 8...

Fibering Trefoil Knot complement.

$K =$  (useful for understanding signature)

$$T^2 \setminus pt \longrightarrow S^3 \setminus K \longrightarrow S^1$$

Option 1 $K = S^3 \cap \{(z,\omega) \in \mathbb{C}^2 \mid z^2 + \omega^3 = 0\}$ (why?)

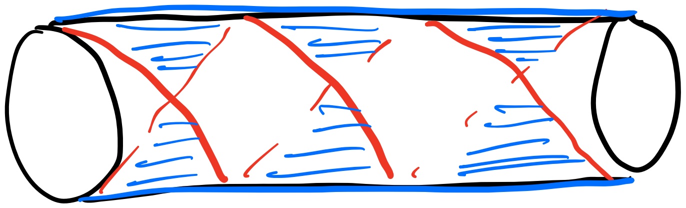
$$S^3 \setminus K \longrightarrow S^1$$

$$(z,w) \longmapsto \frac{z^2+w^3}{|z^2+w^3|}$$

gives fibration. (How to tell what fiber is?)

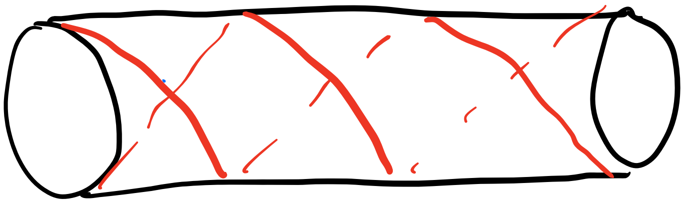
Option 2

$S^3 = S^1 \times D^2 \cup D^2 \times S^1$



Construct family F_θ $\theta \in [0,2\pi]$

each $F_\theta \cong T^2 \setminus D^2$ with $\partial F_\theta = K$.



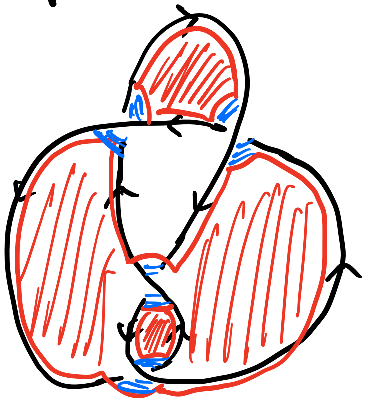
Seifert Surfaces

- Seifert's algorithm

Input: Knot K (planar diagram)

Output: Oriented surface $F \hookrightarrow \mathbb{R}^3$ with $\partial F = K$

By example:



① pick orientation

② create Seifert cycles

③ connect by twisted bands

$$\text{Genus of } F = \frac{1 + \#\text{crossings} - \#\text{Seifert cycles}}{2}$$

eg for fig 8 $\quad$ genus $= \dfrac{1+4-3}{2} = 1$

- Knot genus

$$g(K) := \min\{\text{genus}(F) \mid F \text{ Seifert surface}\}$$

Examples

① $g(K)=0 \iff K$ bounds D^2 in $\mathbb{R}^3$

$\iff K$ unknot

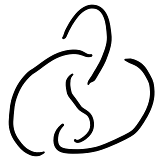
② $g\Big($  $\Big) \leq 1$ by above

fig 8 $\neq$ unknot $\Rightarrow$ genus $= 1$.

③ (Haken) Algorithm to compute $g(K)$ using "normal surface theory"

gives algorithm to recognize unknot

but its run time is exponential in

crossings (impractical)

Prop $g(K \# K') = g(K) + g(K')$

Cor If $K \# K' =$ unknot then

$K = K' =$ unknot

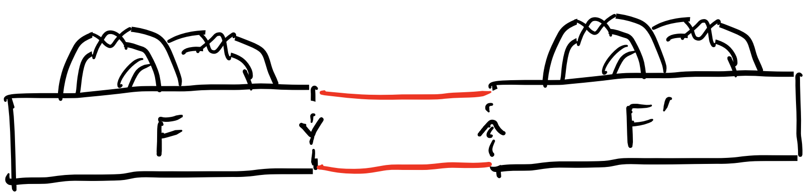
Pf of Cor $0 = g(\text{unknot}) = g(K \# K')$

$= g(K) + g(K')$

$\Rightarrow g(K) = g(K') = 0 \Rightarrow K = K' =$ unknot $\square$

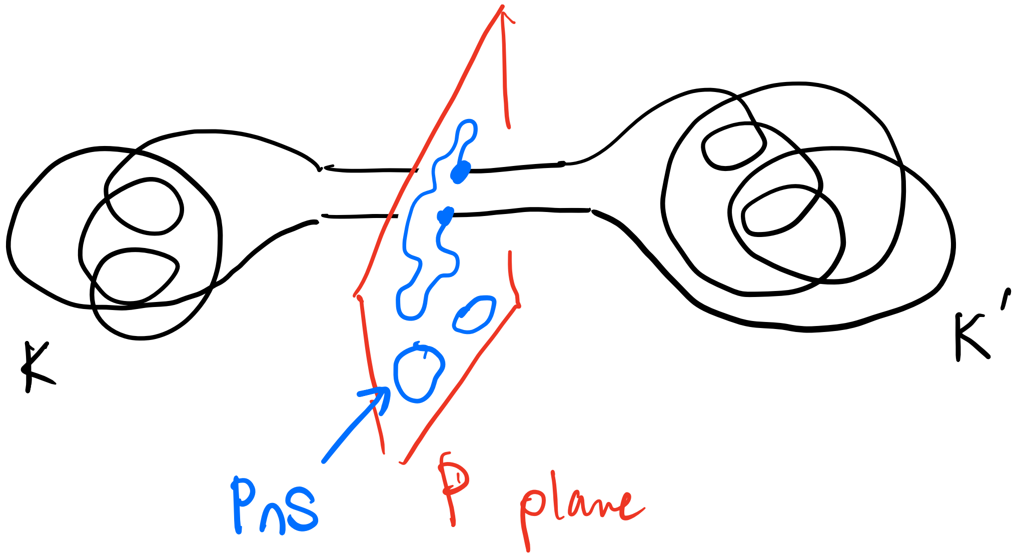
Proof of Prop

- $g(K \# K') \leq g(K) + g(K')$:



Seifert surf for $K \# K'$ of genus $g(K) + g(K')$

- $g(K) + g(K') \leq g(K \# K')$



S = min genus Seifert surface for $K \# K'$

$S \cap P$ = 1-mfld with $\partial \subset K$.

After isotopy $P \cap S$ = single arc

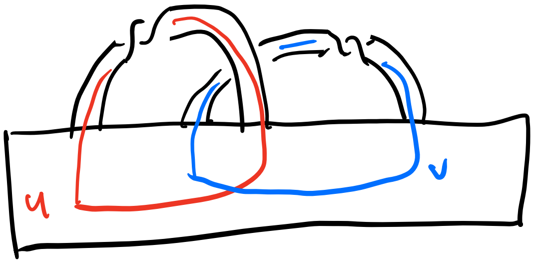
(innermost disk argument)

$\Rightarrow$ $S = F \natural F'$ boundary connected sum of Seifert surfaces for K, K'

$\Rightarrow g(K) + g(K') \leq g(F) + g(F') = g(S) = g(K \# K')$

$\square$

Knot Signature



$K = \partial F$

Seifert asymmetric form

$$\Sigma : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$$\Sigma(u,v) = \mathrm{Link}(u^+, v) \quad \text{(on basis)}$$

$$\Sigma = \begin{array}{c|cc} & u & v \\ u & -1 & 0 \\ v & 1 & 1 \end{array} \qquad B := \Sigma + \Sigma^t = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\mathrm{sig}(K) := \mathrm{sig}(B).$

Thm K slice $\Rightarrow$ $\mathrm{sig}(K) = 0$

Recall K slice if $K \subset S^3$ bounds embedded D^2 in D^4

$\Leftrightarrow$ concordant to unknot $\Leftrightarrow$ $[K] = 0 \in \mathcal{C}$

Cor $\operatorname{sig}(K)$ well defined

Pf of Cor $K = \partial E$, $K = \partial F$ two Seifert surfaces

$E \natural \overline{F}$ Seifert surface for $K \# \overline{K}$

$K \# \overline{K}$ slice (last time) $\xRightarrow{\text{Thm}}$ $\operatorname{sig}(K \# \overline{K}) = 0$

$$0 = \operatorname{sig}(K \# \overline{K}) = \operatorname{sig}\left(B_{K \# \overline{K}}\right)$$

$$= \operatorname{sig}(B_K) + \operatorname{sig}(B_{\overline{K}})$$

$$= \operatorname{sig}(B_K) - \operatorname{sig}(B_K) \qquad \square$$

Pf of Thm

① (Thom) $n \geq 1$ $X^n \subset \mathbb{R}^{n+2}$ or. submfld

$\Rightarrow \exists$ $Y^{n+1} \subset \mathbb{R}^{n+2}$ or submfld $\partial Y = X$

- $m=1$: Seifert surfaces exist
- $m=4$: used for Rokhlin

  $(\operatorname{sig}(M^4)=0 \Rightarrow M^4=\partial W^5)$
- $m=2$: use now

② K slice $K=\partial D^2$ $(D^2,\partial D^2)\hookrightarrow(D^4,S^3)$

Choose Seifert surface $F\subset S^3$

$\overline{F} := F \underset{K}{\cup} D^2$ closed or. surface in $\begin{array}{c} D^4 \\ \cap \\ \mathbb{R}^4 \end{array}$

Thom $\Rightarrow$ $\overline{F}$ bounds a 3-mfld $M\subset\mathbb{R}^4$

Half-lives, half-dies:

$\operatorname{Ker}\left[H_1(\overline{F}) \longrightarrow H_1(M)\right]$ is $\frac{1}{2}$-dim'l

symplectic-isotropic subspace.

$\Rightarrow$ ker is Σ-isotropic (exercise)

$$\Rightarrow \quad B = \begin{pmatrix} 0 & A \\ A & C \end{pmatrix} \Rightarrow \operatorname{sig}(B) = 0 \qquad \square$$

Knot signatures, canonical

Special case: Assume K fibers

$$\begin{tikzcd} F \arrow[r] & S^3 \setminus K \arrow[r] & S^1 \\ & F \times \mathbb{R} \circlearrowright \mathbb{Z} = \langle T \rangle \arrow[u, "\text{reg cover}"] & \end{tikzcd}$$

$T(x,t) = (\phi(x), t+1) \qquad \phi \in \mathrm{Homeo}(F)$ monodromy

$B(u,v) := \langle \phi_*(u), v \rangle - \langle u, \phi_*(v) \rangle$ Symmetric

$\phi_* : H_1(F) \to H_1(F)$

$\langle \cdot , \cdot \rangle : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$ sympl. int. form.

$\mathrm{sig}(K) := \mathrm{sig}(B)$

For general $K \subset S^3$

$$H_1(S^3 \setminus K) \cong \mathbb{Z} \quad \text{(Alexander duality).}$$

so have (canonical) $\mathbb{Z}$-cover

$$\underset{\langle T \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \circlearrowright X \longrightarrow S^3 \setminus n(K)$$

X is "homology surface"

$$H_i(X, \partial X; R) \cong \begin{cases} R & i = 0,2 \\ R^{2g} & i = 1 \\ 0 & \text{else} \end{cases}$$

and there's a sympl. form

$$\langle \cdot, \cdot \rangle : H_1(X, \partial X; R) \times H_1(X, \partial X; R) \to R$$

$$B(u,v) = \langle T_*(u), v \rangle - \langle u, T_*(v) \rangle.$$

$\mathrm{sig}(K) := \mathrm{sig}(B)$ requires no choice.

Bubble method

for Seifert surfaces

More Knot Signatures

$\mathcal{C} = \left\{ \substack{\text{oriented} \\ \text{Knots}} \right\} \Big/ \text{concordance}$

Thm $\mathcal{C}$ is not finitely generated.

In fact $\exists \quad \mathcal{C} \twoheadrightarrow \mathbb{Z}^{\infty}$

Previously:

- defined $\sigma : \mathcal{C} \to \mathbb{Z}$

$K \subset S^3 \rightsquigarrow \underset{\text{Seifert surface}}{F} \rightsquigarrow \overset{\text{Seifert form}}{\Sigma} : H_1(F) \times H_1(F) \to \mathbb{Z}$

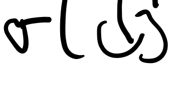
$\Sigma(u,v) = \operatorname{link}(u^+, v)$

$\rightsquigarrow B = \Sigma + \Sigma^t , \quad \sigma(K) := \operatorname{sig}(B)$

- $\sigma(K)$ indep of choices, and is a concordance invariant

- note $\operatorname{Im}(\sigma) = 2\mathbb{Z}$

$\sigma(K) \in 2\mathbb{Z}$ since $\dim H_1(F)$ even

$\sigma($  $) = -2$

Formal process to obtain more signatures:

For $\omega \in S^1 \subset \mathbb{C}$ define Hermitian form

$$B_\omega = (1-\omega)\Sigma + (1-\bar{\omega})\Sigma^t \qquad B_\omega^* = B_\omega$$

$$\sigma_\omega(K) := \operatorname{sig}(B_\omega)$$

also a concordance invar

$$\sigma_\omega : \mathcal{C} \longrightarrow \mathbb{Z}$$

<u>Claim</u> $\{\sigma_\omega \mid \omega = e^{2\pi i/m}, m \geqslant 2\}$ ~~are LI~~

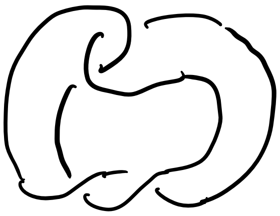
contains ∞ LI subset

To see this consider twist knots

$K_n =$

n full twists.

