Mapping class groups **Spring 2017**

Instructor: Bena Tshishiku

MWF 12-1 SC 304

Course description:

Introduction to topics around the cohomology of mapping class groups of surfaces.

Topics

1. Algebraic structure of Mod_g.
   - Generated by Dehn twists, finitely presented.
   - Compute $H_1(\mathrm{Mod}_g)$, $H_2(\mathrm{Mod}_g)$.
2. Surface bundles.
   - Over the circle: multiple fiberings, Thurston norm.
   - Over surfaces: signature, Atiyah–Kodaira examples, surface subgroups of Mod_g.
   - Characteristic classes: Miller–Morita–Mumford classes.
3. Mumford conjecture.
   - Harer's homological stability theorem.
   - Madsen–Weiss theorem.
   - Homotopy type of diffeomorphism groups, Earle–Eells theorem.
4. Lifting problems for Mod_g.
   - Morita's nonlifting theorem, flat connections on surface bundles.
   - Lifting problems for surface braid groups.
   - Sections of surface bundles and Hain's conjecture.

References:

- Farb–Margalit, *A primer on mapping class groups*
- Hatcher, *A short exposition of the Madsen–Weiss theorem*
- Morita, *Geometry of characteristic classes*

Lectures.

Part I: algebraic structure of Mod_g

1/23: overview, definition of $\mathrm{Mod}(S)$, low-genus examples

1/25: $\mathrm{Mod}(S)$ is finitely generated, Birman exact sequence

1/27: $\mathrm{Mod}(S)$ is finitely generated, curve complex and connectivity

1/30: abelianization of $\mathrm{Mod}(S)$, uniformly perfect groups

2/1: $\mathrm{Mod}(S)$ is finitely presented, finding presentations, Hatcher–Thurston theorem, cut system complex

2/3: $\mathrm{Mod}(S)$ is finitely presented, Morse–Cerf theory, cut system complex is connected

2/6: $\mathrm{Mod}(S)$ is finitely presented, cut system complex is simply connected; Hopf's formula in group homology

2/8: Birman–Hilden and relations in $\mathrm{Mod}(S)$, computing $H_2 \mathrm{Mod}(S)$ with Hopf's formula

2/10: the Euler class in group cohomology and in nature, examples of nontrivial classes in $H^2 \mathrm{Mod}_{g,1}$.

Part II: surface bundles

2/13: introduction, monodromy as complete invariant, trefoil knot is fibered

2/15: surface bundles over S^1, multiple fiberings of the trivial bundle, Goldsmith construction, Stallings criterion for fibered knots

2/17: Thurston norm, definition and properties, norm ball, examples $S \times S^1$

2/22: Thurston norm examples (Hopf and Whitehead links), fiber of fibration is norm minimizing

2/24: Thurston norm and fiberings over S^1, Tischler's theorem

2/27: classifying space $B\,\mathrm{Diff}(S)$, characteristic classes of surface bundles, MMM classes

3/1: interpretations of 1st MMM class

3/3: signature of surface bundles, Hirzebruch criterion for branched covers

3/6: Atiyah–Kodaira constructino of surface bundle over surface with nonzero signature

3/8: surface bundles over surfaces with many fiberings, Salter construction

Part III: cohomology of Mod_g

3/20: Mumford conjecture, applications, precursors, major ingredients of proof

3/22: homological stability, strategy, execution for symmetric groups

3/27: homological stability, equivariant homology and application to computing group homology

3/29: homological stability, spectral sequence argument, application: moduli space $\mathcal{M}_g$ and Mod_g have same rational homology

3/31: homological stability, stability for Mod_g, properties of the arc complex

4/3: homological stability, connectivity of arc complexes

4/5: topology of diffeomorphism groups, diffeomorphisms of spheres, exotic spheres, Smale's theorem on $\mathrm{Diff}(S^2)$

4/7: topology of diffeomorphism groups, proof of Smale's theorem, remarks on generalized Smale conjecture

4/10: topology of diffeomorphism groups, proof of Earle–Eells theorem, topology of space of arcs

Part IV: application

4/12: flat connections on manifold bundles, example: circle bundles and Milnor–Wood inequality

4/14: Milnor–Wood inequality, Sullivan's geometric proof, characteristic classes and flat connections: Chern–Weil theory and bounded cohomology

4/17: bounded cohomology, simplicial volume, Gromov norm proof of Milnor–Wood

4/19: flat surface bundles, homotopy viewpoint on foliations, Bott vanishing theorem

4/21: Bott vanishing theorem, nonflat surface bundles

4/24: Morita m-construction, flatness question for surface bundles over surfaces

4/26: lifting problem for point-pushing subgroup and for braid groups

Office Hours MW 2:30-3:30
SC 525

~~§~~. I. Introduction

$S = S_g$ closed, oriented surface, genus g.

3 spaces

(1) classifying space $BDiff(S)$ for S bundles.

There is a bijection $\left\{ \begin{matrix} \text{smooth fiber bundles} \\ S \to E \to B \end{matrix} \right\} / \text{iso} \longleftrightarrow [B, BDiff(S)]$

(2) Eilenberg-Maclane space $K(Mod_g, 1)$

$Mod_g := \pi_0 Diff(S_g)$ mapping class group.

(3) Moduli space $\mathcal{M}_g = \left(\left\{ \begin{matrix} \text{genus } g \\ \text{Riemann surfaces} \end{matrix} \right\} / \text{iso} \right) =$ moduli space of genus g Riemann surfaces

Problem Compute $H^\cdot(\ \ )$ for any of these spaces.

Fact w/ $\mathbb{Q}$-coeff, $\chi(S) < 0$

$$H^\cdot(BDiff(S)) \underset{①}{\cong} H^\cdot(Mod_g) \cong H^\cdot(\mathcal{M}_g).$$

① $BDiff(S) \sim K(Mod_g, 1)$ h.e. (Earle-Eells)

② $Mod_g \curvearrowright Teich_g \cong \mathbb{R}^{6g-6}$ nicely.

Plan for course describe some of what we know about cohomology ~~and~~ of Mod_g.

and give app. to cc's of surface bundles.

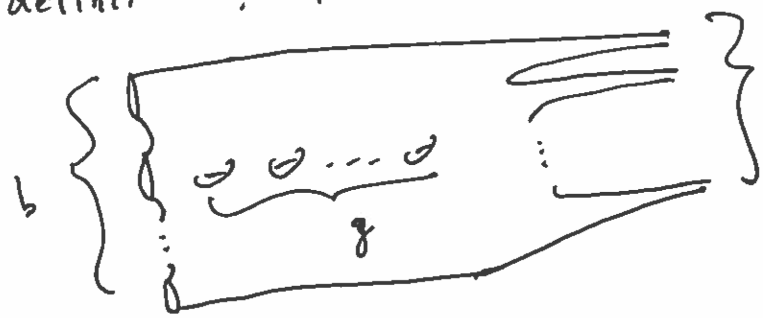
(Hatcher–Thurston) Mod_g is f.p. good warm up / intro to MCGs.

(Harer) $H^1_{\mathbb{Z}}(\mathrm{Mod}_g) = 0 \quad g \geq 3$, $H^2(\mathrm{Mod}_g) = \mathbb{Z} \quad g \geq 4$.

(Morita) $\mathrm{Diff}(S_g) \longrightarrow \mathrm{Mod}_g$ does not split $g \geq 2$.

II. Mod(S): definition & first examples.

$S = S^b_{g,p}$



$\mathrm{Diff}(S)$ — or. pres. diffeos $f : S \to S$, $f|_{\partial S} = id$.

$\triangledown$

$\mathrm{Diff}_0(S)$ — diffeos isotopic to id.

$$\mathrm{Mod}^b_{g,p} \equiv \mathrm{Mod}(S) := \mathrm{Diff}(S)/\mathrm{Diff}_0(S) \simeq \pi_0 \mathrm{Diff}(S).$$

Rmk (variations) could also consider Diff / htpy, Homeo / isotopy, Homeo / htpy.

For surfaces these are all the same.

Examples

(1) Lemma For S one of $D = ⊘, \mathbb{R}^2, S^2$, $\mathrm{Mod}(S) = 1$.

Pf suffices to show every homeo $f : S \to S$ homotopic to id.

- for D or $\mathbb{R}^2$ straight line htpy $f_t = t \cdot id + (1-t) f$
- for S^2 any homotop f so $f(\infty) = \infty$.

(eg by composing w/ rot) $\square$.

to get sth interesting, need some nontrivial π_1

(2) $S = A =$  $= S^1 \times [0,1]$.
$\theta \quad r$

Nontrivial mapping class. $T(\theta, r) = (\theta + 2\pi r, r)$

Rmk: homotopic to id if ∂ not fixed

<u>Prop</u> $\mathrm{Mod}(A) \cong \langle T \rangle \cong \mathbb{Z}$.

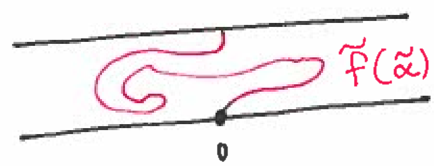
<u>Pf</u> Define

$$\begin{tikzcd} \mathrm{Diff}(A) \arrow[r, "\phi"] \arrow[d] & \mathbb{Z} \\ \mathrm{Mod}(A) \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

$f \longmapsto [\alpha \cdot \overline{f(\alpha)}] \in \pi_1(A) \cong \mathbb{Z}$

(descends to Mod(A))

- surjective since $\phi(T) = 1$
- injective: if $\phi(f) = 1$, lift to $\tilde{f} : \mathbb{R} \times [0,1] \circlearrowleft$
  s.t. $\tilde{f}|_\partial = \mathrm{id}$.



straight-line htpy to id descends to A.

□.

<u>Rmk</u> T is called a <u>Dehn twist</u>.

(3) $S =$ 

$\mathrm{Mod}(S) \cong B_3$ braid group.

σ_1:  σ_2: 

(will discuss this ex. more tomorrow)

(4) $S = T^2 =$ 

<u>Prop</u> $\mathrm{Mod}(T^2) \cong SL_2\mathbb{Z}$.

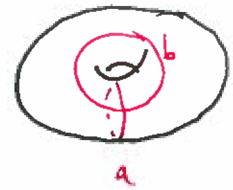
(could argue as before look at action on $\pi_1(T^2) / H_1(T^2)$. For injective lift to $\widetilde{T^2} \cong \mathbb{R}^2$ fixing $\mathbb{Z}^2 < \mathbb{R}^2$. straight-line as before.) Surjective b/c $SL_2\mathbb{Z}$ acts on T^2.

<u>Pf</u> $T^2 = K(\mathbb{Z}^2, 1) \Rightarrow \left\{ \begin{matrix} \text{h.e.} \\ T^2 \to T^2 \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{(outer) automorphisms} \\ \mathbb{Z}^2 \to \mathbb{Z}^2 \end{matrix} \right\} = GL_2\mathbb{Z}$

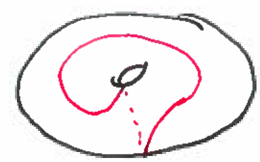
(action on π_1)

$\Rightarrow$ homeo f homotopic to id $\iff$ $f_* : \pi_1(T^2) \circlearrowleft$ is identity. $\square$

<u>Rmk</u> $SL_2\mathbb{Z}$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow$ $\mathrm{Mod}(T^2)$ generated by Dehn twists T_a, T_b 

Check action of T_a on generators for $H_1(T^2)$:

$T_a(a) = a \qquad T_a(b) = a+b$ 

(Dehn twists play role of elementary matrices for MCGs.

<u>Rmk</u> For $g \geq 2$ Mod_g has no other name.
still have $\mathrm{Mod}_g \hookrightarrow \mathrm{Out}(\pi_1(S_g))$
(in fact surj by Dehn-Nielsen-Baer) but ↑ doesn't have a name either.

Next time:
<u>Thm</u> Mod_g is generated by finitely many Dehn twists.

Main ingredients:
(1) Birman exact sequence $1 \to \pi_1(S_g) \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g \to 1$.
(2) Mod_g action on curve complex. $\mathcal{C}(S)$.

Lecture 2

Part I. Finite generation for Mod_g.

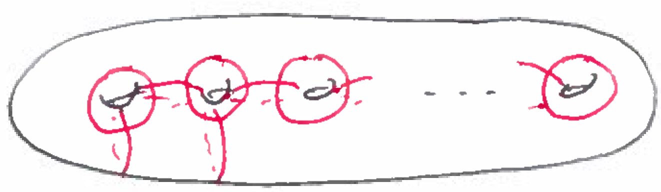
<u>Recall</u> Given simple closed curve (scc) $a \subset S$ (ie embedded circle)

can define Dehn twist $T_a \in \mathrm{Mod}(S)$

<u>Thm</u> (Dehn-Lickorish) $\forall g$ Mod_g is generated by finitely many Dehn twists about <u>nonseparating</u> scc's.
$\hookrightarrow$ $S \setminus a$ connected.