$K_1 =$



trefoil

$K_{-1} =$

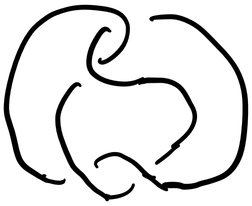


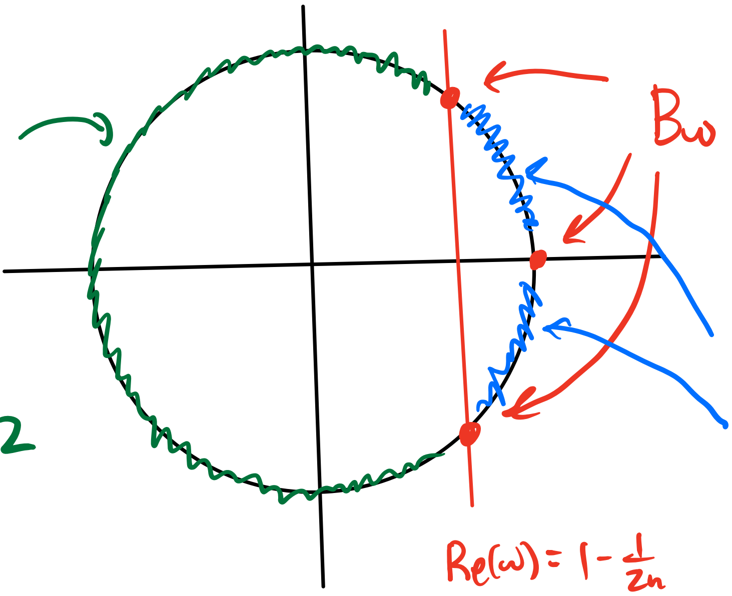
figure 8

Check K_n has Seifert form

$$\Sigma = \begin{pmatrix} -1 & 1 \\ 0 & -n \end{pmatrix}$$ wrt Seifert surface from Seifert's algorithm

$$B_\omega = \begin{pmatrix} \omega+\bar{\omega}-2 & 1-\omega \\ 1-\bar{\omega} & (\omega+\bar{\omega}-2)\cdot n \end{pmatrix}$$

Signatures $\sigma_\omega(K_n)$ $(n>0)$



det $B_\omega > 0$

tr $B_\omega < 0$

$\Rightarrow$

$\sigma_\omega(K) = -2$

B_ω singular

det$(B_\omega) < 0$

$\Rightarrow \sigma_\omega(K) = 0$

$Re(\omega) = 1 - \frac{1}{2n}$

Fix $\omega = e^{2\pi i/m}$, vary $n > 0$. Find

$$\sigma_\omega(K_n) = 0 \quad \text{for} \quad n < \frac{1}{2-\omega-\bar{\omega}} = \frac{1}{2\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}$$

$$\left[\begin{array}{l} B_{\omega} = \begin{pmatrix} \omega+\bar{\omega}-2 & 1-\omega \\ 1-\bar{\omega} & (\omega+\bar{\omega}-2)\cdot n \end{pmatrix} \\ q = \omega+\bar{\omega}-2 \equiv 2\left[\mathrm{Re}(\omega)-1\right] = 2\left[\cos\left(\frac{2\pi}{m}\right)-1\right] \\ \det B_{\omega} = q^2 n + q < 0 \iff n < \frac{-1}{q}. \end{array}\right]$$

inductively build LI set

	σ_{ω_2}	σ_{ω_7}	σ_{ω_9}	$\sigma_{\omega_{11}}$	
K_1	$*$	0	0	0	
K_2	$*$	$*$	0	0	$\cdots$
K_3	$*$	$*$	$*$	0	
K_4	$*$	$*$	$*$	$*$	
	$\vdots$			$\vdots$	

$\square$

Signatures & Conjugacy in $Sp_{2n}(\mathbb{R})$

Last time if $K \subset S^3$ fibers

$$\begin{tikzcd} & F \times \mathbb{R} \arrow[d] \arrow[loop right, "\mathbb{Z} = \langle T \rangle"] & \\ F \arrow[r] & S^3 \setminus K \arrow[r] & S^1 \end{tikzcd}$$

$T(x,t) = (\phi(x), t+1)$

$\phi \in \mathrm{Homeo}(F)$

$$\beta : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$$\beta(u,v) = \langle \phi_*(u), v \rangle - \langle u, \phi_*(v) \rangle$$

$\sigma(K) = \mathrm{sig}(\beta)$.

$\langle \cdot, \cdot \rangle$ = intersection form

$(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle)$ symplectic vector space

Given $A \in Sp_{2n}(\mathbb{R})$ consider

$$\beta_A : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

$$\beta_A(u,v) = \langle Au, v \rangle - \langle u, Av \rangle.$$

symmetric

Exercise sig(β_A) invariant under conjugacy in $Sp_{2n}(\mathbb{R})$.

Ex $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ $A' = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

conjugate in $GL_2(\mathbb{R})$ but not in $\underbrace{SL_2(\mathbb{R})}_{\text{preserves orientation}} \equiv Sp_2(\mathbb{R})$

$$\beta_A = A^tJ - JA = \begin{pmatrix} -2\sin t & 0 \\ 0 & -2\sin t \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\beta_{A'} = \begin{pmatrix} 2\sin t & 0 \\ 0 & 2\sin t \end{pmatrix}$$

$$\text{sig}(\beta_A) = -2 \neq 2 = \text{sig}(\beta_{A'})$$

$\Rightarrow$ A, A' not conjugate.

This invariant doesn't help distinguish

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \& \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

if $t \neq \theta$ in $[0,\pi)$.

<u>ω-signatures</u> Define $H : \mathbb{C}^{2g} \times \mathbb{C}^{2g} \to \mathbb{C}$

$H(u,v) = i\langle u, \bar{v}\rangle$ Hermitian form

$$\left[H(v,u) = i\langle v, \bar{u}\rangle = -i\langle \bar{u}, v\rangle = \overline{i\langle u, \bar{v}\rangle} = \overline{H(u,v)} \right]$$

For $A \in Sp_{2n}(\mathbb{R})$ and $\omega \in \mathbb{C}$ consider

$$E_\omega = \bigcup_{k\geq 1} \ker\left[(A - \omega I)^k \right]$$ characteristic subspace.

$\mathrm{sig}\left(H\big|_{E_\omega} \right)$ conj. invar. of A.

called the <u>ω-signature</u> of A.

$J \rightsquigarrow H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\qquad$ $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

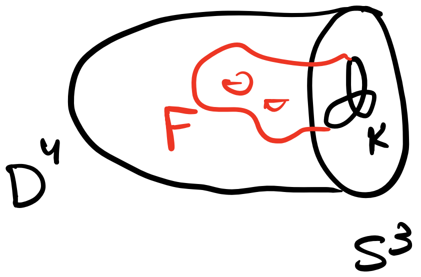
$\omega = e^{it}$ $\qquad$ $E_\omega = \mathbb{C}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$ $\qquad$ $\operatorname{sig}(H|_{E_\omega}) = -1$

$\omega = e^{-it}$ $\qquad$ $E_\omega = \mathbb{C}\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$ $\qquad$ $\operatorname{sig}(H|_{E_\omega}) = +1.$

all other ω-sig's vanish.

3rd definition of $\sigma(K)$ and signature additivity thm

(Kauffman-Taylor)



$M :=$ double cover of D^4 branched over F

$$\sigma(K) := \underbrace{\operatorname{sig}(M)}_{\text{4-mfld signature}}$$

Key to showing this is well-defined:

<u>Thm</u> (Novikov additivity)

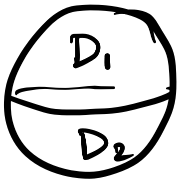
$M^4 = M_1 \cup_N M_2$ where $\partial M_1 = N = \partial M_2$

then $\text{sig}(M) = \text{sig}(M_1) + \text{sig}(M_2)$.

<u>Ex</u> It's important to "glue along full boundary."

eg $M = \mathbb{D}S^2$ unit disk bundle

$$\begin{tikzcd} M = \mathbb{D}S^2 \arrow[d, "\pi"] \\ S^2 = D_1 \cup D_2 \end{tikzcd}$$



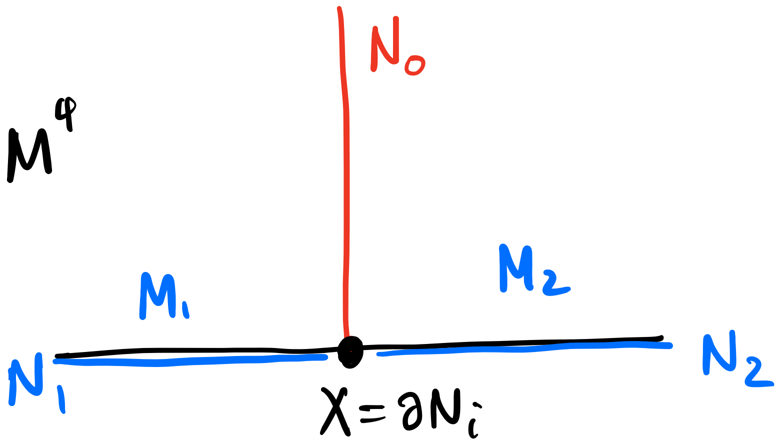
$M = M_1 \cup M_2$ $\quad M_i := \pi^{-1}(D_i) \cong D_i \times \mathbb{D}$

$\cong D^4$

$\text{sig}(M) = -2 \quad \text{sig}(M_i) = 0$.

Here M_1, M_2 glued along subsets of ∂ ...

Thm (Wall nonadditivity)



$$L_i := \ker\left[H_2(X) \longrightarrow H_2(N_i) \right]$$

Lagrangian (half dim'l, $\langle \cdot , \cdot \rangle$ isotropic)

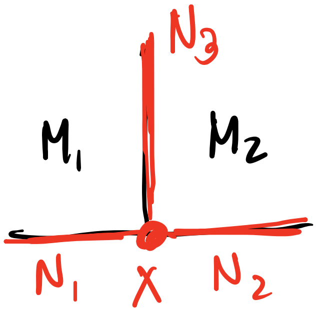
$$\mathrm{Sig}(M) = \mathrm{sig}(M_1) + \mathrm{sig}(M_2) + \underbrace{\mu(L_0, L_1, L_2)}_{\substack{\text{Maslov index} \\ \text{(symplectic invariant)}}}$$

Connect signatures to symplectic geometry.

Maslov index

Motivation: Wall nonadditivity

$M^4 =$



$$L_i = \ker\left[H_1(X;\mathbb{R}) \to H_1(N_i;\mathbb{R})\right]$$

$$\operatorname{sig}(M) = \operatorname{sig}(M_1) + \operatorname{sig}(M_2) + \underbrace{\mu(L_1, L_2, L_3)}_{\text{Maslov index}}$$

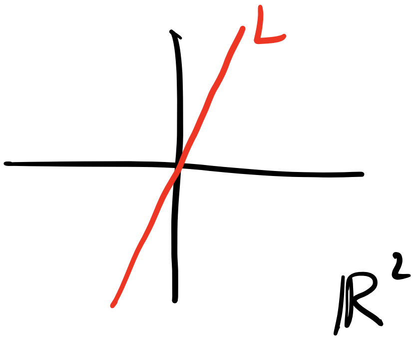
$(\mathbb{R}^{2n}, \omega)$ symplectic vector space

$$\omega(x,y) = x^t J y \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Lagrangian Grassmannian

$$\Lambda_n := \left\{ L \subset \mathbb{R}^{2n} \;\middle|\; \begin{array}{l} \dim L = n \\ \omega(x,y) = 0 \ \forall x,y \in L \end{array} \right\} \subset \mathrm{Gr}_n \mathbb{R}^{2n}$$

Ex $\Lambda_1 = \mathbb{RP}^1 \cong S^1$



In general

$$\Lambda_n \cong Sp_{2n}(\mathbb{R}) / GL_n(\mathbb{R}) \cong U(n)/O(n)$$

and there is a fibration

$$\frac{SU(n)}{SO(n)} \longrightarrow \frac{U(n)}{O(n)} \longrightarrow S^1 \quad \left(\text{Proof later}\right)$$

In particular $S^2 \longrightarrow \Lambda_2 \longrightarrow S^1$

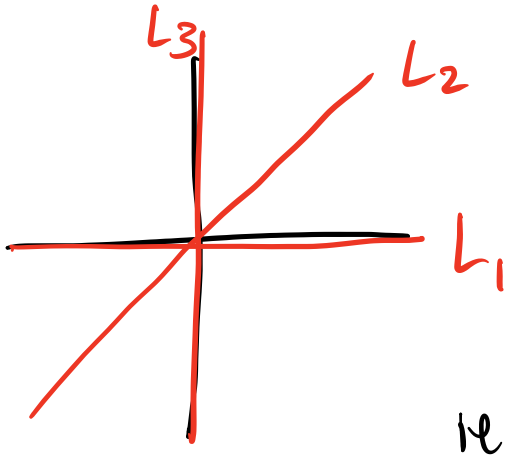
unique nontrivial bundle. (nonorientable 3 mfld)

Defining $\mu : \Lambda_n \times \Lambda_n \times \Lambda_n \longrightarrow \mathbb{Z}$

Special case Suppose

L_1, L_2, L_3 transverse $L_i \cap L_j = \{0\}$

so $R^{2n} = L_i \oplus L_j$.



Can describe L_2 as graph of (unique) linear $f : L_1 \longrightarrow L_3$.

ie $L_2 = \{ x + f(x) : x \in L_1 \}$.

(f inj hence iso since $L_1 \cap L_2 = \{0\}$)

Define $\beta : L_1 \times L_1 \longrightarrow \mathbb{R}$

$$\beta(x,y) = \omega(x, f(y))$$

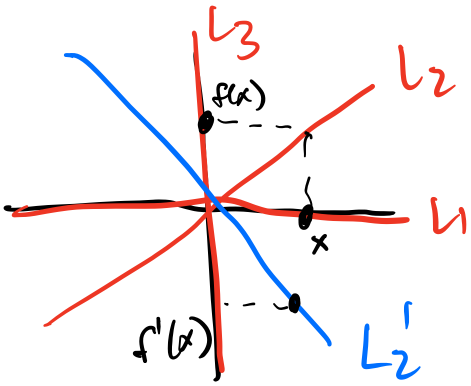
Claim β symmetric.

$$0 \underset{\substack{\uparrow \\ L_2 \text{ Lagrangian}}}{=} \omega(x+f(x), y+f(y)) = \overbrace{\cancel{\omega(x,y)} + \cancel{\omega(f(x),f(y))}}^{L_1,\, L_3 \text{ Lagrangian}}$$
$$+ \omega(x, f(y)) + \omega(f(x), y)$$
$$= \beta(x,y) - \beta(y,x) \checkmark$$

β is nondegenerate b/c $\omega : L_1 \times L_3 \longrightarrow \mathbb{R}$ is nondegen. and f is an iso.

Define $\mu(L_1, L_2, L_3) = \mathrm{sig}(\beta)$.

Ex $(n=1)$



- L_1, L_2, L_3

  $\beta(x,x) > 0 \implies$
  $\mu(L_1, L_2, L_3) = 1$

- L_1, L_2', L_3

  $\beta(x,x) < 0 \implies$
  $\mu(L_1, L_2, L_3) = -1$

As long as L_1, L_2 transverse to L_3 can repeat. $\exists\ f: L_1 \to L_3$ (not nec. iso) s.t. $L_2 = \mathrm{graph}(f) \subset L_1 \oplus L_3 \cong \mathbb{R}^{2n}$

Define β as above and $\mu(L_1, L_2, L_3) = \mathrm{sig}(\beta)$.

Note β may be degenerate but that's okay.

diagonalize $\begin{pmatrix} +I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$ $\mathrm{sig}(\beta) = p-q$.

General definition

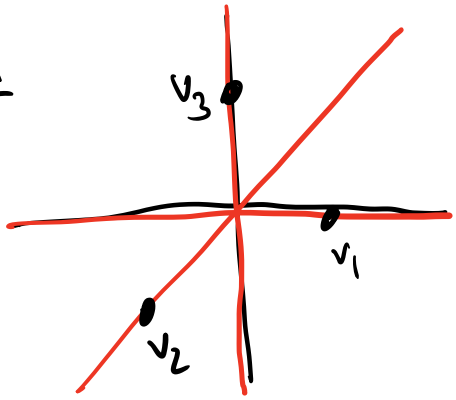
$$V = \{ (v_1, v_2, v_3) \in L_1 \oplus L_2 \oplus L_3 \mid v_1 + v_2 + v_3 = 0 \in \mathbb{R}^{2n} \}$$

$$Q(v_1, v_2, v_3) = \omega(v_1, v_2) \underset{\uparrow}{=} \omega(v_2, v_3) = \omega(v_3, v_1)$$

$$(v_1 = -v_2 - v_3)$$

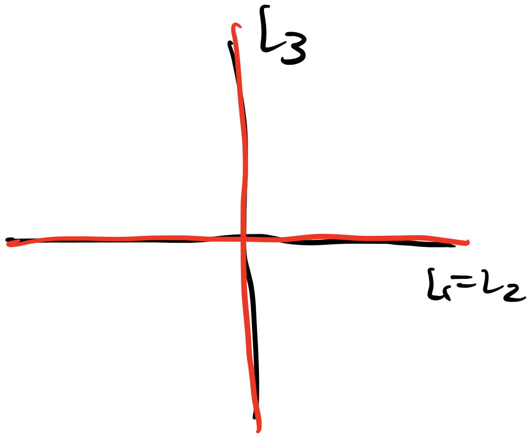
quadratic form. $\mu(L_1, L_2, L_3) := \mathrm{sig}(Q)$.

Ex



V 1-dim'l

$Q(v_1, v_2, v_3) < 0$.

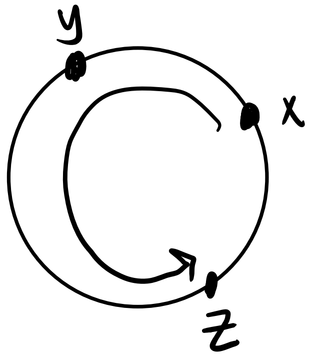


V 2-dim'l

$Q \equiv 0$.

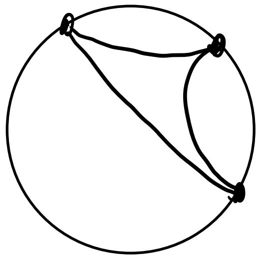
Here $\mu : \Lambda_1 \times \Lambda_1 \times \Lambda_1 \longrightarrow \mathbb{Z}$

agrees w/ the "order cocycle" on $S^1 \cong \Lambda_1$



$$\text{ord}(x,y,z) = \begin{cases} +1 & xyz \text{ ccw} \\ -1 & xyz \text{ cw} \\ 0 & x,y,z \text{ not distinct} \end{cases}$$

Viewing $S^1 = \partial \mathbb{H}^2$

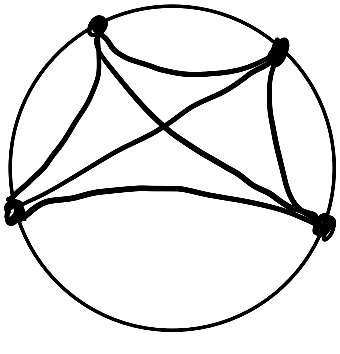


$\text{ord}(x,y,z) = \frac{1}{\pi} \overset{\text{signed}}{\text{area}}(\text{triangle spanned by } x,y,z)$

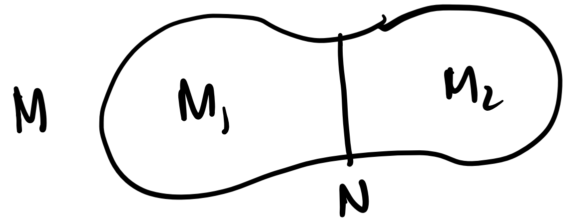
<u>Key Property</u> μ is a cocycle

Given L_1, L_2, L_3, L_4

$\sum (-1)^i \mu(L_1, \ldots, \hat{L_i}, \ldots, L_4) = 0$



Novikov additivity



$\mathrm{sig}(M) = \mathrm{sig}(M_1) + \mathrm{sig}(M_2)$

Warm up Consider Mayer-Vietoris sequence

$$H_2(N) \xrightarrow{(i_1, -i_2)} H_2(M_1) \oplus H_2(M_2) \xrightarrow{j_1 + j_2} H_2(M) \xrightarrow{\partial} H_1(N)$$

If $H_1(N) = H_2(N) = 0$ then

$$H_2(M) \cong H_2(M_1) \oplus H_2(M_2)$$ orthogonal decomp

$\Rightarrow \mathrm{sig}(M) = \mathrm{sig}(M_1) + \mathrm{sig}(M_2)$.

eg if $N = S^3$ (connected sum)

In general

$$H_2(M) \cong \frac{H_2(M_1) \oplus H_2(M_2)}{Im(i_1, -i_2)} \oplus Im(\partial)$$

$$\ldots \cong \frac{H_2(N)}{ker(i_1) + ker(i_2)} \oplus \frac{H_2(M_1)}{Im(i_1)} \oplus \frac{H_2(M_2)}{Im(i_2)} \oplus Im(\partial)$$

<u>Observe</u>

- $Im(i_j)$ degenerate $\Rightarrow$ $sig(M_j) = sig\left(\frac{H_2(M_j)}{Im(i_j)}\right)$

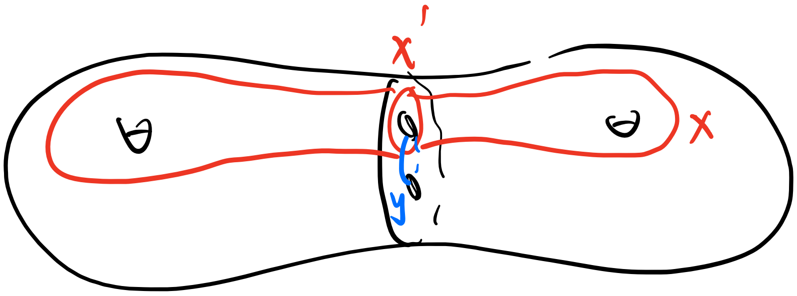
- middle summands orthogonal but not all summands orthogonal.

- WTS can split off copies of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from 1st / last summands.

illustration: Fix $x' = \partial(x)$ N closed, or.

$\Rightarrow \exists\ y \in H_2(N)$ s.t. $x' \cdot y = 1$.

int. form on span(x,y) is $\begin{array}{c|cc} & x & y \\ \hline x & ? & 1 \\ y & 1 & 0 \end{array}$

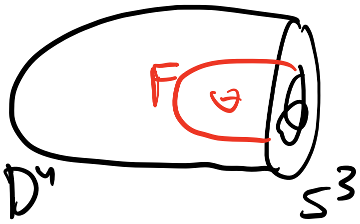


More generally fix basis for $\mathrm{Im}(\partial)$ and take dual basis for $H_2(N)/\ker(i_1)+\ker(i_2)$

on which form is $\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus k}$ ooo $\square$

Application (signature of knots)

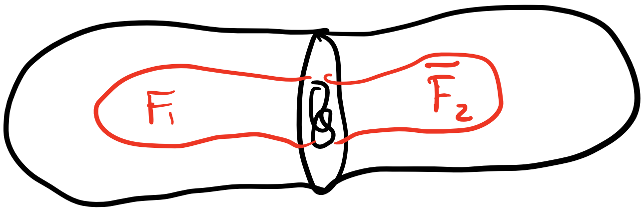
$\sigma(K)$ definition 3:



$M =$ double cover of D^4 branched over F

$\sigma(K) := \mathrm{sig}(M)$

Claim $\sigma(K)$ well defined indep of choice of F.



$$D^4 \cup_{S^3} D^4 = S^4$$

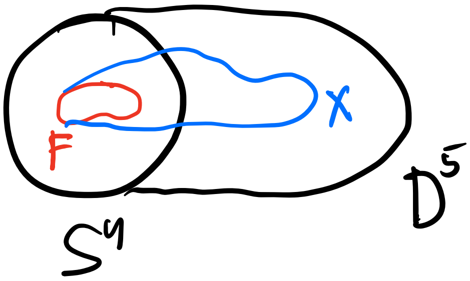
$M :=$ double cover of S^4 branched over $F_1 \cup \overline{F_2}$.

$M = M_1 \underset{N}{\cup} \overline{M_2}$ $N =$ double cover of S^3 branched over K,

Novikov $\operatorname{sig}(M) = \operatorname{sig}(M_1) - \operatorname{sig}(M_2)$

WTS $\operatorname{sig}(M) = 0$.

Equivalently WTS M bounds. (dim 4)



Thom: $\exists\ X^3 \subset S^4$ with $\partial X = F$.

$W^5 =$ double cover of D^5 branched over X.

By construction $\partial W = M$ $\square$.