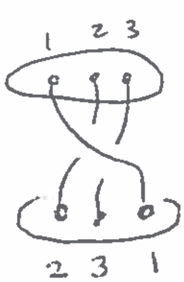
<u>Rmks</u>

- Analogous to $SL_n\mathbb{Z} = \langle$ elementary matrices $\rangle$
- Humphries generators



<u>Warmup</u>: Pure braid group is finitely generated.

<u>Defn</u> P_n braids whose endpoints are not permuted.



$$\begin{array}{c} P_n \\ \wedge \\ B_n := \pi_1(\mathrm{Conf}_n(D)) \end{array} \qquad \mathrm{Conf}_n(D) = \left\{ \begin{array}{l} n \text{ distinct} \\ \text{points in } \mathring{D} \end{array} \right\}$$

<u>Prop</u> P_n is f.g.

Pf (induction on n)   Base case: $P_1 = 1$.

$F: P_n \twoheadrightarrow P_{n-1}$ forget nth strand.

$\ker(F)$: last strand wraps around others
$\cong$
$\pi_1(D \setminus (n-1)\text{points}) \simeq F_{n-1}$

Note A, C f.g. groups and $1 \to A \to B \to C \to 1$ then B f.g. $\square$.

Rmk (Artin braid combing) $P_n \simeq F_{n-1} \rtimes P_{n-1}$ inductively

solves word problem for P_n (write braid as concatenation of braids \
1st: last strand wraps around 1st $n-1$
2nd: 2nd to last strand wraps around 1st $n-2$
...

word prob easy in free gp

Rmk For DL Thm have extra task to show can take special generators.
Will still use inductive strategy above

Thm Pf outline $PMod_{g,n} < Mod_{g,n}$ punctures not permuted.

(1) Puncture induction   (today)

$PMod_{g,n}$ $\underbrace{\text{gen by fin. many Dehn twists about nonsep scc's}}_{fg(+)}$ $\Longrightarrow$ $PMod_{g,n+1}$ fg(+).

(2) genus induction $PMod_{g,2}$ fg(+) $\Longrightarrow$ $PMod_{g+1}$ fg(+).
(next time $Mod(S) \curvearrowright \mathcal{C}(S)$)
gives induction $SL_2\mathbb{Z} = Mod_1 \rightsquigarrow PMod_{1,2} \rightsquigarrow Mod_2 \rightsquigarrow$ etc.

Rmk Argument illustrates: even if primarily care about closed surfaces sometimes puncture ...

II. Birman exact sequence & puncture induction

<u>Rmk</u> (punctures vs marked pts) S closed, $X \subset S$ finite

$\mathrm{Mod}(S,X) := \pi_0 \mathrm{Homeo}(S,X)$ ⟶ homeos $f: S \to S$ s.t. $f(X) = X$.

Then $\mathrm{Mod}(S,X) \simeq \mathrm{Mod}(S \setminus X)$ fill in homeo of $S \setminus X$ at punctures

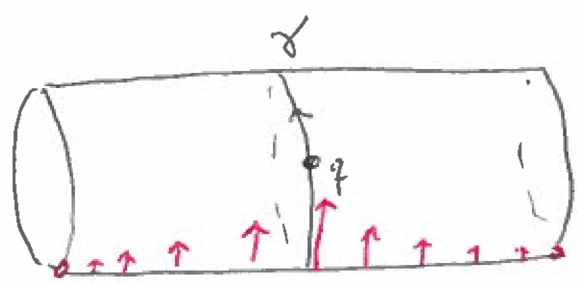
<u>Thm</u> (BES) S closed, $\chi(S) < 0$, $q \in S$. There is exact seq.

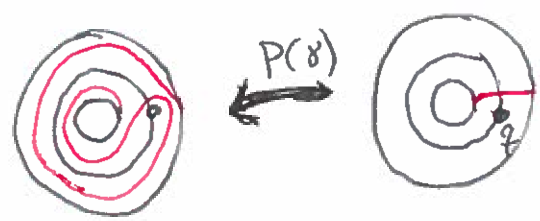
$$1 \longrightarrow \pi_1(S,q) \xrightarrow{P} \mathrm{Mod}(S,q) \xrightarrow{F} \mathrm{Mod}(S) \longrightarrow 1$$

(an element of kernel traces a loop in S) as before.

- F is forgetful map.
- P is "point-pushing"

Define P(γ) as time-1 map of flow of v.f.



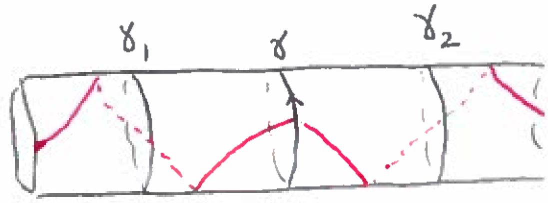


<u>Note</u> (1) $F \circ P(\gamma) = 1$ by defn.

(2) $P(\gamma)$ not obviously well defined in $\mathrm{Mod}(S,q)$ (picked ~~a~~ rep for γ, ~~isotop~~ v.f.)

(3) $P(\gamma)$ is composition of Dehn twists

$$P(\gamma) \sim T_{\gamma_1} \circ T_{\gamma_2}^{-1}$$



<u>Proof of BES</u> • Consider evaluation map $\eta : \mathrm{Diff}(S) \to S$, $f \mapsto f(q)$

<u>Claim 1</u> This defines a fiber bundle

$$\mathrm{Diff}(S,q) \longrightarrow \mathrm{Diff}(S) \longrightarrow S$$

• LES in homotopy $\pi_1 \mathrm{Diff}(S) \to \pi_1(S) \xrightarrow{\delta} \mathrm{Mod}(S,q) \to \mathrm{Mod}(S) \to 1$

<u>Claim 2</u> δ injective.

want to explain proof of claims we'll meet similar seqs later. Treat this one carefully so you can believe later ones...

<u>Pf of Claim 1</u> Need: $\forall\, x \in S$ $\exists$ nbhd $U \ni x$ and homeo

$$\phi : U \times \mathrm{Diff}(S,q) \longrightarrow \eta^{-1}(U).$$

- Reduction 1: each fiber has free right $\mathrm{Diff}(S,q)$ action $\Rightarrow$ enough to find section $\sigma : U \to \mathrm{Diff}(S)$ (so $\sigma(u)(q) = u$)

  Then take $\phi(u,f) = \sigma(u) \circ f$.

  (section $\longleftrightarrow$ to identify fiber w/ Diff ~~identifying id~~ ~~in each fiber~~ choose id.

- Reduction 2: total space $\mathrm{Diff}(S)$ is gp so enough to do this for $x = q$.

- Exercise: $D \subset \mathbb{R}^2$ unit disk. Construct $\sigma : D \to \mathrm{Diff}_c(\mathbb{R}^2)$ s.t. $\sigma(z)(0) = z$.

  $\sigma(z) : x \mapsto x + \rho(x) z$

  $\rho \equiv 1$ on $D(1)$

  $\rho \equiv 0$ on $D(10)$

  $\rho' \ll 1$ everywhere.

Pf of Claim 2

Cheap proof: (Earle-Eells) $\chi(S)<0 \Rightarrow \mathrm{Diff}_0(S)\sim * \Rightarrow \pi_1 \mathrm{Diff}(S)=0$.

Alternative: Understand $\delta: \pi_1(S)\longrightarrow \mathrm{Mod}(S,q)$. (need this any way to identify seq w/ BES

Given $\gamma:[0,1]\longrightarrow S$ repping $[\gamma]\in\pi_1(S)$

$$\begin{tikzcd} & \mathrm{Diff}(S) \arrow[d, "\eta"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde\gamma"] & S \end{tikzcd}$$

Choose $\tilde\gamma$ so $\tilde\gamma(0)=id$.

Then $\delta([\gamma])$ = component of $\tilde\gamma(1)$ in $\mathrm{Diff}(S,q)$.

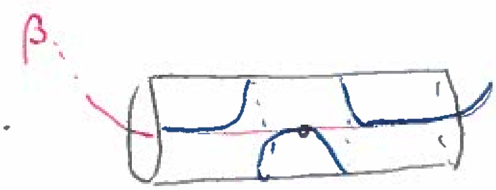
$\mathrm{Diff}(S,q)\cap\mathrm{Diff}_0$

<u>Note</u> $\tilde\gamma$ is point-pushing isotopy $\tilde\gamma(t)(q)=\gamma(t)$.

To see δ injective check that

$\rho: \pi_1(S)\longrightarrow \mathrm{Mod}(S,q)\longrightarrow \mathrm{Aut}(\pi_1(S))$ is inner aut gp.

$\chi(S)<0 \Rightarrow Z(\pi_1(S))=1 \Rightarrow \rho$ injective.



$\beta\longmapsto\gamma^{-1}\beta\gamma$

□.

<u>Rmks</u>

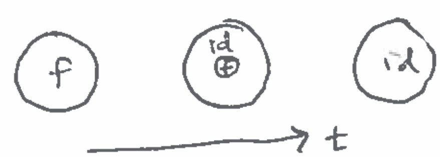
(1) Similar arg gives $1\longrightarrow\pi_1(S_{g,n})\longrightarrow \mathrm{PMod}_{g,n+1}\longrightarrow\mathrm{PMod}_{g,n}\longrightarrow 1$ (rigorously)

Note This recovers $1\longrightarrow F_n\longrightarrow P_{n+1}\longrightarrow P_n\longrightarrow 1$

(2) similar arg applied to $\mathrm{Homeo}(D, n\text{ pts})\longrightarrow\mathrm{Homeo}(D)\longrightarrow\mathrm{Conf}_n(D)$

gives $\pi_1 \cancel{\mathrm{Homeo}(D)}^{0}\longrightarrow B_n\longrightarrow \mathrm{Mod}'_{0,n}\longrightarrow \pi_0\cancel{\mathrm{Homeo}(D)}^{0}$.

Homeo(D) def. ret. to id: f_t:



Alexander trick.

(3) <u>Cor</u> $\mathrm{PMod}_{g,n}$ fg(+) $\Longrightarrow$ $\mathrm{PMod}_{g,n+1}$ fg(+)

Follows from BES and

- for $\gamma \in \pi_1(S)$ rep'd by $\overset{\text{nes.}}{\text{scc}}$ $P(\gamma)$ is prod of Dehn twists
- Dehn twist $T_a \in \mathrm{PMod}_{g,n}$ lifts to DT in $\mathrm{PMod}_{g,n+1}$.

Lecture 3

I. Generating Mod_g (part 2)

Thm (Dehn–Lickorish) Mod_g is fg(+) (gen. by finitely many DT about nonseparating s.c.c.'s)

- Last time: Puncture induction: $g \geq 1$, $n \geq 0$. $\mathrm{PMod}_{g,n}$ fg(+) $\Longrightarrow$ $\mathrm{PMod}_{g,n+1}$ fg(+).

$$\cancel{\text{map}}\ \pi_1(S_{g,n}) \to \mathrm{PMod}_{g,n+1} \to \mathrm{PMod}_{g,n} \to 1.$$

only need this much of seq.

- Today: genus induction. For $g \geq 1$ $\mathrm{PMod}_{g,2}$ fg(+) $\Rightarrow$ Mod_{g+1} fg(+).

Idea (natural to consider stabilizer of scc.)

For ~~[illegible]~~ α: isotopy class of scc.

(1) Define $\mathrm{Mod}(S,\alpha) < \mathrm{Mod}(S)$ mapping classes w/ $\phi(\alpha)=\alpha$.

Forgetful map $1 \to \langle T_\alpha \rangle \to \mathrm{Mod}(S,\alpha) \to \mathrm{Mod}(S \setminus \alpha) \to 1$

(Similar to BES)

(2) How far is $\mathrm{Mod}(S,\alpha)$ from all of $\mathrm{Mod}(S)$?

From $\mathrm{Mod}(S) \curvearrowright \mathcal{C}(S)$ curve complex will see

$$\mathrm{Mod}(S) = \langle \mathrm{Mod}(S,\alpha), T_\beta \rangle \quad \text{for some scc } \beta.$$

II. Curve complex & some geometric group theory.

Assume $\chi(S) < 0$.

Defn For isotopy classes of ~~simple~~ closed curves α, β

$$i(\alpha,\beta) := \min_{\substack{a \in \alpha \\ b \in \beta \\ a \pitchfork b}} |a \cap b|$$

<u>Defn</u> $N_g = N(S_g)$ graph w/ vertices : $\alpha \subset S$ isotopy class of nonsep. scc

edges : $\alpha \bullet\!\!-\!\!\bullet \beta$ if $i(\alpha,\beta)=1$.

<u>Note</u> $\mathrm{Mod}_g \curvearrowright N_g$

<u>Lemma</u> (basic lemma from geometric group theory)

- X proper, geodesic connected metric space
- $G \curvearrowright X$ by proper, by isometries, X/G compact

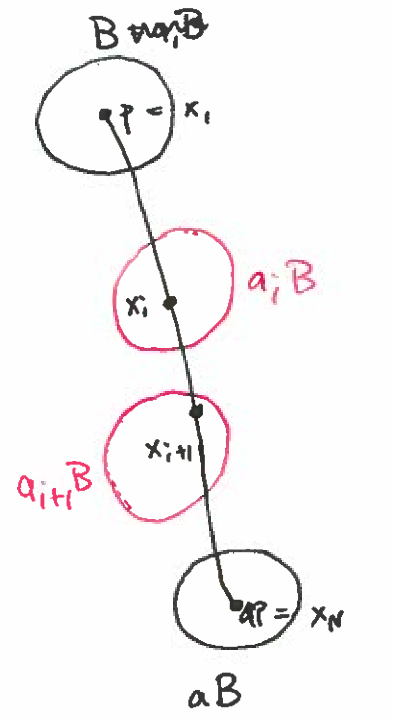
Then G is f.g. Moreover given compact $B \subset X$ whose translates cover X,

$$G = \langle S \rangle \qquad S = \{ h \in G \mid hB \cap B \neq \phi \}.$$

<u>Pf</u> Take $B = B(R,p)$ $R = \mathrm{diam}(X/G)$ any $p \in X$.

- S finite by properness (of X & $G \curvearrowright X$)
- To see $G = \langle S \rangle$.

Define $r = \inf\limits_{g \in G \setminus S} d(B, gB)$.



Given $a \in G$, choose

- path γ $p \to ap$
- $x_1, \dots, x_N$ on γ $d(x_i, x_{i+1}) < r$
- $a_i \in G$ $x_i \in a_i B$. ($a_1 = id$, $a_N = a$)

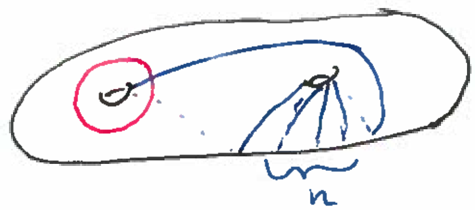
Then $a = (a_1^{-1} a_2)(a_2^{-1} a_3) \cdots (a_{N-1}^{-1} a_N)$

and $a_i^{-1} a_{i+1} \in S$ b/c $d(B, a_i^{-1} a_{i+1} B) = d(a_i B, a_{i+1} B) < r$.

$\Rightarrow a \in \langle S \rangle$. $\square$

Rmks

(1) N_g not proper: vertices have infinite deg



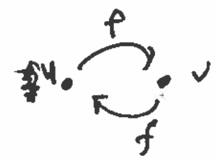
(2) $\mathrm{Mod}_g \curvearrowright N_g$ not proper:

$\mathrm{Mod}(S_g,\alpha) \twoheadrightarrow \mathrm{Mod}(S_g \setminus \alpha)$ not finite.

Lemma' (Exercise) $G \overset{\text{simplicial}}{\curvearrowright} X$ connected simplicial complex

If (i) G transitive on ~~edges~~ vertices & edges

(ii) edge inversion: for (u,v) edge $\exists\ f \in G$ st.



(apply lemma to half edge

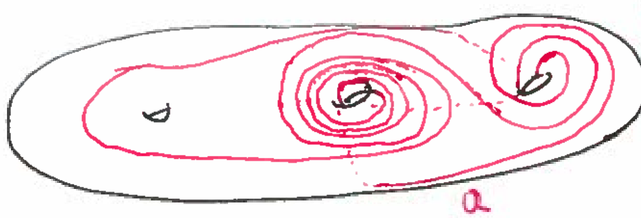
) since action is simplicial $r>0$.

Then $G = \langle G_u, f \rangle$

Summary to express $\mathrm{Mod}_g = \langle \mathrm{Mod}(S,\alpha), f \rangle$ using N_g need transitivity, edge inversion, N_g connected.

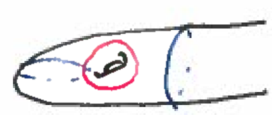
Mod$_g \curvearrowright N_g$

- Transitive on ~~edges~~ vertices: Classification of surfaces. Cpt surface S det. by $\chi(S)$, $\#\,\partial$ comp

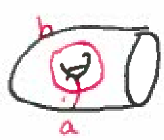


why is there a homeo $f(a)=a'$?

similar for edges: $i(a,b)=1 \implies S_g \setminus N(a \cup b) \sim S_{g-1,1}$



- Edge inversion



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2\mathbb{Z}$ swaps a,b. Equivalently check: $T_aT_bT_a : \begin{cases} b \mapsto a \\ a \mapsto b \end{cases}$

4

Prop $g \geq 2$ N_g connected.

Cor $\mathrm{Mod}_g = \langle \mathrm{Mod}(S_g, \alpha), T_\beta \rangle$ $i(\alpha,\beta) =$
finishes proof.

Pf inductive argument
given two loops, try to surger to lessen $i(\alpha,\beta)$

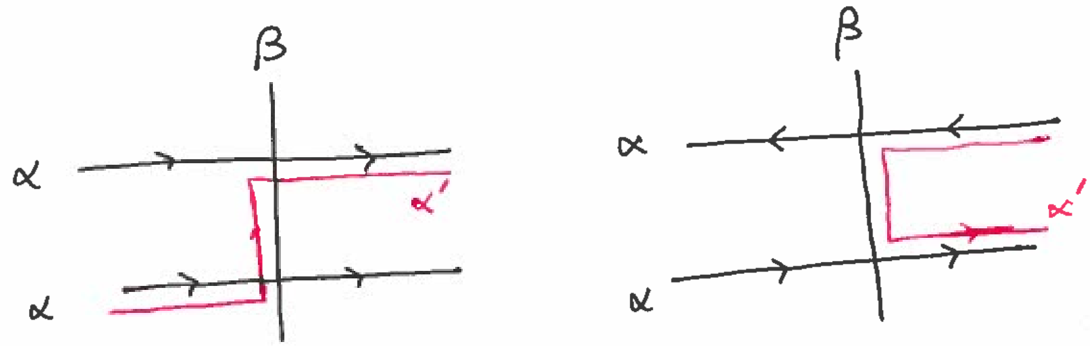
Step 1 ~~Lemma~~ X_g = graph w/ vertices: all scc's (possibly separating)
edges: $i(\alpha,\beta) \leq 1$.

Lemma X_g connected. $(g \geq 2)$

Pf induction on $i(\alpha,\beta)$.

If $i(\alpha,\beta) \geq 2$ want to find α' so $i(\alpha,\alpha') < i(\alpha,\beta)$
$i(\alpha',\beta) \leq i(\alpha,\beta)$.

Surgery:



$i(\alpha,\alpha') = 1$
$i(\alpha',\beta) \leq i(\alpha,\beta) - 1$.

$i(\alpha',\alpha) = 0$
$i(\alpha',\beta) \leq i(\alpha,\beta) - 2$

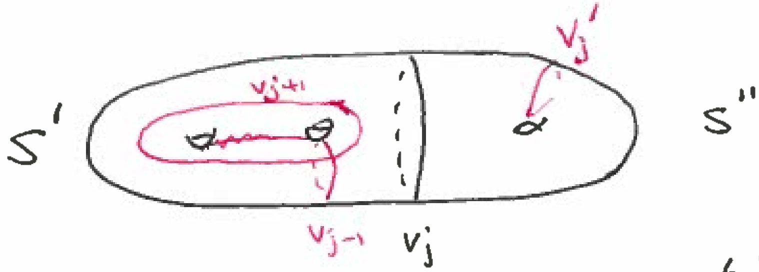
(请 α' not null homotopic b/c cw α,β in min pos.)

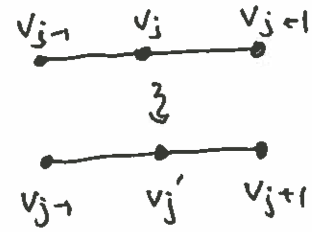
Note α' might separate so only get path in X_g. $\square$.

Step 2 Construct path in N_g from path $\alpha = v_0, v_1, \ldots, v_N = \beta$ in X_g (endpts in N_g)

2 main issues

(i) Some v_j may separate.



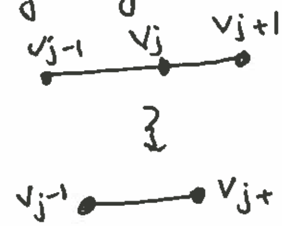


Case 1 $\quad v_{j\pm 1}$ on same ~~sides~~ component of $S\setminus v_j$ (say S'):

Choose v_j' non sep on S''

Case 2 $\quad v_{j\pm 1}$ on diff components:

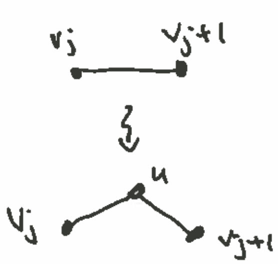
Then $i(v_{j-1}, v_{j+1}) = 0$ Connected by edge.



(ii) may have $i(v_j, v_{j+1}) = 0$.

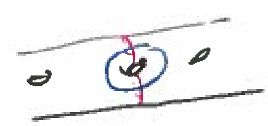
but can always find u s.t. $i(v_j, u) = 1$

$i(u, v_{j+1}) = 1$.



Rmk There are cases based on topology of $S \setminus v_j \cup v_{j+1}$ but all handled similar.

eg.



$\square$

~~Bounding Cohomology~~ Next week

Thm (Harer) $\quad H_1 \mathrm{Mod}_g = 0 \quad g \geq 3$

Thm (Hatcher–Thurston) (algorithm for) finite presentation for Mod_g.

- via Morse–Cerf theory
- leads to computation of $H_2 \mathrm{Mod}_g$ (by Hopf's formula)

Lecture 4

I. Abelianization of $\Gamma_g := \mathrm{Mod}_g$.

<u>Recall</u> For group G $\quad G^{ab} = G/[G,G]$.

<u>Thm</u> (Mumford, Birman, Powell, Harer) For $g \geq 3$ $\quad \Gamma_g^{ab} = 0$.

Mumford: $\mathbb{Z}/10 \twoheadrightarrow \Gamma_g^{ab}$ $\quad g \geq 3$
Birman, Powell: $\Gamma_g^{ab} = 0$ $\quad g \geq 3$
Harer: simple proof

<u>Rmks</u> (1) $\Gamma_1^{ab} \simeq \mathbb{Z}/12 \qquad \Gamma_2^{ab} \simeq \mathbb{Z}/10$

(2) Proof also gives $\left(\mathrm{PMod}_{g,n}^b\right)^{ab} = 0 \qquad g \geq 3$

Note $\quad (\mathrm{Mod}_{g,n})^{ab} \neq 0 \quad$ for $n \geq 2$ since $\mathrm{Mod}_{g,n} \twoheadrightarrow S_n \twoheadrightarrow \mathbb{Z}/2$.

(3) $G^{ab} \simeq H_1(K(G,1)) \equiv H_1(G)$

<u>Cor</u> $\mathrm{Mod}_g^1 \longrightarrow \mathrm{Mod}_{g+1}^1$ induces iso on H_1 $\quad g \geq 3$.

Work from last week gives

<u>Lemma</u> For $g \geq 0$ $\quad \Gamma_g^{ab}$ is cyclic.

<u>Pf</u> • Dehn–Lickorish $\quad \Gamma_g = \langle T_{a_1}, \ldots, T_{a_k} \rangle \quad a_i \subset S_g$ nonseparating scc.

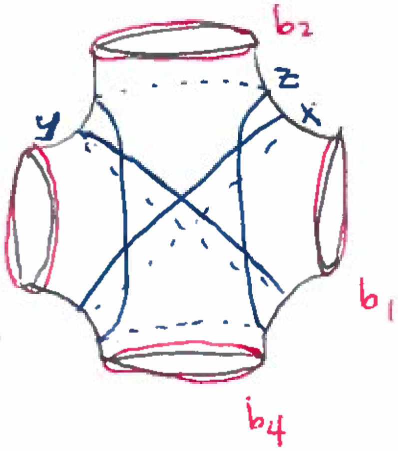
• Observe: if $a, b \subset S_g$ ns scc's then T_a, T_b conjugate in Γ_g.

For $\phi \in \Gamma_g$ w/ $\phi(a) = b$, $\quad \phi T_a \phi^{-1} = T_{\phi(a)} = T_b$. (conjugation changes names)

$\Rightarrow$ under $\pi : \Gamma_g \longrightarrow \Gamma_g^{ab} \qquad \pi(T_{a_i}) = \pi(T_{a_j})$

$\Rightarrow \Gamma_g^{ab} = \langle \pi(T_{a_1}) \rangle$. $\qquad \square$

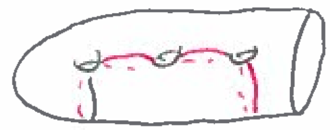
II. Lantern relation



In $\mathrm{Mod}(S_0^4)$

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}$$

- Lanterns appear in genus ≥ 3



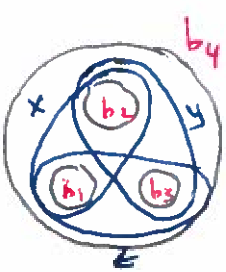
i.e. $\mathrm{Mod}(S_0^4) \hookrightarrow \Gamma_g \quad g \geq 3$

<u>Cor / Pf of Thm</u> Lantern rel. $\Longrightarrow 3\,\pi(T_a) = 4\,\pi(T_a) \Rightarrow \pi(T_a) = 0.$

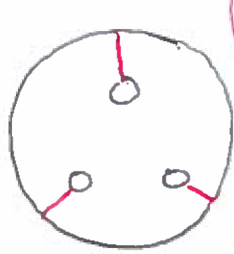
Proof 1 of lantern relation

<u>Lemma</u> (Alexander method) $\phi \in \mathrm{Mod}(S)$ determined by action on collection of curves/arcs whose complement is a disk.