Last time $(\mathbb{R}^{2n}, \omega)$ symplectic v-space. Λ_n = Lagrangian Grassmannian

Maslov index $\mu : (\Lambda_n)^{\times 3} \longrightarrow \mathbb{Z}$

$Q : V \longrightarrow \mathbb{R}$

$\mu(L_1, L_2, L_3) = \mathrm{sig}(Q)$

$Q(v_1, v_2, v_3) = \omega(v_1, v_2)$

$V(L_1, L_2, L_3) = \{(v_1, v_2, v_3) \in L_1 \oplus L_2 \oplus L_3 \mid \sum v_i = 0 \text{ in } \mathbb{R}^{2n}\}$

If L_1, L_2 transverse to L_3

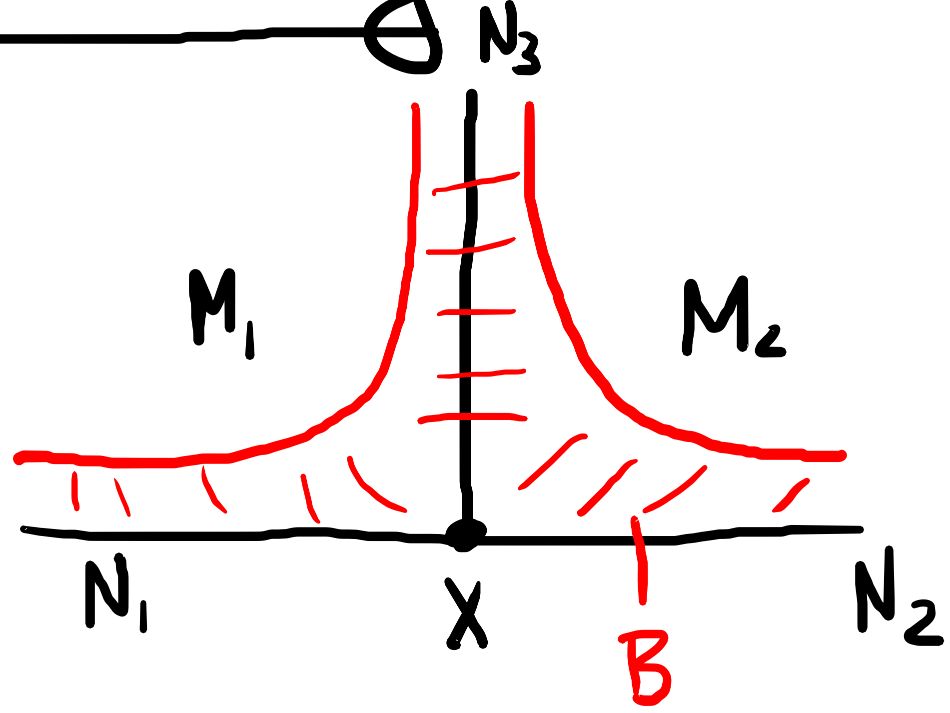
write $(v_1, v_2, v_3) = (v_1, -(v_1' + v_3'), v_3)$ $L_2 = \mathrm{graph}(f : L_1 \longrightarrow L_3)$

$L_1 \xrightarrow{\cong} V$ $Q(v_1) = \omega(v_1, f(v_1))$ quadratic form on L_1

$v_1 \longmapsto (v_1, -(v_1 + f(v_1)), f(v_1))$

Wall nonadditivity

$M^4 =$



$$L_i = \operatorname{Ker}\left[H_1(X;\mathbb{R}) \longrightarrow H_1(N_i;\mathbb{R})\right]$$

$$\operatorname{sig}(M) = \operatorname{sig}(M_1) + \operatorname{sig}(M_2) + \mu(L_1, L_2, L_3)$$

Proof idea ① $Z :=$ Thickening of $N_1 \cup N_2 \cup N_3$

Novikov: $\operatorname{sig}(M) = \operatorname{sig}(M_1 \cup M_2) + \operatorname{sig}(Z)$.

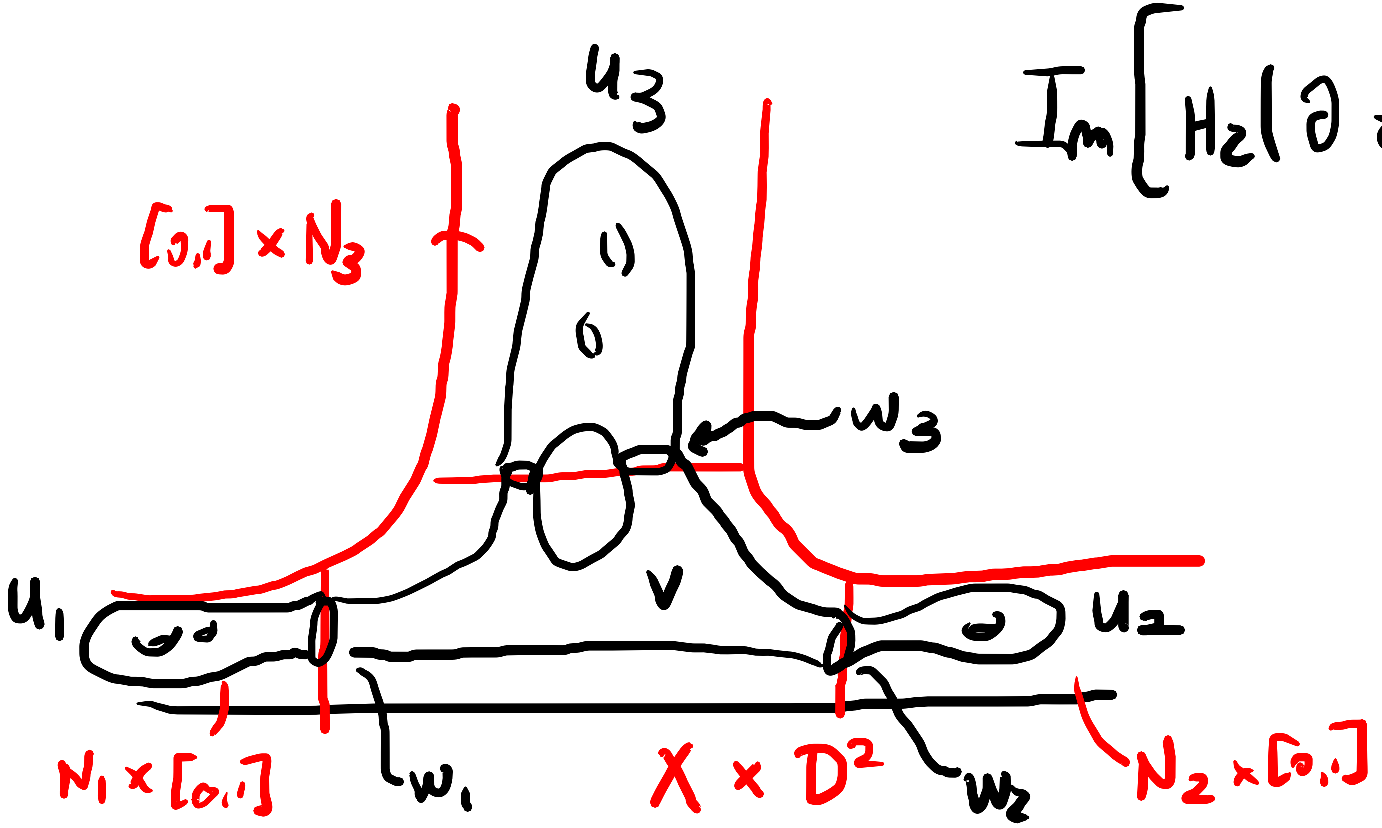
Rmk Here M has ∂.

Key is that $(M_1 \cup M_2) \cup Z$ glued along closed manifold.

(think back to proof...)

② $\mathrm{sig}(Z) = \mu(L_1, L_2, L_3)$ (explain idea/convention)

$\mathrm{Im}\left[H_2(\partial Z) \longrightarrow H_2(Z)\right]$ isotropic.



WT understand int. form on cycles $u \in H_2(Z)$ where

$$u = u_1 + u_2 + u_3 + v$$

$$\partial v = -(\partial u_1 + \partial u_2 + \partial u_3)$$

if $w_i = \partial u_i$ then $w_i \in \ker\left[H_1(X) \longrightarrow H_1(N_i)\right]$.

and $w_1 + w_2 + w_3 = 0$ in $H_1(X)$.

Check $u \cdot u = Q(w_1, w_2, w_3)$ (up to a multiple) □

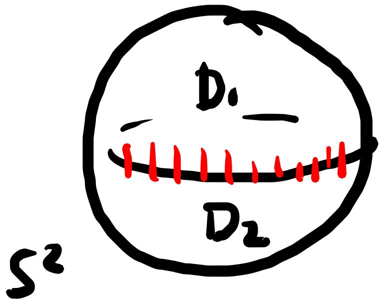
Example

unit disk in Hopf bundle ↘

$$\begin{tikzcd} M \arrow[d, "\pi"'] & = \underbrace{\pi^{-1}(D_1)}_{M_1} \cup \underbrace{\pi^{-1}(D_2)}_{M_2} \\ S^2 & = D_1 \cup D_2 \end{tikzcd}$$

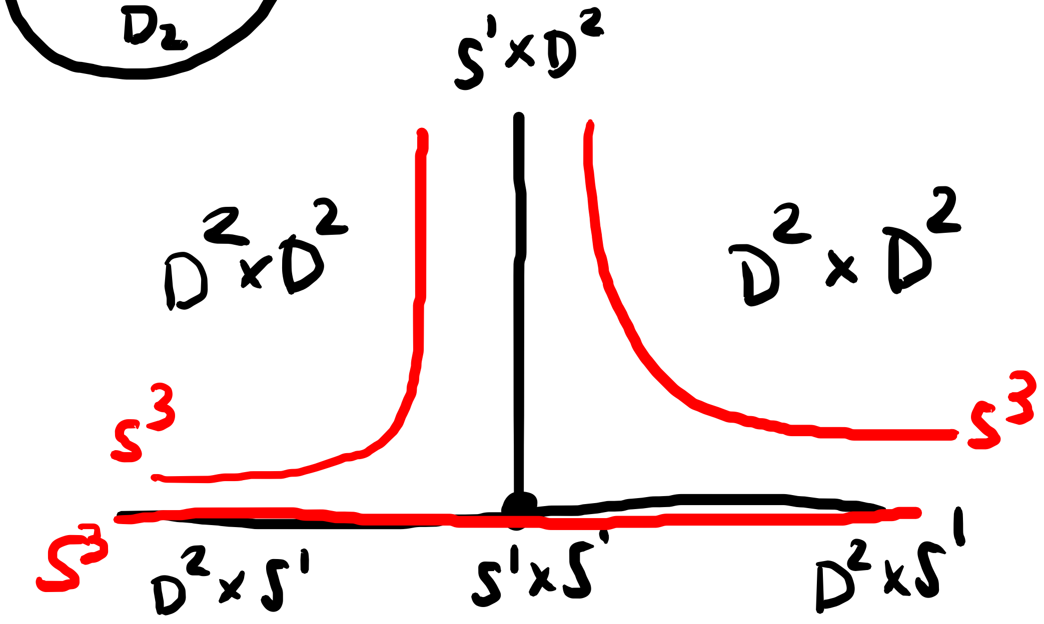
$M_i \cong D^2 \times D^2$

glued along $S^1 \times D^2 \subset \partial M_i \cong S^3$

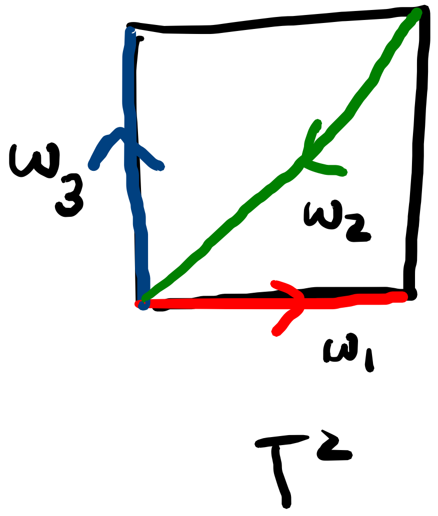


$$\overbrace{D_2 \times (S^1 \times D^1)}^{\text{nbhd of } \partial D \text{ in fibers}} \cup D_1 \times (S^1 \times D^1) \cup \underset{\nwarrow\, S^1 \times I}{A} \times D^2$$

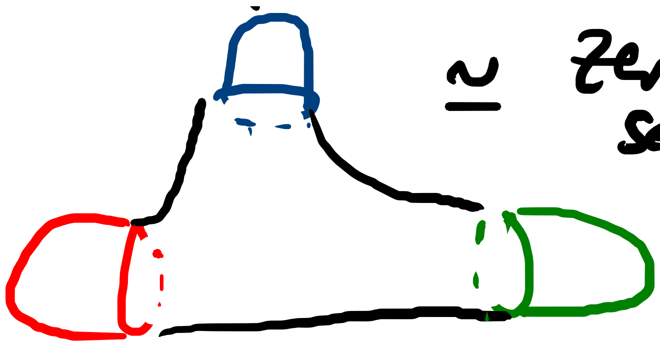
$D_2 \times \theta \qquad D_1 \times \theta'$



3 copies of Heegaard splitting for S^3



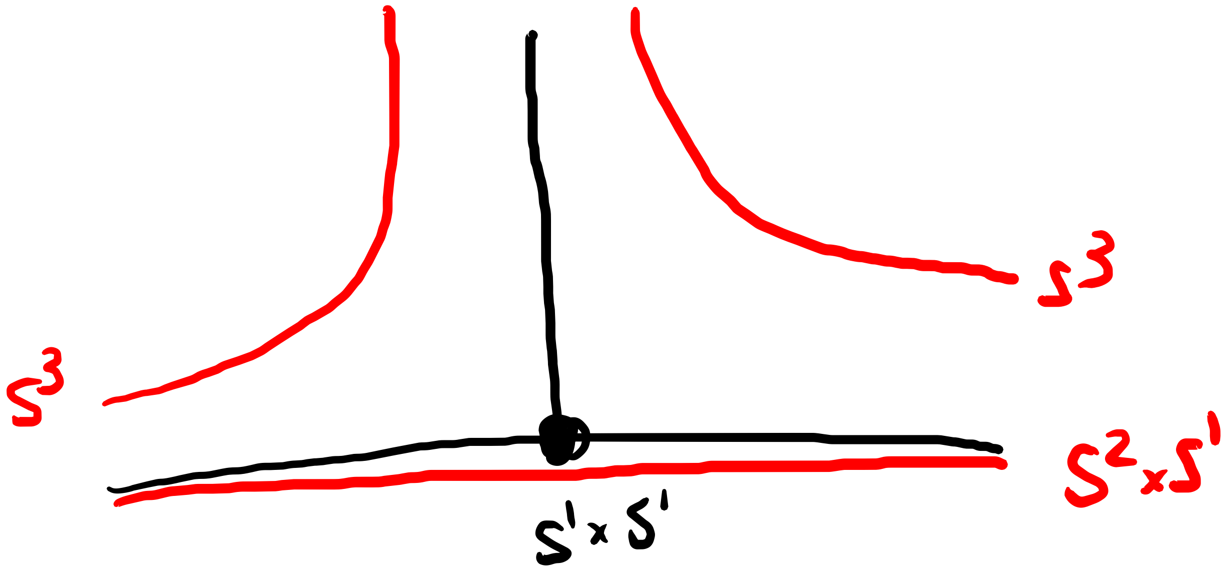
$Q(w_1, w_2, w_3) \neq 0 \leftrightarrow$

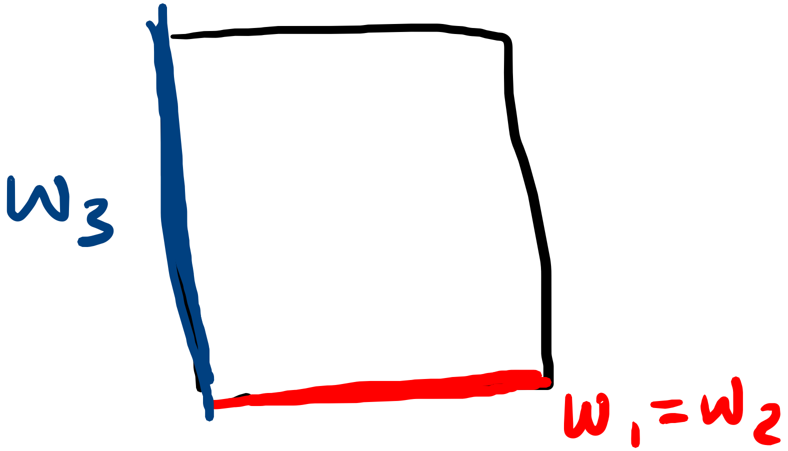


$\cong$ zero section

nontrivial self int.