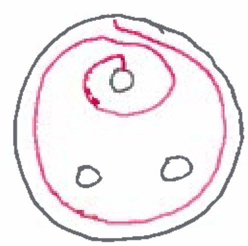
(fine point: curves have to be in min pos. no triple intersections!)



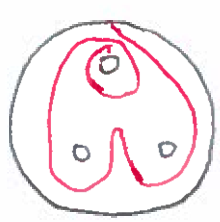
(explain lemma for this example)



$\xmapsto{T_{b_1} \cdots T_{b_4}}$



$\downarrow T_x T_y T_z.$



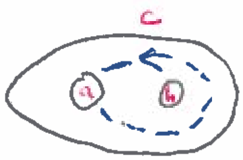
(do this carefully explaining how to compute action of DT by ~~[illegible]~~ resolving crossings!)

Similar for other arcs. ~~etc.~~

$\square$

Rmk By Alexander method, relations in Mod(S) are easy to verify but in practice hard to discover. (Hatcher Thurston next time)

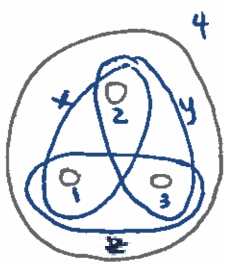
Proof 2 of Lantern relation (conceptual)

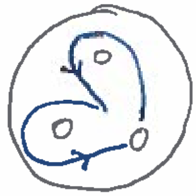
- On 

consider $P^c_{ab} \in \mathrm{Mod}(S_0^3)$ "push a around b inside c"

(be careful not to rotate a when pushing think ~~do-si-do~~ do-si-do)

- $\mathrm{Mod}(S_0^3) = \langle T_a, T_b, T_c \rangle$ Check $P^c_{ab} = T_c^{-1} \circ T_b \circ T_a$

- For 


$P^z_{b_3 b_1} \circ P^y_{b_3 b_2} = P^{b_4}_{b_3 x}$

$$\Rightarrow T_z^{-1} T_{b_1} T_{b_3} T_y^{-1} T_{b_2} T_{b_3} = T_{b_4}^{-1} T_{b_x} T_{b_3}$$

$$\Rightarrow T_{b_1} T_{b_2} T_{b_3} T_{b_4} = T_x T_y T_z .$$

□.

Note: T_{b_i}'s commute & T_x, T_y, T_z commute w/ T_{b_i}'s

II. Uniformly perfect groups and group actions on S^1

<u>Defn</u> • A group G s.t. $G = [G,G]$ is called <u>perfect</u>

• $\forall g \in G$ can write $g = \prod_{i=1}^{N} [a_i, b_i]$

smallest possible N is called <u>commutator length</u> $cl(g)$.

• if $\exists k$ st. $cl(g) \leq k$ $\forall g \in G$, G called uniformly perfect

<u>Q</u>: Is Mod_g uniformly perfect?

<u>Examples</u> $SL_n\mathbb{Z}$ $n \geq 3$, $Sp_{2n}\mathbb{R}$, $\mathrm{Homeo}_c(\mathbb{R}^n)$, $\mathrm{Homeo}(S^n)$

<u>Application</u> (Dynamics of circle homeos)

illustrate briefly one ex. of how uniform perfection comes in in low-dim top/dynamics

• $1 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}(S^1) \to \mathrm{Homeo}(S^1) \to 1.$ $(*)$

$$\widetilde{\mathrm{Homeo}}(S^1) = \{ \underset{\text{homeo}}{F: \mathbb{R} \to \mathbb{R}} \mid F(t+1) = F(t) + 1 \}.$$

<u>Defn</u> (Poincaré) translation number $\tau(F) = \lim_{n \to \infty} \frac{F^n(0)}{n} \in \mathbb{R}.$

$\rightsquigarrow$ conjugacy invariant $r: \mathrm{Homeo}(S^1) \to \mathbb{R}/\mathbb{Z}$ "rotation number"

$$f \longmapsto \tau(\tilde{f}) \bmod \mathbb{Z}$$

(Poincaré): $r(f) \in \mathbb{Q}/\mathbb{Z} \Leftrightarrow f$ has periodic orbit. $r(f)$ irrational $\Leftrightarrow$ of conjugate to irrat. rotation

$\rightsquigarrow$ <u>bounded</u> 2-cocycle $[e] \in H^2_b(\mathrm{Homeo}(S^1); \mathbb{R})$

$$e(f,g) = \tau(\tilde{f}\tilde{g}) - \tau(\tilde{f}) - \tau(\tilde{g})$$

refinement of class in $H^2(\mathrm{Homeo}(S^1); \mathbb{Z})$ of extension $(*)$

• $\mathrm{Homeo}(S^1)$ uniformly perfect $\Rightarrow$ $H^2_b(\mathrm{Homeo}(S^1); \mathbb{Z}) \simeq H^2(\mathrm{Homeo}(S^1); \mathbb{Z})$

So Euler class has unique lift to ~~bounded~~ H^2_b.

Thm (Ghys) Given $\rho_i : \Gamma \to \mathrm{Homeo}(S^1)$ $\quad i=1,2$

$$\rho_1^*(e) = \rho_2^*(e) \quad (\text{in } H^2_b(\Gamma;\mathbb{Z})) \iff \rho_1, \rho_2 \text{ (semi) conjugate.}$$

Rmk For $\Gamma = \mathbb{Z}$ $\quad \rho^*(e) \in H^2_b(\mathbb{Z};\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ is rotation number $r(\rho(1))$.

(

Thm (Endo-Kotschick) Mod_g is not uniformly perfect.

Pf idea Fix separating $a \subset S_g$

Strategy: show $\mathrm{cl}(T_a^k) \to \infty$.

- writing $T_a^k = \prod_{i=1}^{N} [a_i, b_i] \longleftrightarrow \pi_1(S_{N,k}) \to \mathrm{Mod}_g$.

$\longleftrightarrow$ Lefschetz fibration $\begin{matrix} S_g \to X_k \\ \downarrow \\ S_N \end{matrix}$ $\Big|$ X_k is 4-dim'l symplectic mfld.

- Sieberg-Witten theory (Taubes): $c_1^2(X_k) \geq 0 \quad \forall k$.
- Computation: if $\mathrm{cl}(T_a^k)$ bounded then

$$c_1^2(X_k) = 3\,\mathrm{sig}(X_k) + 2\chi(X_k) < 0 \text{ for } k \text{ large.} \quad \times$$

$\square$

Lecture 5

I. Finding presentations

Example $\Gamma = SL_2\mathbb{Z} \curvearrowright \mathbb{H}^2$
$\cup$
X connected graph

Generators

- pick fund dom $\Gamma \curvearrowright X$

$\Rightarrow \Gamma = \langle S \rangle$ $\quad S = \{ g \in G \mid gF \cap F \neq \phi \}$

$= G_i \cup G_\zeta \cup G_\infty$ (vert. stabilizer

- $G_i = \langle \underset{A}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \rangle \simeq \mathbb{Z}/4$ $\quad G_\zeta = \langle \underset{B}{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}} \rangle \simeq \mathbb{Z}/6.$

$G_\infty = \langle \underset{C}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \underset{D}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} \rangle \simeq \mathbb{Z} \times \mathbb{Z}/2$

Relations

- edge stabilizers $\quad G_{(\zeta,\infty)} = \langle \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \rangle = G_{(\zeta, i)}$

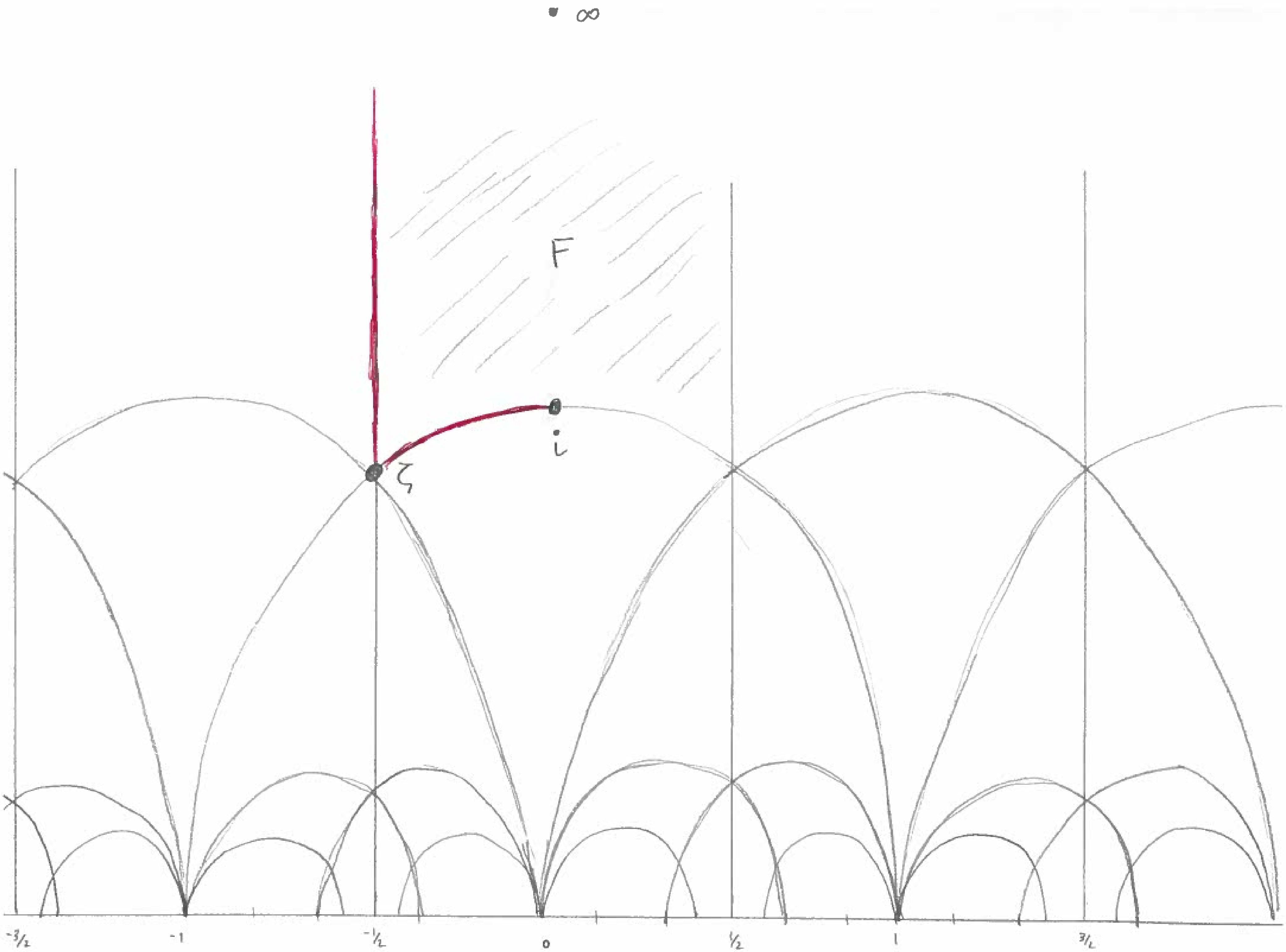
$\Rightarrow A^2 = B^3 = D.$

- face relation $\quad AB = C.$

$\Rightarrow G = \langle A, B, C, D \mid \begin{matrix} A^4 = 1 = B^6 \\ A^2 = B^3 = D \end{matrix}, AB = C \rangle \cong \langle A, B \mid \begin{matrix} A^4 = 1 = B^6 \\ A^2 = B^3 \end{matrix} \rangle$

$\simeq \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6.$

Cor $(\mathrm{Mod}_1)^{ab} \simeq \mathbb{Z}/12 = \langle \overline{AB} \rangle.$



∞
F
i
ζ
-3/2
-1
-1/2
0
1/2
1
3/2

General algorithm for finding presentations

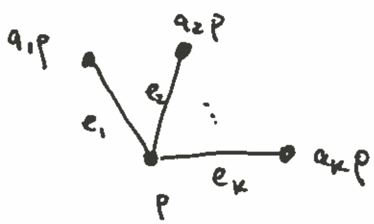
<u>Setup</u> • X 2-dim'l polyhedral complex, $\pi_1(X)=0$.

• $G \curvearrowright X$ by cellular homeos. Assume

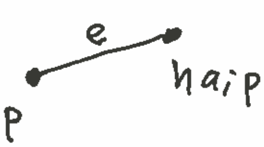
- $G \curvearrowright X^{(0)}$ transitive. Fix $p \in X^{(0)}$. $H := G_p$
- $E :=$ edges meeting p $\quad H \curvearrowright E, F$.
  $F :=$ faces meeting p

Generating G

• choose reps $e_1, \dots, e_k$ for E/H and choose $a_1, \dots, a_k$ s.t.



<u>Note</u> any $e \in E$ has form

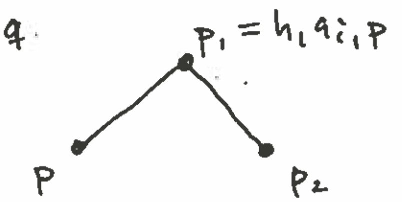


some $h \in H$, $i = 1, \dots, k$.

(any q w/ $d(p,q)=1$ has $q = h a_i p$ since $e_1, \dots, e_k$ are coset reps)

• for any edge path $p \to q$ ~~[illegible]~~, get word w in $\{a_1, \dots, a_k\} \cup H$ with $wp = q$.

<u>ex</u>



$h_1 a_{i_1}$ sends nbhd of p to nbhd of p_1 so $\exists$ $h_2 a_{i_2}$ s.t.

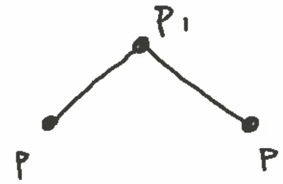
$h_1 a_{i_1}(h_2 a_{i_2} p) = p_2$.

• for $g \in G$ choose path $p \to gp$ $\rightsquigarrow$ $w \in \langle a_i, H \rangle$ with $wp = gp$.

$\Rightarrow$ $w^{-1}g \in H$ $\Rightarrow$ $g \in \langle a_1, \dots, a_k, H \rangle$.

Relations

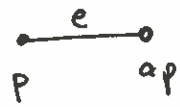
(1) back tracking



$\Rightarrow$ $h_1 a_{i_1} h_2 a_{i_2} p = p$

$\Rightarrow$ $\boxed{h_1 a_{i_1} h_2 a_{i_2} \in H}$

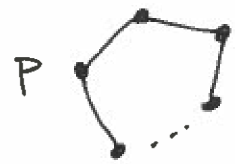
(2) edge stabilizers For



if $te = e$ (fixing both vertices)

then $tap = ap \Rightarrow \boxed{a^{-1} t a \in H}$

(3) faces



$\boxed{h_1 a_{i_1} h_2 a_{i_2} \cdots h_r a_{i_r} \in H}$

<u>Basic Fact</u> This gives presentation of G up to

(i) presentation of H

(ii) ~~identification~~ expressing of each relation in terms generators of H.

In particular G is f.p. if

(a) X/G compact (so (1) & (3) give fin. many relations)

(b) vertex stab H is f.p.

(c) each edge stab. f.g. (so 2 gives fin many relations)

<u>Rmk</u> This latter fact can be shown abstractly. Consider $EG = \widetilde{K(G,1)} \sim *$.

$$EG \to \frac{EG \times X}{G} \xrightarrow{\pi} X/G$$

<u>Note</u> • $\pi^{-1}(\text{vertex } v) \simeq K(G_v, 1)$ similar for edges / faces.

$G/G_v \to \frac{EG \times G/G_v}{G} \to BG$

• $\pi_1\left(\frac{EG \times X}{G}\right) = \pi_1(BG) = G$ since $\pi_1(X) =$

Build model of $\frac{EG \times X}{G}$ w/ finite 2-skeleton from

$K(G_v, 1)$, $\Delta^1 \times K(G_e, 1)$, $\Delta^2 \times K(G_f, 1)$ each of which has finite 2-sk. by ass.

II. Presentation for Mod_g.

Thm (Hatcher–Thurston) An explicit pres of Mod_g can be derived from $\mathrm{Mod}_g \curvearrowright X_g$ cut system complex.

Rmks (1) Worked out ~~explicitly~~ by Wajnryb

(2) Mod_g f.p. known earlier (McCool, Deligne–Mumford)

McCool – algebraic methods ; algebraic geo arg: Deligne–Mumford compactification $\Rightarrow$ $\mathcal{M}_g$ is quasiproj variety ; also has smooth finite cover qed.

Defn A cut system on S_g is a collection $\{c_1, \ldots, c_g\}$ of disjoint scc's s.t. $S_g \setminus \bigcup c_i \cong S^2 \setminus (2g \text{ points})$

The isotopy class of $\{c_1, \ldots, c_g\}$ denoted $\langle c_1, \ldots, c_g \rangle$.

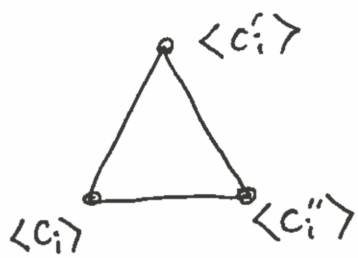
Defn A simple move between cut systems

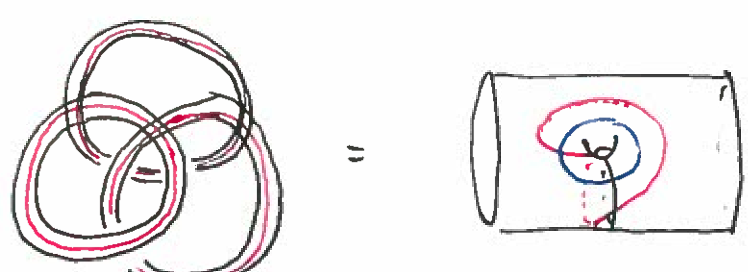
$$\{c_1, \ldots, c_i, \ldots, c_g\} \longrightarrow \{c_1, \ldots, c_i', \ldots, c_g\}$$

with $i(c_i, c_i') = 1$.

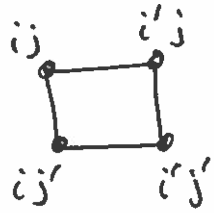
Defn simple relations

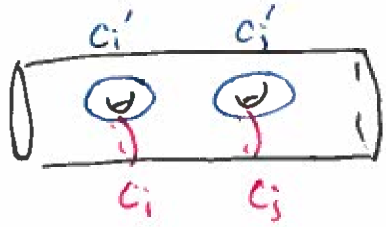
(1)



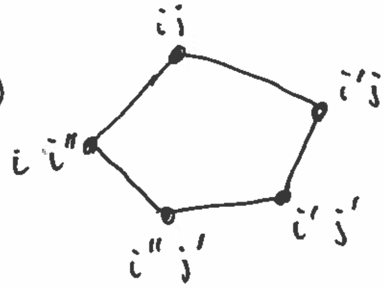


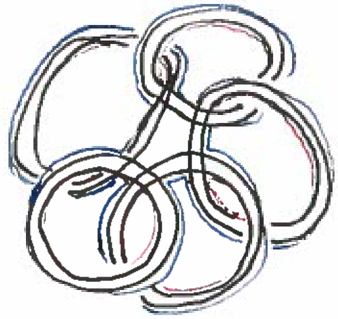
(2)

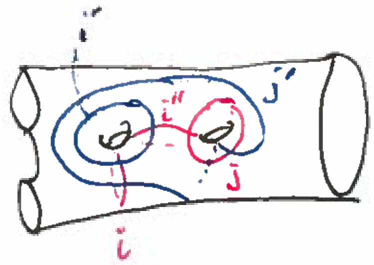




(3)







<u>Defn</u> Cut system complex X_g

vertices - cut systems up to isotopy

edges - simple moves   faces - simple relations.

To use $X = X_g$ to show $\Gamma = \mathrm{Mod}_g$ f.p. need.

Vertex stab f.p. } Birman exact seq. type argument.
Edge stab f.g. }

X/Γ Compact } classification of surfaces
$\Gamma \curvearrowright X^{(0)}$ Transitive }

X Simply connected } Morse-Cerf theory.

I. Presenting Mod_g (part 2) Lecture 6

Last time

(1) defined cut system complex X_g

vertices - cut systems   edges - simple moves btwn cut sys's   faces - $\triangle, \square, \pentagon$, corresp. to certain edge paths

Example $g=1$

vertex = scc

edge = (α, β) s.t. $i(\alpha,\beta)=1$

face = (α,β,γ) s.t. each pair has $i(-,-)=1$

$$\{\text{vertices}\} \longleftrightarrow \mathbb{Q} \cup \infty$$

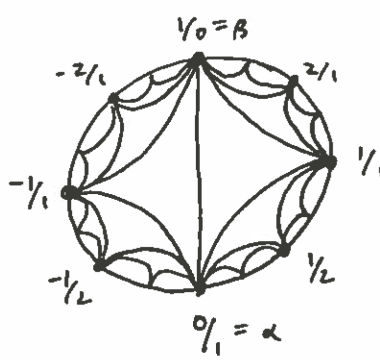
$$\gamma \longmapsto \frac{\hat{i}(\gamma,\hat{\alpha})}{\hat{i}(\gamma,\hat{\beta})}$$

$\hat{i}$ = alg. int. #

$\hat{\alpha}$ = α w/ or

(quotient doesn't dep on or on γ)

X_1 = Farey complex



(Rmk: Same pic as last class. $SL_2\mathbb{Z}$ acts.)

(2) To use $X = X_g$ to present $\Gamma = \mathrm{Mod}_g$ need.

finite pres for Vertex stab.

finite gen set for Edge stab.

X/Γ Compact

$\Gamma \curvearrowright X^{(0)}$ Transitive

X Simply connected

c.o.s.

Exercise Use classification of surfaces to show there are finitely many orbits of triangles.

vertex stabilizer Γ_v

$$\mathbb{Z} \to \underbrace{\mathbb{Z}^g \times P_{2g-1}}_{\substack{\text{if curves not}\\ \text{permuted, cut}\\ \text{along open annuli}}} \longrightarrow \overset{①}{\Gamma_v} \longrightarrow \overset{②}{\underbrace{\Big(\bigoplus_g \mathbb{Z}/2\Big) \rtimes \mathrm{Sym}(g)}_{\substack{\text{permutations of}\\ \{c_1,\dots,c_g\} \text{ w/ sign.}\\ (= \pi_0 \mathrm{Diff}(\amalg_g S^1))}}} \longrightarrow 1$$

$\mathbb{Z} \to T_{\partial D_{2g-1}} \circ T_{\partial D}^{-1}$

Γ_v = mapping classes pres. $v = \langle c_1, \dots, c_g \rangle$

→ - pure braids of disks $D_1, \dots, D_{2g-1}$

- Dehn twists $T_{\partial D_{2i-1}}$ $\left(T_{\partial D_{2i-1}} \circ T_{\partial D_{2i}}^{-1} = 1 \text{ in } \Gamma_v \right)$

can derive finite presentation of Γ_v from this

Main task **Thm** (Hatcher-Thurston) X_g simply connected.

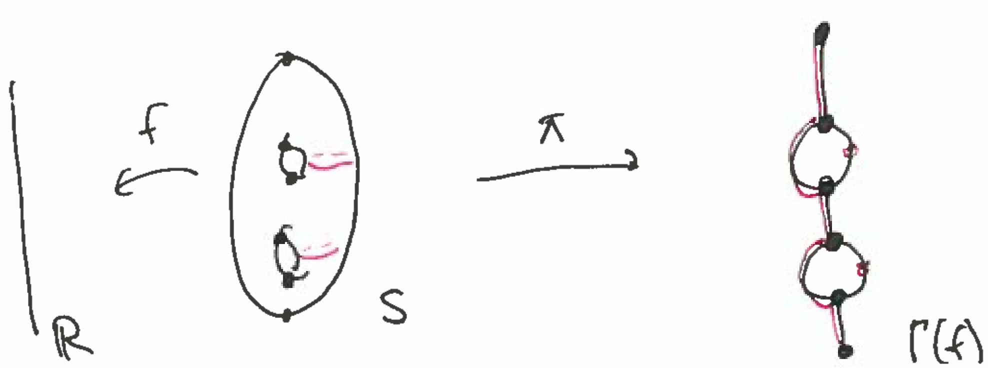
II. Morse-Cerf theory

Observation Cut systems arise from "marked" Morse fxns.

$f : S \to \mathbb{R}$ generic Morse (crit pts nondegenerate and have distinct values)

Defn (Reeb graph) $\Gamma(f) = S/\sim$ $\quad x \sim y$ if in same component of $f^{-1}(f(x))$

has str. of graph w/ vertices $\leftrightarrow$ critical pts



Rmk Maximal tree $T \subset \Gamma(f)$ determines cut sys.
& every cut system arises like this (map to std cut sys & take height)

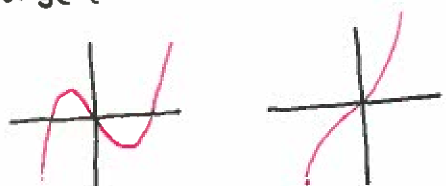
Strategy for Thm lift problem to $C^\infty(S_g)$

$$\begin{array}{c} C^\infty(S_g) \\ \cup \\ \mathrm{Morse}(S_g) \twoheadrightarrow X_g^{(0)} \end{array}$$

e.g. to show X_g connected,
given $v_0, v_1 \in X_g^{(0)}$
- lift to $(f_i, T_i \subset \Gamma(f_i))$ $i=0,1$
- choose generic path $f_t \subset C^\infty(S_g)$
- study non-Morse pts ~~the obstruction~~ and extract path in X_g.