Ex $M = S^2 \times D^2 = M_1 \times M_2$ $\partial M = S^2 \times S^1$





$\mathrm{sig}(M) = 0.$

Ex (trisections)

Morse 2-function $M^4 \to D^2$

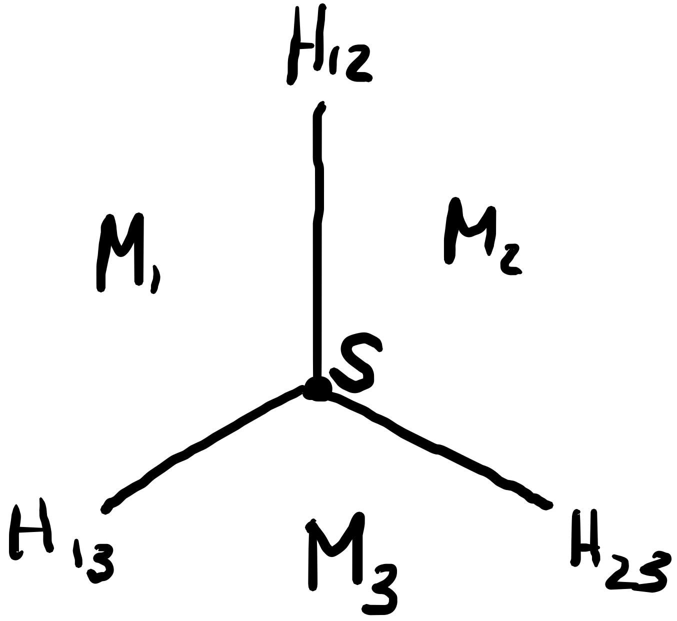
trisection

Every closed oriented M^4 has a trisection

$M = M_1 \cup M_2 \cup M_3$ $\qquad M_i \cong S^1 \times D^3$

$H_{ij} := M_i \cap M_j$ = genus g handlebody

$S = M_1 \cap M_2 \cap M_3$ genus g surface.



$H_{ij} \cup H_{jk}$ = Heegaard splitting of $\partial M_j \cong S^1 \times S^2$

Novikov + Wall $\quad \mathrm{sig}(M) = \mathrm{sig}(M_1 \cup M_2) + \cancel{\mathrm{sig}(M_3)}^{0}$

$$= \cancel{\mathrm{sig}(M_1)} + \cancel{\mathrm{sig}(M_2)} + \mu(L_{12}, L_{23}, L_{31})$$

$L_{ij} = \ker[H_1(S) \to H_1(H_{ij})]$

Signature Cocycles

$$\mu : (\Lambda_n)^{\times 3} \longrightarrow \{-n, \dots, n\} \subset \mathbb{Z}$$

<u>Elementary Properties</u>

- $Sp_{2n}(\mathbb{R})$-invariant $\quad \mu(gL_1, gL_2, gL_3) = \mu(L_1, L_2, L_3)$

- $\mu(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)}) = \text{sign}(\sigma)\, \mu(L_1, L_2, L_3) \quad \sigma \in Sym(3)$

- cocycle

$$0 = \delta\mu(L_1, L_2, L_3, L_4) \equiv \sum_{i=1}^{4} (-1)^i \mu(\dots \hat{L_i} \dots)$$

Symplectic cocycle

$\hat{\mu} : (Sp_{2n}(\mathbb{R}))^{\times 3} \longrightarrow \mathbb{Z}$

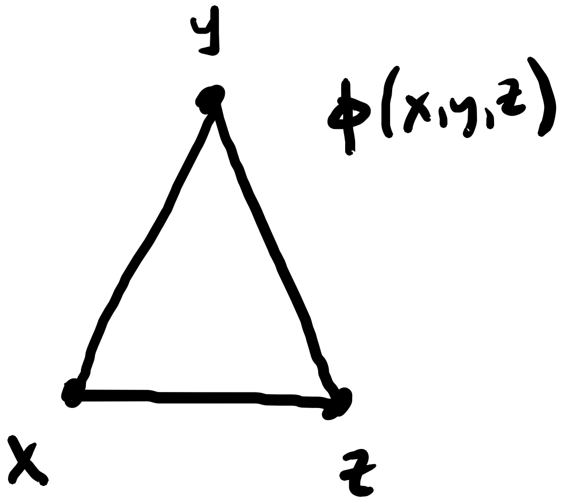
$[\hat{\mu}] \in H^2(Sp_{2n}(\mathbb{R}); \mathbb{Z})$

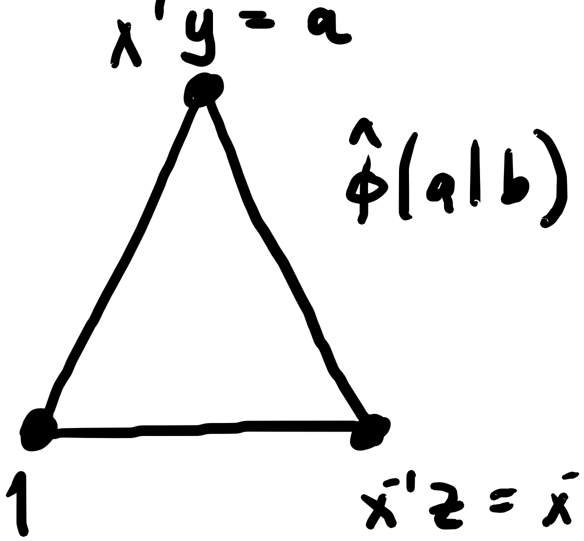
Group cohomology

$C^k(G; \mathbb{Z}) = \{\phi : G^{k+1} \longrightarrow \mathbb{Z}\}$

translation invar

$\delta : C^k \longrightarrow C^{k+1}$

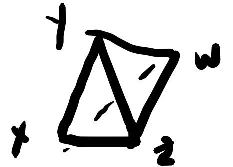


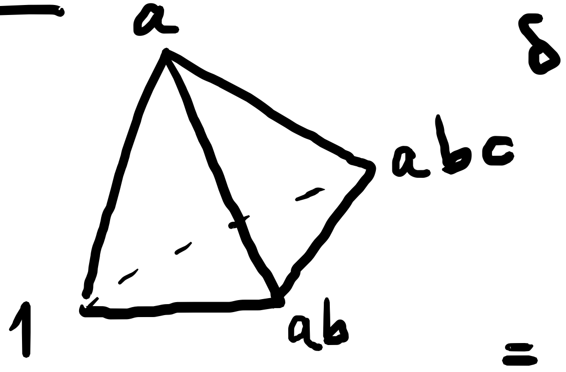


$x^{-1}z = x^{-1}y\,y^{-1}z = ab$

$x,y,z \in G$

cocycle





$\delta\phi(a|b|c)$

$$\delta\phi(1,a,ab,abc) = \phi(a,ab,abc) - \phi(1,ab,abc) + \phi(1,a,abc) - \phi(1,a,ab)$$

$$= \phi(b|c) - \phi(ab|c) + \phi(a|bc) - \phi(a|b)$$

Other cocycles

$c \in H^2(Sp_{2n}\mathbb{R})$

① Central extensions.

② Kähler form

③ Signature cocycle

① Central extensions. For any group G.

$$H^2(G;\mathbb{Z}) \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{central extensions} \\ 1 \to \mathbb{Z} \to \tilde{G} \to G \to 1 \end{array} \right\} / \sim$$

For $G = Sp_{2n}(\mathbb{R})$ $\pi_1(Sp_{2n}\mathbb{R}) \cong \pi_1(U(n)) \cong \mathbb{Z}$

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, equal] & \tilde{G} \arrow[r] & G \arrow[r] & 1 \arrow[r, rightsquigarrow] & \phi \in H^2(G;\mathbb{Z}) \arrow[dd] \\
& \mathbb{Z} & \text{univ. cover} & & & \\
& \prod[\tilde{a}_i, \tilde{b}_i] \arrow[u, phantom, sloped, "\in"] & \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g \tilde{b}_g & \pi_1(S_g) \arrow[uu] & & Hom(H_2(G), \mathbb{Z}) \\
& & & \langle a_1 b_1 \ \ a_g b_g \mid \prod[a_i, b_i] = 1 \rangle \arrow[u, phantom, sloped, "\subseteq"] & & c \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\phi(c) = \prod[\tilde{a}_i, \tilde{b}_i] \in \mathbb{Z}$ $\boxed{\begin{array}{l} \text{eg } \pi_1(S_g) \longrightarrow SL_2\mathbb{R} \\ \text{discrete faithful } \ \phi(c) = \chi(S_g). \end{array}}$ $\pi_1(S_g) \longrightarrow G.$

② Kähler form

$X = Sp_{2n}(\mathbb{R}) / U(n) \cong$ Siegel upper half space $\mathfrak{h}_g = \left\{ A \in GL_n\mathbb{C} \text{ symmetric}, \ \mathrm{Im}(A) > 0 \right\}$

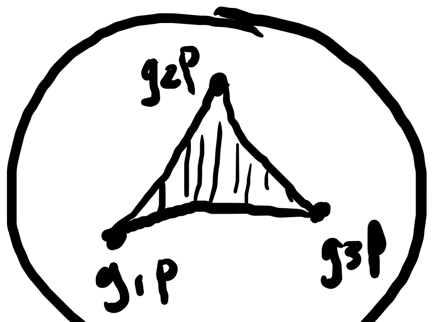
complex mfld, Riemannian sym. space.

$\omega \in \Omega^2(X)$ symplectic form (Kähler form) closed, Sp-invar

For $p \in X$ define $\hat{\omega} : (Sp_{2n}\mathbb{R})^3 \longrightarrow \mathbb{R}$

$$(g_1, g_2, g_3) \longmapsto \int_{\Delta(g_1,g_2,g_3)} \omega$$

order cocycle area ...



$\rightsquigarrow [\hat{\omega}] \in H^2(Sp_{2n}\mathbb{R}; \mathbb{R})$.

compare to class from last time.

③ Signature cocycle on $Sp_{2g}(\mathbb{Z})$

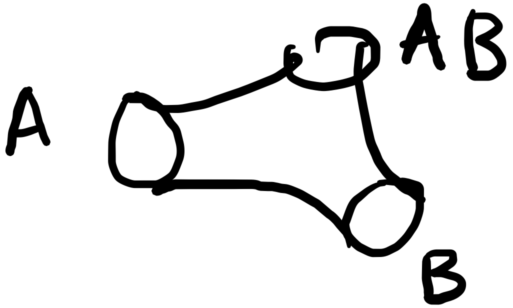
For $a, b \in Sp_{2g}(\mathbb{Z})$

- $\mathrm{Mod}(S_g) \twoheadrightarrow Sp_{2g}(\mathbb{Z})$ surjective (Meeks–Patrusky)

  choose lifts $A, B \in \mathrm{Mod}(S_g)$

- Build S_g-bundle over pant

$S_g \longrightarrow E_{A,B} \longrightarrow$



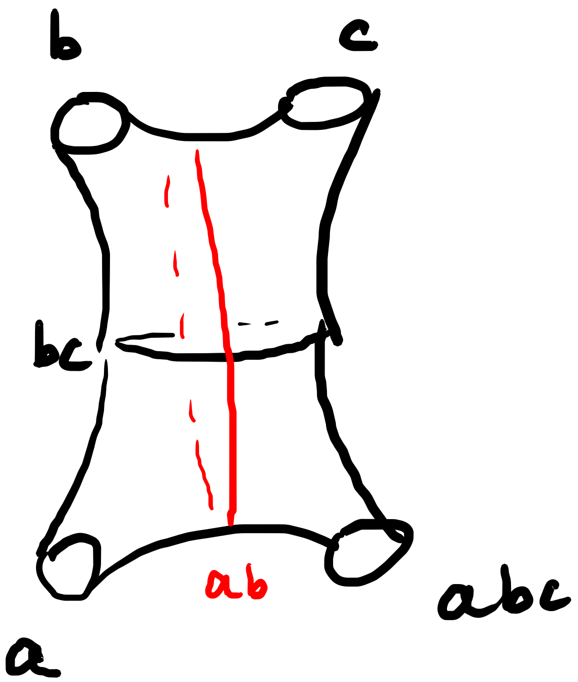
$\sigma(a,b) := \mathrm{sig}(E_{A,B})$

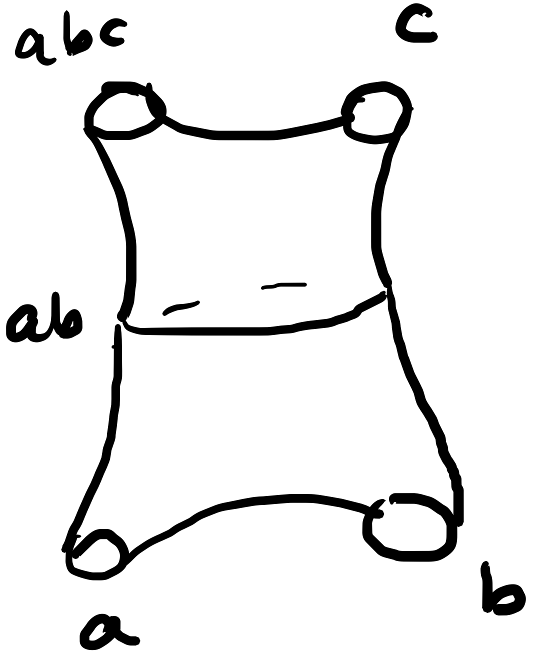
indep of choice of lifts...