Main feature to understand/exploit Morse(S_g) not connected.

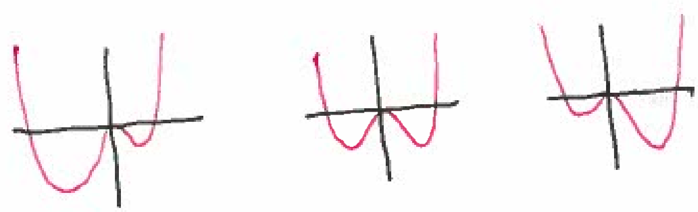
Examples ① Morse$(\mathbb{R})$ not connected



$x^3 - x$ $\qquad$ $x^3 + x$

path $t \mapsto x^3 - tx$ has deg. c.p. at $t=0$
(This path is generic)

② c.p.s may cross



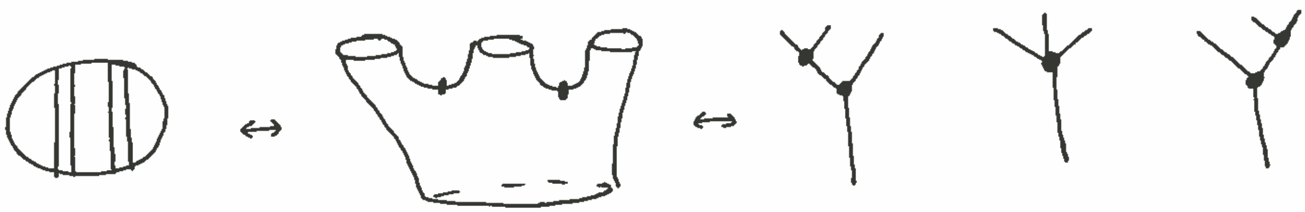
Thm (Cerf) f_0, f_1 Morse, $f_t \subset C^\infty(S_g)$ generic. Then f_t Morse w/ exceptions $0 < t_1 < \cdots < t_r < 1$ of form
(i) (birth/death) f_{t_i} has deg. c.p. $p \in S$ s.t. near p, t_i
$$f(x,y) = x^3 \pm (t - t_i)x \pm y^2$$
(ii) (crossing) f_{t_i} has cp's p, q w/ same value.

Degeneracies and their Reeb graphs

<u>Birth / Death</u>

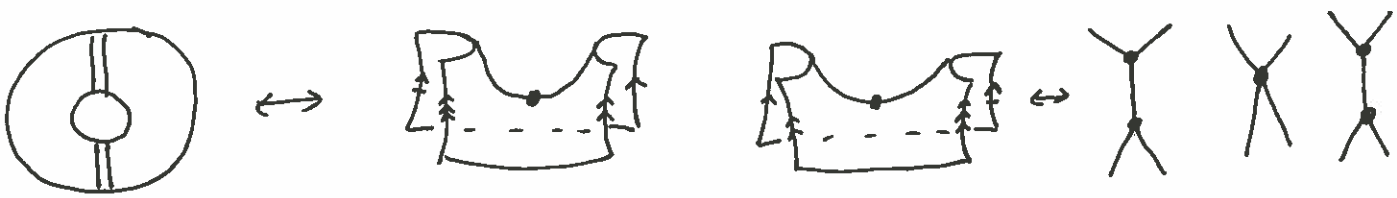
<u>Crossing</u> (one type for each way to attach 1-handles to collection of circles with connected result)

(i)



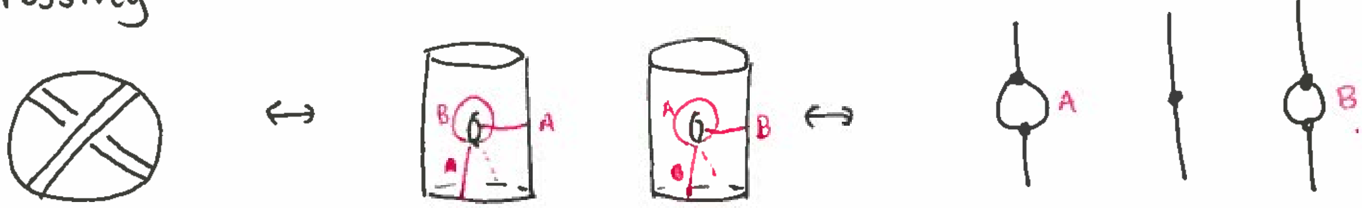
(ii)

(iii)



(iv)

(v) essential crossing



Degeneracies (handout)

Important pt At essential crossing cut sys's corresp. to

$T_{t_i \pm \varepsilon} \subset \Gamma(f_{t_i \pm \varepsilon})$ differ by simple move.

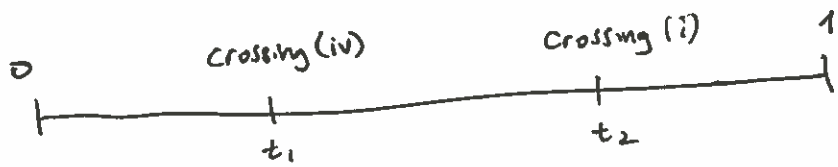
Thm X_g connected

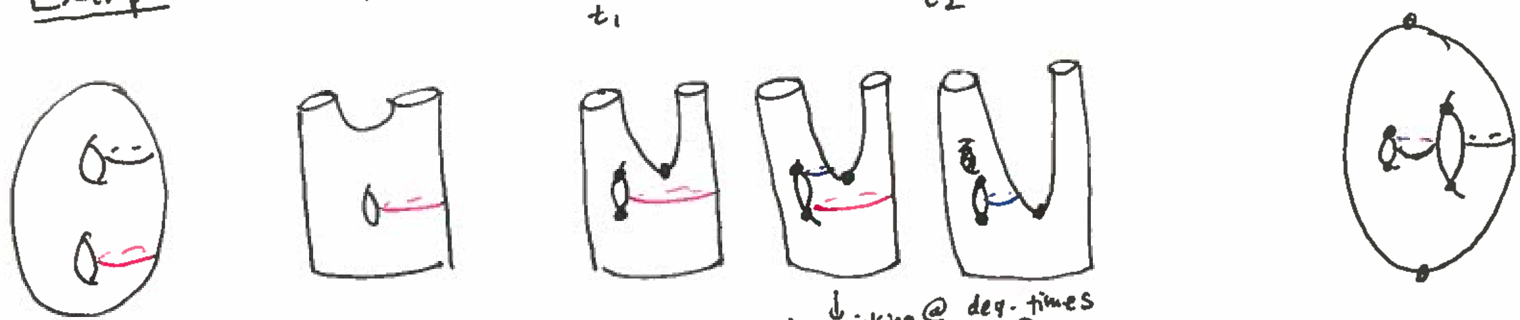
Pf Sketch Given $v_0, v_1 \in X_g^{(0)}$

(1) Lift to $(f_i, T_i \subset \Gamma(f_i))$ $i = 0,1$, extend to f_t generic

(2) Extract edge path

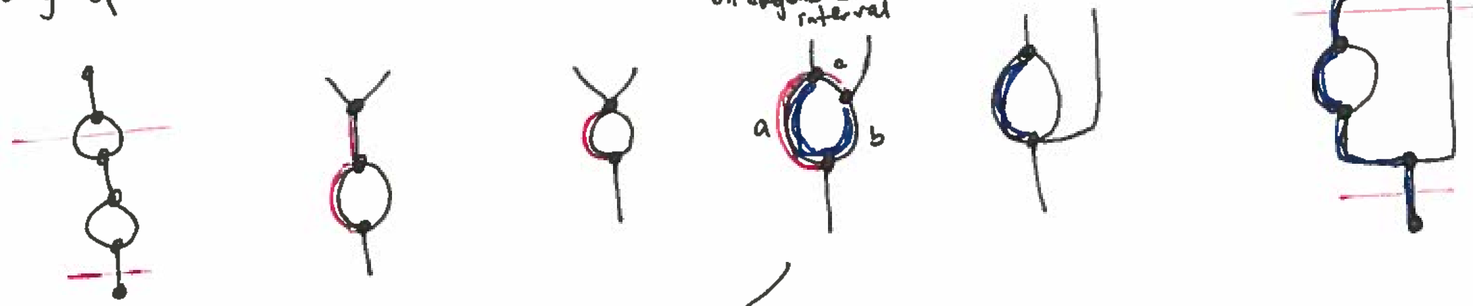
Example





(after picking @ deg. times scc's / cut systems) don't agree! on interval between.

Reeb graphs

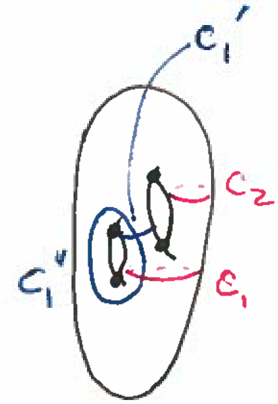


trees differ by elementary move $T \mapsto T + b - c$

For trees differing by elementary move,

cut systems differ by path of length 2:

lift cycle $a \cup b \cup c$ to $C_1'' \subset S$



Then $\langle C_1, C_2\rangle$ — $\langle C_1'', C_2\rangle$ — $\langle C_1', C_2\rangle$

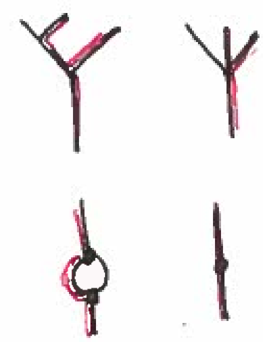
get path in X_g.

In general (for general path $f_t\}$ w/ degeneracies $t_1, \dots, t_r$)

(A) choose max tree $T_{t_i} \subset \Gamma(f_{t_i})$ $i = 1, \dots, r$

(B) extend to nbhd of t_i in obvious way

(add collapsed edge to tree)



cut systems for $T_{t_i \pm \varepsilon}$ nonisotopic

iff t_i is essential crossing in which

case corresp. cut systems differ by simple move.

(C) in between t_i & t_{i+1} trees differ by elementary

move $\Rightarrow$ cut systems differ by sequence of (2)

simple moves.

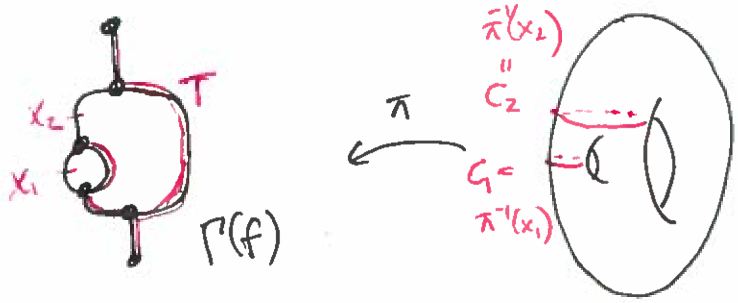
$\square$.

I. Finishing Hatcher–Thurston. $\boxed{\text{Lecture 7}}$

Recap Goal: cut sys X_g is simply connected.

Last time

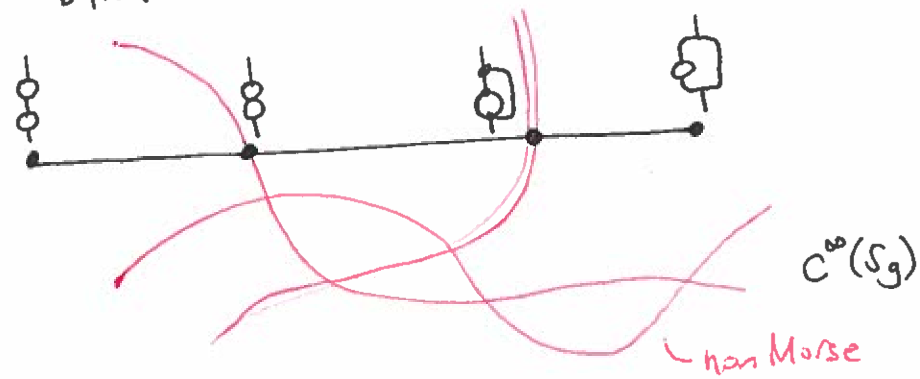
- marked Morse function $(f, T) \rightsquigarrow$ cut system.



(to go from $\Gamma(f)$ to S thicken $\Gamma(f)$ in $\mathbb{R}^3$)

- Morse–Cerf theory: describes degeneracies of generic path f_t btwn Morse functions.

- X_g connected



Remarks on showing $\pi_1(X_g) = 0$. (similar to $\pi_0(X_g) = 0$)

WTS any ~~[illegible]~~ loop in X_g can be filled w/ some combo of cells $\Delta, \square, ⬠$ (ie the 2-cells used to define X_g)

(1) Given loop in X_g lift to $C^\infty(S)$, fill w/ generic disk.

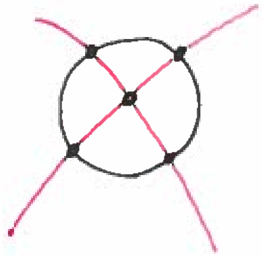
(2) Morse–Cerf: describes degeneracies of generic 2-param family f_{tu}

eg. (*) may have time (t_0, u_0) where index 1 cps p, p' have same value, q, q' — " —

(**) may have time where index-1 cp's p, p', p'' — " —

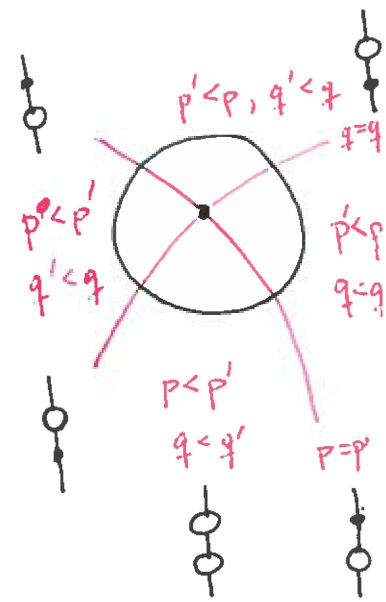
(in total 6 cases + subcases based on type of degeneracies)

(3) Key:



$C^\infty(S)$

look at loop in X_g induced by loop around "codim 2" degeneracy. Show it bounds polygon △, □ or ⬠ (since ∃ finitely many types of deg. there are finitely many cases)



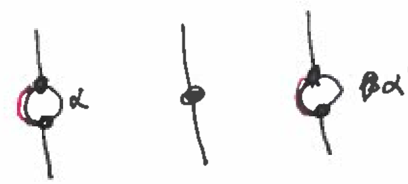
2 representative examples of codim 2 degeneracies

(1) 2 pair of index-1 c.p.'s have same value

ie 2 crossing degeneracies happen simultaneously

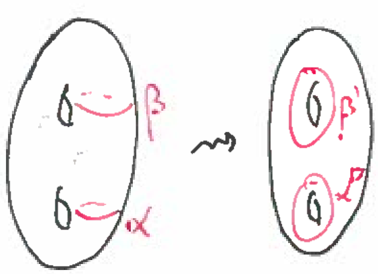
Ex: 2 essential crossings.

Recall Effect of essential crossing on cut system is simple move



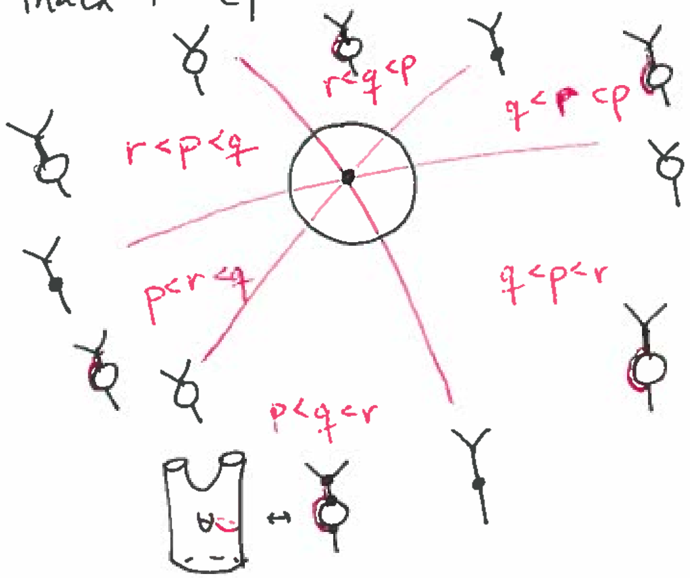
$i(\alpha, \alpha') =$

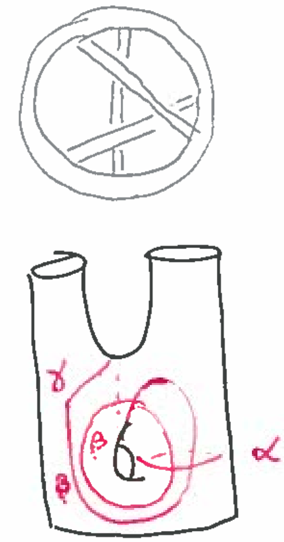
Corresponding ~~path~~ loop in X_g bounds □



$$\begin{array}{ccc} \langle \alpha, \beta \rangle & \text{———} & \langle \alpha', \beta \rangle \\ | & & | \\ \langle \alpha, \beta' \rangle & \text{———} & \langle \alpha', \beta' \rangle \end{array}$$

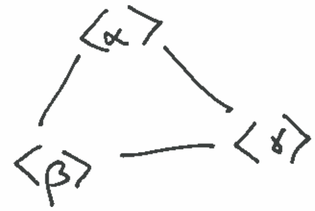
(2) 3 index-1 cps have same value





Ex.

This loop projects to



in X_g. □

- This completes proof that $\mathrm{Mod}_g \curvearrowright$ ~~B~~ X_g can be used to produce f.p.
- Wajnryb ~~[illegible]~~ carried this out (discuss the relations next time)
- Explain how presentation can be used to compute $H_2 \mathrm{Mod}_g$.
- Today Hopf formula: general fact from gp hom. that ~~expressed~~ ~~relations~~ expresses $H_2 G$ in terms of finite ~~the~~ presentation

II. Presentations & $H_2(G)$.

- $G \overset{=\langle S \mid R\rangle}{\;}$ finitely presented. $\quad F := \langle S \rangle$ free group. $\quad N := \langle\langle R \rangle\rangle \triangleleft F$ (so $G \simeq F/N$).

Thm (Hopf) $H_2(G;\mathbb{Z}) \simeq \dfrac{N \cap [F,F]}{[N,F]}$. "relations that are commutators modulo trivial ones."

Rmk presentation gives beginning instructions for building $K(G,1)$

from $w \in N \cap [F,F]$ can give map $S_g \to K(G,1)$
$\overset{\shortparallel}{\prod_1^g [a_i, b_i]}$ and hence $[S_g] \in H_2(K(G,1))$.

Rmk HF follows from 5 term exact seq for SES

$$1 \to N \to F \to G \to 1.$$

$$H_2F \to H_2G \to (H_1N)_G \to H_1F \to H_1G \to 0.$$

- $H_2F = 0$ (F free group) $\quad H_1F = F/[F,F]$

- **Claim** $(H_1N)_G \simeq N/[N,F]$

F preserves $[N,N]$ so F acts on $N/[N,N]$. $N<F$ acts trivial so $G \curvearrowright N/[N,N]$.

pf: $H_1N = N/[N,N]$

- $F \curvearrowright N$ by conj $\rightsquigarrow G \curvearrowright \dfrac{N}{[N,N]}$
- coinvariants adds relations $[fnf^{-1}] = [n]$ $\quad f \in F$, $n \in N$.

$\dfrac{N}{[N,F]} \simeq (H_1N)_F \simeq (H_1N)_G$

• then $H_2(G) = \ker\left(\frac{N}{[N,F]} \longrightarrow \frac{F}{[F,F]}\right) = \frac{N \cap [F,F]}{[N,F]}$

coset of n in ker $\Leftrightarrow$ lands in $[F,F]$

<u>Example</u> $G = H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}$ Heisenberg group.

$$= \langle x,y,z \mid \underbrace{[x,z]}_{A} = 1,\ \underbrace{[y,z]}_{B} = 1,\ \underbrace{[x,y]z^{-1}}_{C} = 1 \rangle$$

$x = (1,0,0)$ $y = (0,1,0)$ $z = (0,0,1)$ w/ coords (a,b,c).

<u>Use Hopf to show</u> $H_2(G)$ has rank ≤ 2

• N normally generated by $R = \{A,B,C\}$.

$\Rightarrow N/[N,F]$ generated by cosets of A,B,C.

ie $u[N,F] \in N/[N,F]$ can be written as $u = A^j B^k C^l$

<u>Q</u>: When $u \in N \cap [F,F]$ (up to $[N,F]$)?

(ie when $u[N,F] = v[N,F]$ where $v \in [F,F]$)

Necessary condition: total exponent of x,y,z in u must be zero.

– total exp of z in A is 0

in B is 0

in C is -1

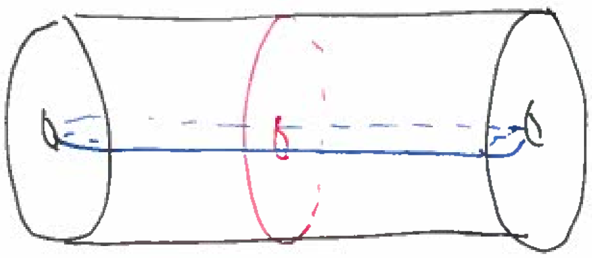
$\Rightarrow$ if $u[N,F] \in N \cap [F,F]/[N,F]$ then $l = 0$.

$\Rightarrow H_2G$ is quotient of $\mathbb{Z}^2$ (note both $A,B \in N \cap [F,F]$)

(not clear there aren't more relations btween A, B...)

In fact $H_2(G) \simeq \mathbb{Z}^2$.

<u>Rmk</u> $G = \pi_1(X)$ where $\begin{array}{c} T^2 \longrightarrow X \\ \downarrow \\ S^1 \end{array}$ X = mapping torus of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.



two tori giving generators of $H_2(G) \simeq H_2(X)$.

Lecture 8

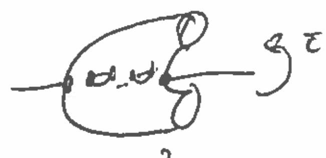
Today Apply Hopf formula to Mod_g.

Cor (of next time) $0 \neq H^2\,\mathrm{Mod}_g \simeq H^2\,\mathrm{BDiff}(S_g)$ nontrivial cc.

I. Relations in braid groups & Mod_g.

- hyperelliptic involution

S_g^1



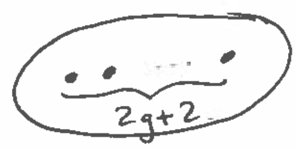
S_g^2

- Symmetric mapping class group $\mathrm{SMod}(S) < \mathrm{Mod}(S)$

  mapping classes commuting w/ τ.

- $\mathrm{SMod}(S) \xrightarrow{\pi} \mathrm{Mod}(S/\tau)$ = braid group.

$S_g^1/\tau =$ 

$S_g^2/\tau =$ 

Thm (Birman-Hilden) $\mathrm{SMod}_g^1 \simeq B_{2g+1}$, $\mathrm{SMod}_g^2 \simeq B_{2g+2}$.

π is isomorphism.

$\Rightarrow$ relations in B_n give relations in Mod_g.

(1) braid relation "half twist" σ lifts to Dehn twist T_a.

$T_a(x) = y$.

$\sigma\tau\sigma = \tau\sigma\tau$

braid relation lifts to $\boxed{T_a T_b T_a = T_b T_a T_b}$

(2) Chain relation. $(\sigma\tau)^3 = T_\delta$ (generates center of B_3.)

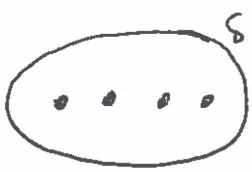
T_δ lifts to "half twist" of ∂. (hold ∂ fixed and twist surface by 180°)

does not lift to DT

T_δ^2 lifts to T_d. $\rightsquigarrow$ $\boxed{(T_a T_b)^6 = T_d}$

2-chain.

version on S_1^2



$(\sigma_1\sigma_2\sigma_3)^4 = T_\delta$

$\boxed{(T_a T_b T_c)^4 = T_d T_e}$

3-chain.

<u>Thm</u> (Wajnryb) Mod_g^1 has presentation

$$\langle a_0, \dots, a_{2g} \mid D_{ij}, B_{i,j}, C, L \rangle.$$

- $a_i = T_{c_i}$ Dehn twist about nonsep scc.
- D_{ij} disjointness $[a_i, a_j] = 1$ if $i(c_i, c_j) = 0$.
- B_{ij} braid - C 3 chain - L lantern.

II. ~~$H_2 Mod_g^1$~~ Hopf & $H_2 Mod_g^1$

<u>Thm</u> $H_2 Mod_g^1$ cyclic $g \geq 4$. (in fact $= \mathbb{Z}$ homological stability)

(in general hard to apply Hopf - kind of amazing that it works for Mod_g)

<u>Proof</u> (Pitsch)

<u>Recall</u> (Hopf) $G = \langle S \mid R \rangle$ $F = \langle S \rangle$ $N = \langle\langle R \rangle\rangle$

$$H_2 G = N \cap [F,F] / [N,F].$$

<u>Observation</u> (from yesterday)

$$\frac{N \cap [F,F]}{[N,F]} < \frac{N}{[N,F]}$$

abelian group generated by (cosets of) ~~...~~ elements of R.