<u>Claim</u> σ is a cocycle

[inhomogeneous $(x,y,z) \longleftrightarrow (1, x^{-1}y, x^{-1}z) = (1, \underbrace{x^{-1}y}_{a}, \underbrace{x^{-1}y\, y^{-1}z}_{ab})$]

WTS $$0 = \delta\sigma(a,b,c) = \sigma(b,c) - \sigma(ab,c) + \sigma(a,bc) - \sigma(a,b)$$





Novikov $$\sigma(b,c) + \sigma(a,bc) = \mathrm{sig}\left(E_{a,b,c,abc}\right)$$

$$= \sigma(ab,c) + \sigma(a,b).$$

□

Fact $H^2(Sp_{2n}(\mathbb{R});\mathbb{R}) \cong \mathbb{R}$ so all cocycles are (basically) the same!

More Maslov index

(Rich connections)

Wall nonadditivity

Several points of view

① $\mu : (\Lambda_n)^3 \longrightarrow \mathbb{Z}$ Maslov cocycle

$\rightsquigarrow \hat{\mu} : (Sp_{2n}\mathbb{R})^3 \longrightarrow \mathbb{Z} \rightsquigarrow [\hat{\mu}] \in H^2(Sp_{2n}(\mathbb{R}) ; \mathbb{Z})$

② Central extension $1 \to \mathbb{Z} \to \widetilde{Sp_{2n}\mathbb{R}} \longrightarrow Sp_{2n}(\mathbb{R}) \longrightarrow 1$

$\rightsquigarrow c \in H^2(Sp_{2n}(\mathbb{R}) ; \mathbb{Z})$

③ $\xi \in H^1(\Lambda_n ; \mathbb{Z})$ Connect to dynamics

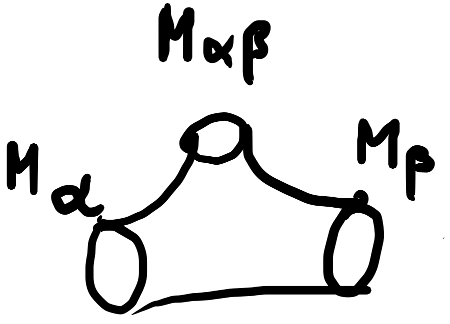
④ Meyer Signature cocycle Connect to topology of surface bundles

Meyer Signature cocycle

$S_g =$ 

$\mathrm{Mod}(S_g) = \mathrm{Homeo}(S_g) / \text{isotopy} \equiv \pi_0 \mathrm{Homeo}(S_g)$

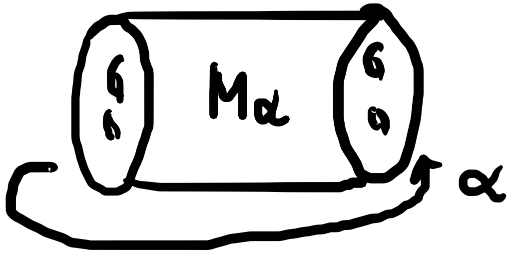
$\sigma : \mathrm{Mod}(S_g) \times \mathrm{Mod}(S_g) \longrightarrow \mathbb{Z}$

For $\alpha, \beta \in \mathrm{Mod}(S_g)$ $\exists!$ $S_g \to E_{\alpha\beta} \longrightarrow$ 

$S^2 \setminus \bigcup_3 D^2$

$\sigma(\alpha, \beta) := \mathrm{sig}(E_{\alpha\beta})$

Construction of $E_{\alpha,\beta}$



$S_g \to M_\alpha \to S^1$

Claim σ is a 2-cocycle (inhomogeneous)

$[\sigma] \in H^2(\mathrm{Mod}(S_g); \mathbb{Z})$

Last time Elements of $H^2(G, \mathbb{Z})$ rep'd by $\phi : G^3 \longrightarrow \mathbb{Z}$

$\phi(ga_1, ga_2, ga_3) = \phi(a_1, a_2, a_3)$ & $0 = \delta\phi(a_1, \dots, a_4) = \sum (-1)^i \phi(\dots, \hat{a}_i, \dots)$

ϕ
(Homogeneous 2-cocycle)

$$\phi(a_1, a_2, a_3) = \phi(1, \underbrace{a_1^{-1}a_2}_{x}, a_1^{-1}a_2 \cdot \underbrace{a_2^{-1}a_3}_{y}) =: \hat{\phi}(x, y)$$

$$\delta\phi = 0 \iff \delta\hat{\phi}(x, y, z) = \hat{\phi}(y, z) - \hat{\phi}(xy, z) + \hat{\phi}(x, yz) - \phi(x, y)$$

$\hat{\phi}$
(inhomogeneous 2-cocycle)

Facts about σ

(3) $\sigma(\alpha,\beta)$ depends only on action of α,β on $H_1(S_g)$

$\mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$. ie σ descends to

$\bar{\sigma}: Sp_{2g}(\mathbb{Z}) \times Sp_{2g}(\mathbb{Z}) \longrightarrow \mathbb{Z}$ (Maslov index)

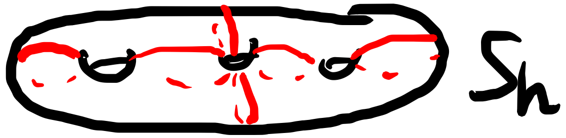
(2) $g=1$ $\quad \mathrm{Mod}(T^2) \cong Sp_2(\mathbb{Z}) \cong SL_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}$

$[\sigma] = 4 \in H^2(SL_2(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$

(1) σ measures signature of surface bundles over surfaces

(Novikov)

$S_g \to E \to S_h$



$S_h \to K(\mathrm{Mod}(S_g),1)$

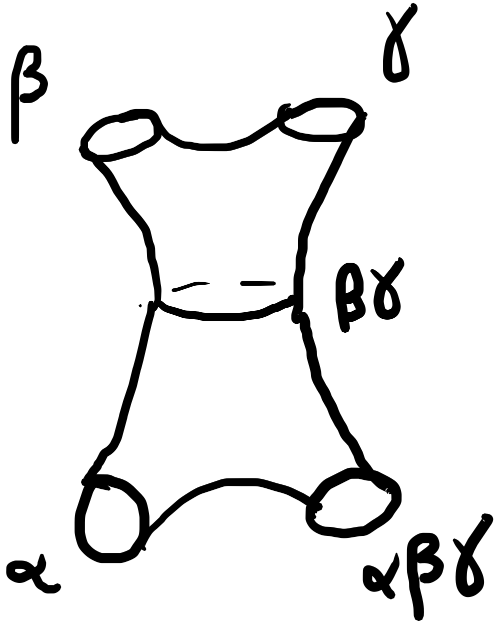
Thm (Atiyah, Kodaira) $\exists\ S_g \to E \to S_h$ w/ $\mathrm{sig}(E) \neq 0$ $(g \geq 3)$

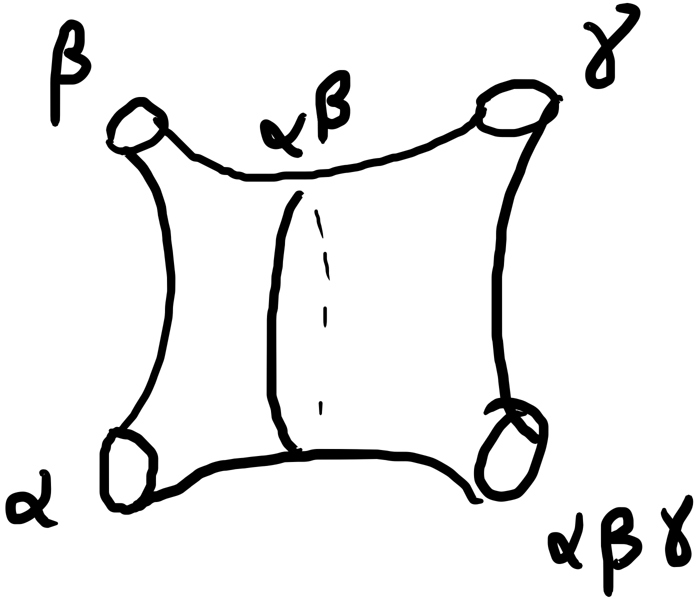
(1.5) **Thm** (Harer) $g \geq 3$ $H^2(\mathrm{Mod}(S_g);\mathbb{Q}) \cong \mathbb{Q}$ ($\Rightarrow$ gen by σ)

Proof of Claim Novikov! Fix $\alpha, \beta, \gamma \in \mathrm{Mod}(S_g)$

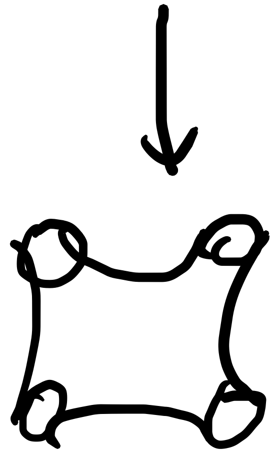
WTS $\sigma(\beta,\gamma) + \sigma(\alpha,\beta\gamma) = \sigma(\alpha,\beta) + \sigma(\alpha\beta,\gamma)$

Both LHS & RHS are signature of (total space) of bundle





$S_g \to E_{\alpha,\beta,\gamma,\alpha\beta\gamma}$



Topology of Λ_n

① $\Lambda_n \cong Sp_{2n}(\mathbb{R}) / GL_n(\mathbb{R}) \cong U(n)/O(n)$

② $$\frac{SU(n)}{SO(n)} \longrightarrow \frac{U(n)}{O(n)} \longrightarrow S^1$$ fibration

$\Rightarrow \pi_1(\Lambda_n) \cong \mathbb{Z}$. Generator of $H^1(\Lambda_n; \mathbb{Z})$

encarnation of Maslov class.

①

$$\mathbb{R}^{2n} = \underbrace{\langle e_1,\dots,e_n, f_1,\dots,f_n\rangle}_{\text{Sympl. basis}}$$

$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ complex structure $GL_n\mathbb{C}$

$\omega(x,y) = x^t J y$ symplectic form $Sp_{2n}\mathbb{R}$

$g(x,y) = \omega(Jx,y)$ inner product $O(2n)$

$= x^t y$

Check $\underbrace{U(n) \curvearrowright \Lambda_n \text{ transitive}}$, $\text{Stab}_{U(n)}(\underbrace{\langle e_1,\dots,e_n\rangle}_{L_0}) \cong O(n)$.

Given $L \in \Lambda_n$ choose ONB $u_1,\dots,u_n$.

Then $u_1,\dots,u_n, Ju_1,\dots,Ju_n$ symplectic basis

matrix A w/ these columns in $Sp_{2n}(\mathbb{R}) \cap O(2n) \cong U(n)$

and $A(L_0) = L$.

redundancy is choice of ONB of L ie $O(n)$

②
$$\begin{tikzcd} U(n) \arrow[r, "\det^2"] \arrow[d] & S^1 \\ \Lambda_n = U(n)/O(n) \arrow[ur, "\det^2"'] & \end{tikzcd}$$

<u>Check</u>

$SU(n)$ acts transitively on fiber $SU^{\pm}(n)/O(n)$ Stabilizer $SO(n)$.

$$\det^2 : \Lambda_n \longrightarrow S^1 \rightsquigarrow m \in H^1(\Lambda_n ; \mathbb{Z})$$ (Brown representability)

Any loop $\gamma \cong S^1 \subset \Lambda_n$ has a "Maslov index" $m(\gamma) \in \mathbb{Z}$.

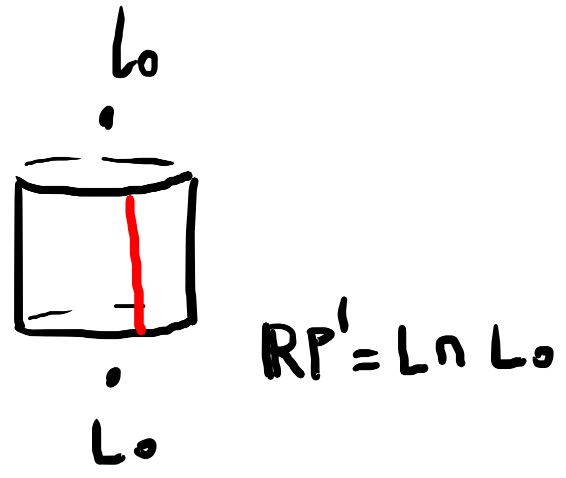
Ex $n=2$ $\qquad Sp_4(\mathbb{R})$

$$S^2 \cong S^3/S^1 \cong \frac{SU(2)}{SO(2)} \longrightarrow \Lambda_2 \xrightarrow{\det^2} S^1$$

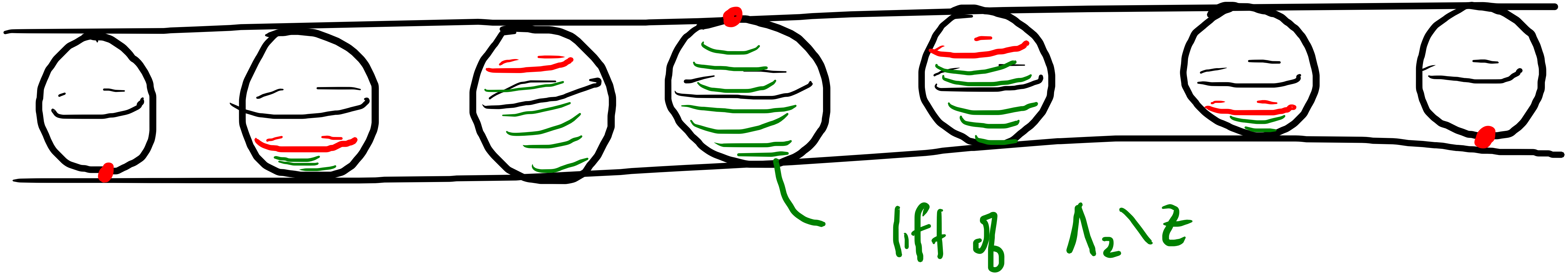
- **Exercise** Monodromy is antipodal ie $\Lambda_2 \cong \dfrac{S^2 \times [0,1]}{(x,0)\sim(-x,1)}$

- Fix $\overset{\langle e_1,e_2\rangle}{\overset{\shortparallel}{L_0}} \in \Lambda_n$ decompose

$$\Lambda_n = \underbrace{\left\{ \begin{matrix} L\cap L_0 = \{0\} \\ \text{transverse} \end{matrix} \right\}}_{\substack{\text{quadratic forms on any} \\ \text{fixed } L \pitchfork L_0. \\ \cong \mathbb{R}^3 \longleftrightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix}}} \cup \underbrace{\left\{ \begin{matrix} L\cap L_0 \neq \{0\} \\ \text{nontransverse} \end{matrix} \right\}}_{\cong S^2/0=\infty}$$



Picture of $\tilde{\Lambda}_2 \cong S^2 \times \mathbb{R}$ $\tilde{Z}$



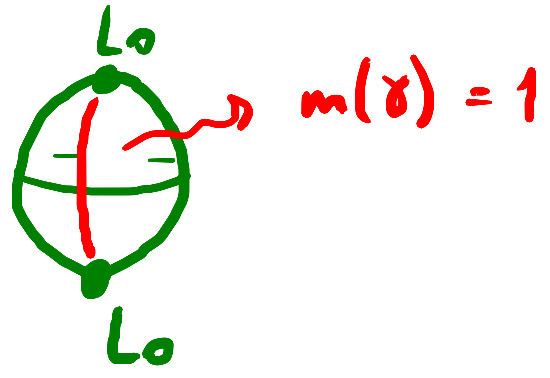
$[Z] \in H_2(\Lambda_2)$ dual to $[m] \in H^1(\Lambda_2)$ (Arnold)

ie for $\gamma \in H_1(\Lambda_n)$ $m(\gamma) = \gamma \cdot Z$.

Next Unify different definitions of Maslov; applications to dynamics

Ex $n=2$ $Sp_4(\mathbb{R})$

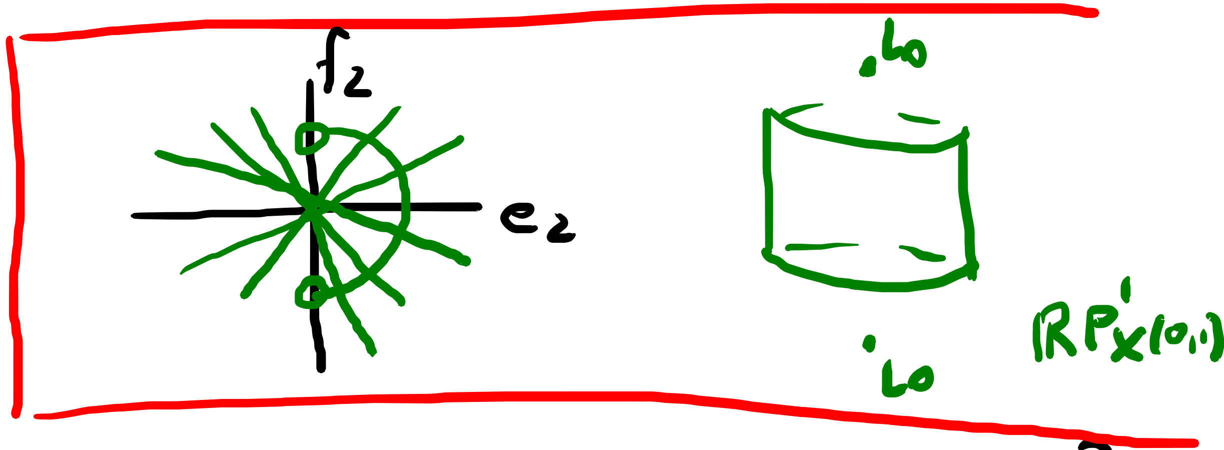
$$S^2 \cong S^3/S^1 \cong \frac{SU(2)}{SO(2)} \longrightarrow \Lambda_2 \xrightarrow{\det^2} S^1$$



Fix $L_0 \in \Lambda_2$. Consider $Z = \{ L \in \Lambda_n \mid L \cap L_0 \neq 0 \}$.

$L_0 = \langle e_1, e_2 \rangle$

$L \in Z \setminus \{L_0\}$ $L \cap L_0 \subset L_0$ line.



eg $L \cap L_0 = \langle e_1 \rangle$. what are all Lagrangians containing $\langle e_1 \rangle$?

$$0 = \omega(e_1, ae_1 + be_2 + cf_1 + df_2) = c$$
$$0 = g(e_1, ae_1 + be_2 + cf_1 + df_2) = a$$

$d \neq 0$. $L = \langle e_1, be_2 + df_2 \rangle$

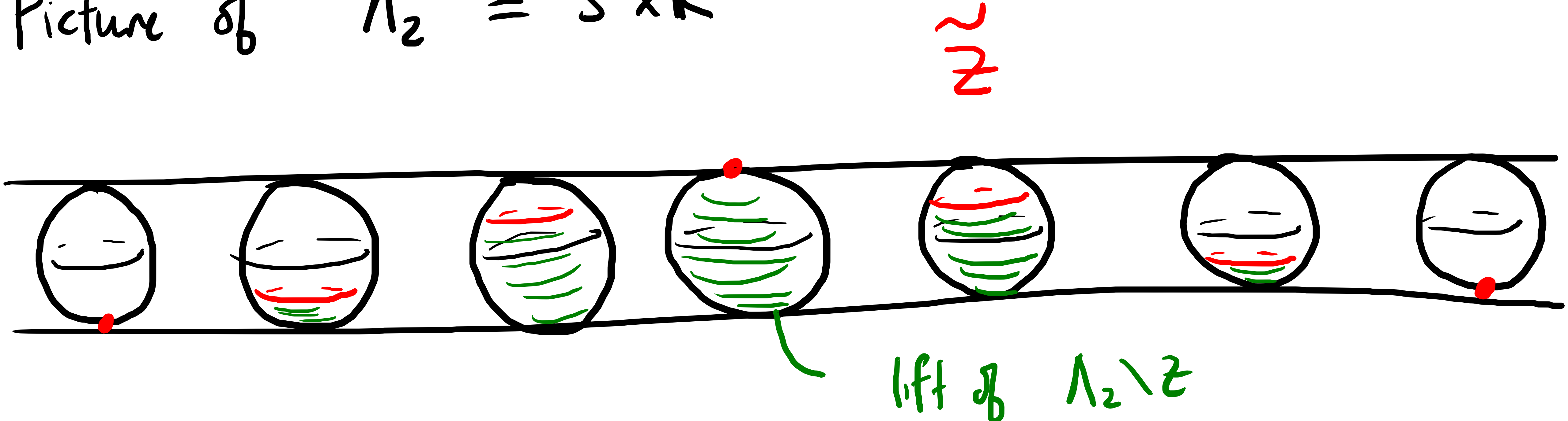
$$Z \cong S^2/(0=\infty) \sim S^2 \vee S^1$$

$$\Lambda_2 \setminus Z \cong \{ \text{Lagrangians transverse to } L_0 \}$$

$$\cong \{ \text{quadratic forms on any fixed } L \pitchfork L_0 \}$$ (as in defn of Maslov

$$\cong \mathbb{R}^3$$

Picture of $\tilde{\Lambda}_2 \cong S^2 \times \mathbb{R}$



Λ_1 story

$\circlearrowright$

$SL_2(\mathbb{R})$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{SL_2\mathbb{R}} \arrow[r] \arrow[d] & SL_2\mathbb{R} \supset SO(2) \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Homeo}(\mathbb{R}) \arrow[r] \arrow[d, "\text{orbit}"] & \mathrm{Homeo}(S^1) \arrow[d, "\text{orbit}"] \\
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1
\end{tikzcd}$$

Λ_1 story

$\circlearrowright$

$SL_2(\mathbb{R})$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{Sp_{2n}\mathbb{R}} \arrow[r] \arrow[d] & Sp_{2n}(\mathbb{R}) \supset U(n) \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Homeo}(\tilde{\Lambda}_n) \arrow[r] \arrow[d, "\mathrm{orbit}"] & \mathrm{Homeo}(\Lambda_n) \arrow[d, "\mathrm{orbit}"] \\
\mathbb{Z} \arrow[r] \arrow[d, equal] & \tilde{\Lambda}_n \arrow[r] \arrow[d, "\widetilde{\det}^2"] & \Lambda_n \arrow[d, "\det^2"] \\
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1
\end{tikzcd}$$

Λ_n story

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1 \\
\mathbb{Z} \arrow[u, equal] \arrow[r] & \widetilde{\Lambda}_n \arrow[u] \arrow[r] & \Lambda_n = U(n)/O(n) \arrow[u, "\det^2"'] \\
\mathbb{Z} \arrow[u, equal] \arrow[r] & \widetilde{Sp_{2n}\mathbb{R}} \arrow[u] \arrow[r] & Sp_{2n}\mathbb{R} \arrow[u, "\text{orbit}"'] \arrow[d, phantom, "\cup"] \\
 & & U(n)
\end{tikzcd}$$

Fixed points of surface diffeomorphisms

S oriented surface. $f : S \to S$ diffeomorphism

Q: Does f have a fixed point? $(f(x) = x)$

Examples

- rotations on $S^1 \times S^1$ are fixed point free.

$$r(\theta, \varphi) = (\theta + c, \varphi)$$

- Brouwer: every $f : D^2 \to D^2$ has a fixed point.
  $\uparrow$ closed disk

- Lefschetz: $f : S \to S$ (continuous)

  has fixed point if

$$\underbrace{\sum_{i=0}^{2} (-1)^i \operatorname{Trace}\left[f_* : H_i(S;\mathbb{Q}) \to H_i(S;\mathbb{Q})\right]}_{\overset{!!}{\Lambda_f} \text{ Lefschetz number}} \neq 0$$

Consequently $g \simeq f$ (homotopic) also has a fixed pt.

Eg $f: S^2 \to S^2$

$$\Lambda_f = 1 + \operatorname{Tr}\left[f_* : H_2(S^2) \to H_2(S^2)\right]$$

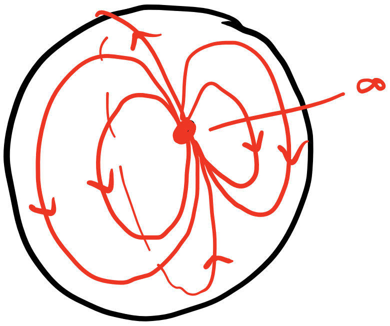
$$= 1 + \deg(f)$$

f or. pres. diffeo $\Rightarrow$ $\Lambda_f = 2$

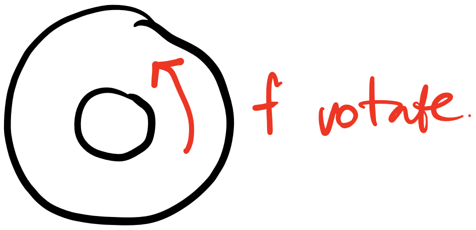
$\Rightarrow$ f has at least one fixed point

This is sharp: $f: z \mapsto z+1$ on $\mathbb{C}$ extends to diffeo (biholo) of $\hat{\mathbb{C}} \cong S^2$

f has exactly one fixed pt. $f(\infty) = \infty$.



Ex $A = S^1 \times I$



$f: A \to A$

or. pres. diffeo $\Lambda_f = 0$ Lefschetz silent.

Indeed f need not have any fixed point.

Remarkable trend: area preserving diffeos tend to have more fixed guaranteed fixed points.

Example 1 $S = S^2$

Thm (Nikishin, Simon 1974) $f: S^2 \to S^2$ area pres. diffeo (homeo) $\Rightarrow$

f has $\geqslant 2$ fixed points

eg $f \in \mathrm{Isom}^+(S^2) = SO(3)$ has 2 fixed pts

by linear algebra

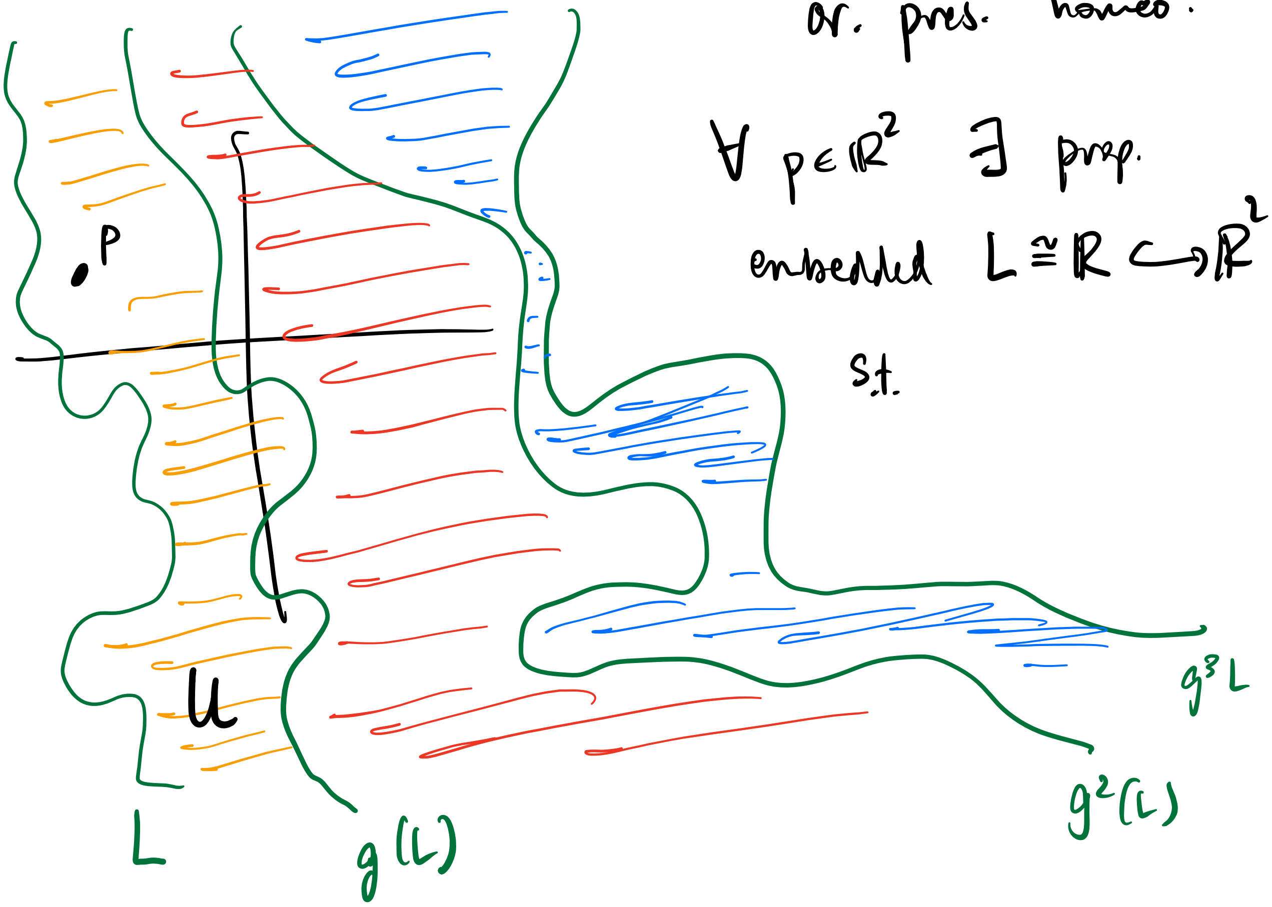
For general case:

Brouwer translation Thm $g: \mathbb{R}^2 \to \mathbb{R}^2$

or. pres. homeo.

$\forall\ p \in \mathbb{R}^2\ \exists$ prop. embedded $L \cong \mathbb{R} \hookrightarrow \mathbb{R}^2$

s.t.



$$\begin{tikzcd}
(\mathbb{R}^2, 0) \arrow[r, hook, "\text{embedding}", "\Phi"'] \arrow[d, "\tau"', "(x,y)\mapsto(x+1,y)"'] & (\mathbb{R}^2, p) \arrow[d, "g"] \\
(\mathbb{R}^2, 0) \arrow[r, hook, "\Phi"'] & (\mathbb{R}^2, p)
\end{tikzcd}$$

$g \circ \Phi = \Phi \circ \tau$

Pf of Thm By Lefschetz f has a fixed pt. "∞". Apply Brouwer translation to $f|_{S^2 \setminus \infty \cong \mathbb{R}^2}$ $\Rightarrow \exists$ open U s.t.

$U, f(U), f^2(U), \ldots$ disjoint

Since f preserves area, $\text{area}(U) > 0$, $\text{area}(S^2) < \infty$, this is a contradiction. $\square$

Example 2 $S = S^1 \times [0,1]$ annulus.

Here an area preserving diffeo (still) need not have any fixed points (rotate)

However...

Thm (Poincare, Birkhoff 1913)

$f : A := S^1 \times [0,1] \circlearrowleft$ orientation & area preserving

and $f|_{S^1 \times 0}$ CW rotation

$f|_{S^1 \times 1}$ CCW rotation

Then f has ≥ 2 fixed points.

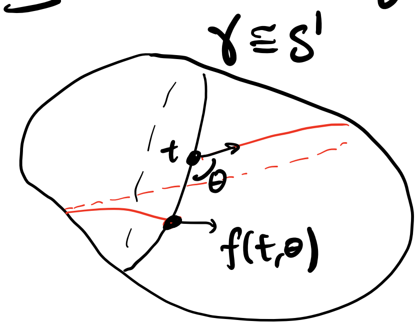
Why Poincare cared: celestial mechanics

3-body problem, want periodic solutions

Mathematical formulation

Riemannian 2-spheres (S^2, g)

Q: Does (S^2, g) have a closed geodesic?

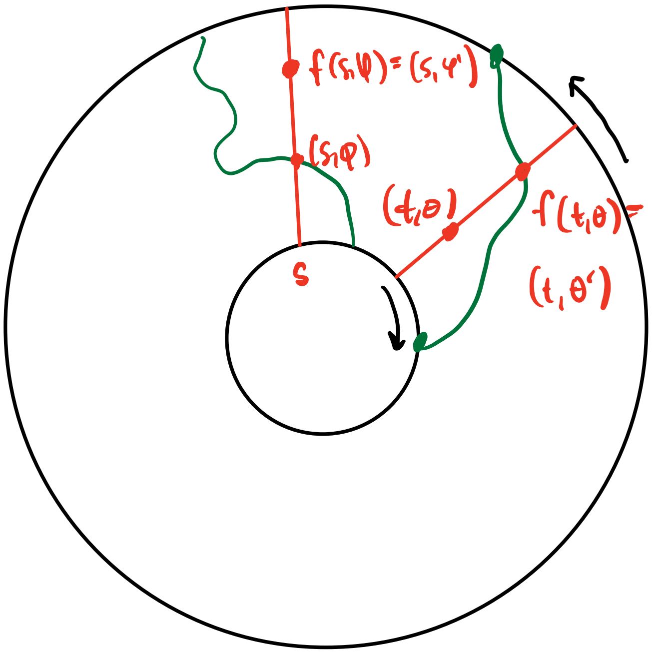


For $t \in \gamma \cong S^1$, $\theta \in (0, \pi)$

$f(t, \theta) =$ value of 1st return map f extends to $A \longrightarrow A$ w/ properties of the theorem.

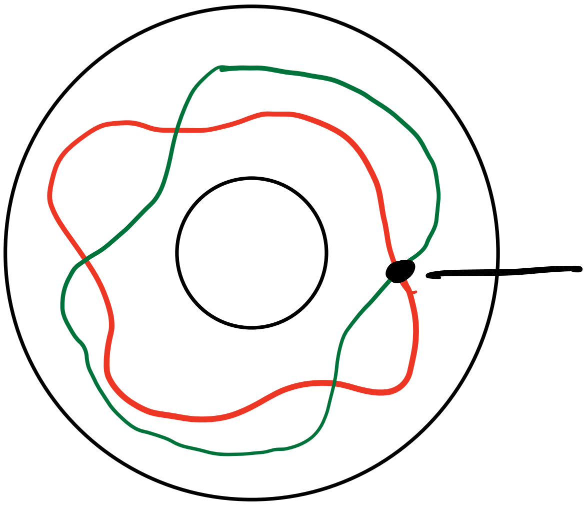
Fixed point of f $\leftrightsquigarrow$ simple closed geodesic transverse to γ

A proof idea



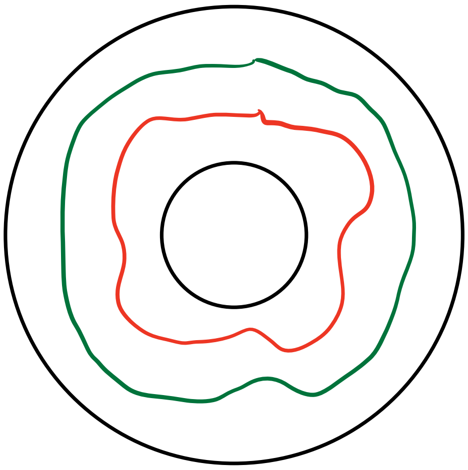
For each $t \in S^1$ take θ, θ' st.

$f(t,\theta)=(t,\theta')$



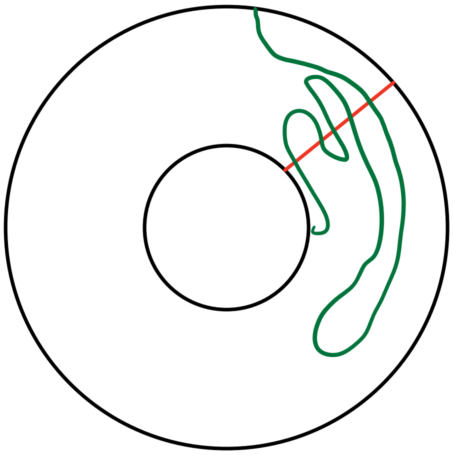
fixed point of f.

Problem: what if red / green nested?



This can't happen b/c f is area preserving.

Problem For fixed t, θ, θ' not uniquely determined



fix (not so easy) by Birkhoff

see also Graven-Hubbard 2021

Arnold called this Thm the "seed of symplectic topology"

Arnold conjecture

(M,ω) symplectic manifold (eg surface w/ area form)

$f: M \longrightarrow M$ symplectic diffeo $f^*\omega = \omega$ (area preserving)

that's also Hamiltonian.

Then $\# \{ f(x) = x \} \geq \underbrace{\sum \text{Betti \#'s of } M}_{\text{topology}}$.

Eg an area preserving diffeo of T^2 has ≥ 4 fixed points.

Proof by Floer (birth of Floer homology)

Hamiltonian diffeo:

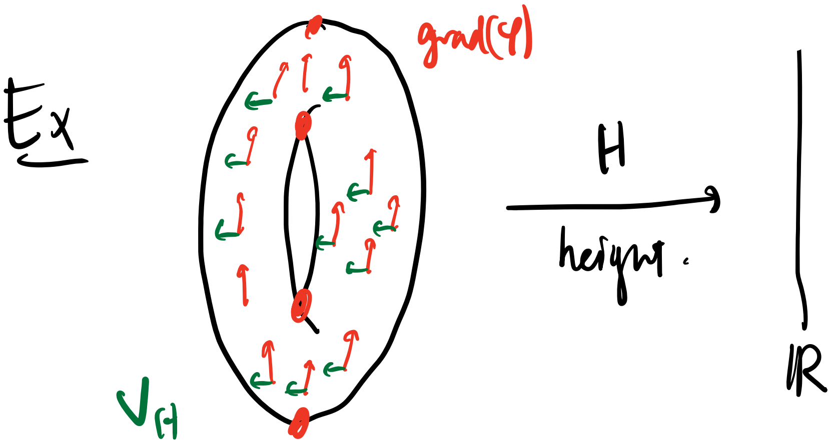
Given $H: M \longrightarrow \mathbb{R}$ ⇝ $d\varphi$ 1-form

$\exists!$ vector field V_H (Hamiltonian v.f.)

s.t. $dH(u) = \omega(u, V)$ $\forall u$.

⇝ flow f_t^H f_1^H = Hamiltonian diffeo.

Ex



Here f_1^H has as many fixed points as critical points of H.

More generally Arnold predicts same for time dependent Hamiltonian vector field

$H: M \times \mathbb{R} \longrightarrow \mathbb{R}$ periodic

$H(x, t+1) = H(x,t)$.

Key: Conley-Zehnder index (related to Maslov)

For $H: M^{2n} \times \mathbb{R} \to \mathbb{R}$

and fixed point $f_1^H(x) = x$.

consider path $t \overset{\gamma}{\longmapsto} df_H^t(x) \in Sp_{2n}(\mathbb{R})$

<u>Recall</u> $Sp_{2n}\mathbb{R} \longrightarrow \Lambda_n \longrightarrow S^1$

a loop in $Sp_{2n}\mathbb{R}$ has a Maslov index

But generally γ not a loop...