For $G = Mod_g^1$ if $u[N,F] \in N/[N,F]$ can write

$$u = \prod D_{ij}^{n_{ij}} \prod_{i=1}^{2g-1} B_{i,i+1}^{n_i} B_{04}^{n_0} C^{n_c} L^{n_L}$$

for some $n_{ij}, n_i, n_0, n_c, n_L \in \mathbb{Z}$.

$\Rightarrow$ rank $H_2 G \leq g(2g-1) + (2g-1) + 3.$ (~~...~~ already kind of interesting)

Goal reduce this to rank ≤ 1.

<u>Step 1</u> Show $n_{ij} = 0$ for each ij.

- For $[g,h] \in N \cap [F,F]$ denote $\{g,h\}$ image in $\frac{N \cap [F,F]}{[N,F]}$.

<u>Claim</u> For commuting nonsep DT's a, b $\{a,b\} = 0$.

(<u>Rmk</u> A pair of commuting elements always gives $T^2 \to K(G,1)$.
Claim here is that these tori always nullhomologous)

<u>Observations</u>.

(i) if $g \in \mathrm{Mod}_g$ commutes w/ h, k then

$$\{g, hk\} = \{g,h\} + \{g,k\}.$$

This follows from general relation $[g,hk] = [g,h][g,k]^h$

(ii) $\{g, h^{-1}\} = -\{g,h\}$. (Note $[g,h^{-1}]^{ghg^{-1}} = [h,g] = [g,h]^{-1}$)

<u>Pf of claim</u>. Cut S along curve of a.

Mod^3_{g-1} perfect $(g-1 \geq 3) \Rightarrow b = \prod [x_i, y_i]$

$x_i, y_i \in \mathrm{Mod}^3_{g-1} \Rightarrow x_i, y_i$ commute w/ a.

$$\Rightarrow \{a,b\} = \{a, \prod [x_i, y_i]\} = \sum \{a, [x_i, y_i]\}$$

$$= \sum \{a, x_i\} + \{a, y_i\} - \{a, x_i\} - \{a, y_i\} = 0.$$

<u>Step 2</u>. Counting total exponents.

$$u = \prod_{i=1}^{2g-1} B_{i,i+1}^{n_i} \; B_{0,4}^{n_0} \; C^{n_C} \; L^{n_L} \qquad u[N,F] \in \frac{N \cap [F,F]}{[N,F]} < \frac{N}{[N,F]}$$

$\Rightarrow$ total exponent of a_i in u is zero $i = 0, \dots, 2g$.

(here we have $2g+2$ variables (the n_i, n_0, n_C, n_L) and $2g+1$ relations (for $a_0, \dots, a_{2g}$). If this syst. has full rank we win. But this is a separate computation $\forall g \dots$)

<u>Check</u> a_{2g} appears only in $B_{2g-1,2g}$ and has exponent 1 $\Rightarrow n_{2g-1} = 0$

$$B_{2g-1,2g} = a_{2g} a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} a_{2g-1}^{-1}$$

Similarly $n_i = 0$ $i \geq 5$.

$$\Rightarrow \quad u = B_{04}^{n_0} B_{12}^{n_1} B_{23}^{n_2} B_{34}^{n_3} B_{45}^{n_4} C^{n_C} L^{n_L}$$

(Now reduced to single comp. $\forall g$)

These relations only involve $a_0, \dots, a_5$.

$\rightsquigarrow$ matrix eqn. $\quad A_{6\times 7} \begin{pmatrix} n_0 \\ \vdots \\ n_4 \\ n_C \\ n_L \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

Compute $\operatorname{rank}(A) = 6$. $\Rightarrow$! solution up to scaling

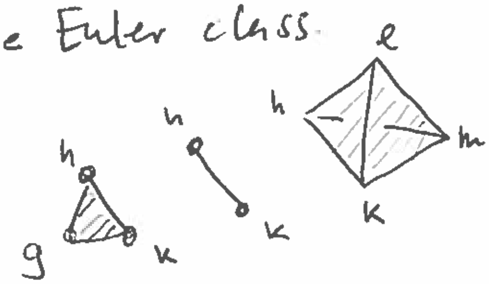
$$H_2 \operatorname{Mod}_g^1 = \langle u_0 \rangle \qquad u_0 = B_{04}^{-18} B_{12}^{6} B_{23}^{2} B_{34}^{8} B_{45}^{-10} C L^{10} \qquad \square$$

<u>Rmk</u> similar arg for Mod_g (include hyperelliptic relation)

<u>Next time</u>. nontrivial elts of $H^2 \operatorname{Mod}_{g,1} \simeq \mathbb{Z}^2$ geometrically.

Lecture 9

I. Group cohomology & the Euler class



- G group.
- $EG = \bigcup G^{k+1} \times \Delta^k / \sim$

  contractible.
- $G \curvearrowright EG$ freely $\quad X = EG/G \sim K(G,1)$.
- cellular cochain $\phi \in C^k(X)$ can be written
  - homogeneously $\quad \phi : G^{k+1} \longrightarrow \mathbb{Z}$ s.t.

    $\phi(hg_0, \dots, hg_k) = \phi(g_0, \dots, g_k)$, $\quad \delta\phi = \sum (-1)^i \phi(g_0, \dots, \hat{g}_i, \dots, g_k$
  - inhomogeneously $\quad \bar{\phi}(a_1, \dots, a_k) := \phi(1, a_1, a_1 a_2, \dots, a_1 a_2 \cdots a_k)$

    $$\delta\bar{\phi}(a_1, \dots, a_{k+1}) = \phi(a_2, \dots, a_{k+1}) - \phi(a_1 a_2, a_3, \dots, a_{k+1}) + \cdots$$
    $$\pm \phi(a_1, \dots, a_k a_{k+1}) \mp \phi(a_1, \dots, a_k)$$

<u>Ex</u> inhomogeneous 1-cocycle $\phi : G \longrightarrow \mathbb{Z}$.

$0 = \delta\phi(a,b) = \phi(b) - \phi(ab) + \phi(a)$ $\quad$ homomorphism.

<u>Ex</u> 2 cocycles arise from central extensions

$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \Gamma \xrightarrow{p} G \longrightarrow 1$ $\quad$ (central means $i(\mathbb{Z}) < \mathrm{Center}(\Gamma)$)

<u>Ex</u> (i) oriented circle bundle $S^1 \longrightarrow E \longrightarrow S_{g \geq 1}$ induces

$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \pi_1(S_g) \longrightarrow 1.$ $\quad$ central.

(ii) $H_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$

$$0 \to \mathbb{Z} \to H_3 \longrightarrow \mathbb{Z}^2 \to 0$$
$$(a,b,c) \longmapsto (a,b)$$

<u>Euler class</u> of $0 \to \mathbb{Z} \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$.

- $s : G \longrightarrow \Gamma$ set-theoretic section ($p \circ s = id$)
- $\phi : G \times G \longrightarrow \mathbb{Z}$ $\qquad \phi(g,h) = s(g)\, s(h)\, s(gh)^{-1} \in i(\mathbb{Z})$.

  2 cocycle $\qquad \phi(h,k) - \phi(gh,k) + \phi(g,hk) - \phi(g,h) = 0$

  $\forall\, g, h, k \in G$.
- $e(\Gamma) := [\phi] \in H^2(G)$ (indep of s).

  $e(\Gamma) = 0 \iff \Gamma \cong \mathbb{Z} \times G$.

<u>Fact</u> $\left\{ \begin{matrix} \text{central extensions} \\ 0 \to \mathbb{Z} \to \Gamma \to G \to 1 \end{matrix} \right\} / \sim \quad \xleftrightarrow{1-1} \quad H^2(G; \mathbb{Z})$.

<u>Rmk</u> $S^1 \to E \to S_g$. $\qquad$ group Euler = topology Euler

$$H^2(\pi_1(S_g)) \simeq H^2(S_g).$$

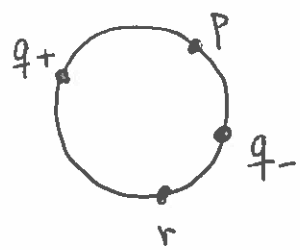
II. Euler class in nature.

<u>Circle Homeos</u> $\widetilde{\text{Homeo}}(S^1) = \{ \underset{\text{homeo}}{f : \mathbb{R} \to \mathbb{R}} \mid f(t+1) = f(t)+1 \}$

$$0 \to \mathbb{Z} \to \widetilde{\text{Homeo}}(S^1) \longrightarrow \text{Homeo}(S^1) \to 1.$$

$\rightsquigarrow \ e \in H^2(\text{Homeo}(S^1))$.

• cocycle representative



$$\text{ord}(p,q,r) = \begin{cases} 1 & \text{ordered ccw} \\ -1 & \text{ordered cw} \\ 0 & \text{two coincide.} \end{cases}$$

$\text{ord}(p, q_\pm, r) = \pm 1.$

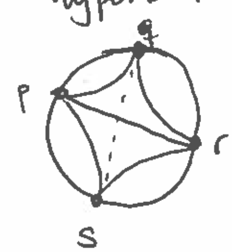
Fix $* \in S^1$. Define $\psi(g,h) = \text{ord}(*, f*, fg*)$ $f, g \in \text{Homeo}(S$

cocycle $[\psi] = e$.

<u>Rmk.</u> $PSL_2\mathbb{R} < \text{Homeo}(S^1)$ $\quad e \in H^2(PSL_2\mathbb{R})$ related to hyperbolic area form

$$\text{ord}(p,q,r) = \frac{1}{\pi} \text{Area}(\Delta(p,q,r))$$

(cocycle relation apparent)



<u>Hermitian Lie groups</u> $Sp_{2n}\mathbb{R}$

- $\mathbb{R}^{2n}$ inner prod $g(\cdot,\cdot)$, cplx str $J^2 = -\text{id}$, sympl. form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$
- $Sp_{2n}\mathbb{R} < GL_{2n}\mathbb{R}$ subgp pres. ω $\quad$ (std. choices $\rightsquigarrow$ $A^t J A = J$, $J = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}$)
- For any top. gp. $\quad 0 \to \pi_1 G \longrightarrow \tilde{G} \longrightarrow G \to 1.$ $\quad$ Central ext.

- $\underset{G=}{Sp_{2n}\mathbb{R}} \sim$ max cpt $\underset{K=}{U(n)} = GL_n\mathbb{C} \cap O(2n) \cap Sp_{2n}\mathbb{R}$.

$\Rightarrow \pi_1(G) \cong \pi_1(K) = \mathbb{Z}$.

$\rightsquigarrow \mu \in H^2(Sp_{2n}\mathbb{R})$.

- Cocycle representative

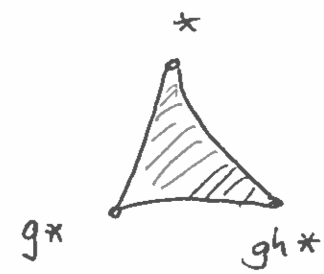
Symmetric space $X = G/K \cong \left\{ A+iB \,\middle|\, \begin{array}{l} A,B \in M_n(\mathbb{R}) \text{ symmetric} \\ B \text{ pos. def.} \end{array} \right\}$.

Siegel upper half space ($n=1 \Rightarrow \mathbb{H}^2$) complex mfld, Riem npc.

Kähler form $k \in \Omega^2(X)^G$

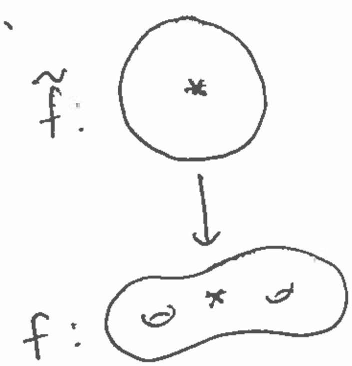
- Fix $* \in X$ define $\psi : G \times G \longrightarrow \mathbb{R}$.

$$\psi(g,h) = \int_{\Delta(g,h)} k$$



$[\psi] = \mu$ in $H^2(Sp_{2n}\mathbb{R};\mathbb{R})$

III. 2 cocycles on $\mathrm{Mod}_{g,1}$ $g \geq 2$.



(1) (Nielsen) $\mathrm{Mod}_{g,1} \hookrightarrow \mathrm{Homeo}(S^1)$.
$[f] \longmapsto \partial\tilde{f}$

$\rightsquigarrow e \in H^2\,\mathrm{Mod}_{g,1}$

corresponds to extension $0 \to \underset{\langle T_c \rangle}{\overset{\shortparallel}{\mathbb{Z}}} \to \mathrm{Mod}_g^1 \longrightarrow \mathrm{Mod}_{g,1} \to$

(2) $\mathrm{Mod}_g \curvearrowright H_1(S;\mathbb{Z})$ preserve intersection form.

$\rightsquigarrow \mathrm{Mod}_g \longrightarrow Sp_{2g}\mathbb{Z} \longrightarrow Sp_{2g}\mathbb{R}$.

$\rightsquigarrow \mu \in H^2 \mathrm{Mod}_g$ $\qquad 1 \to \pi_1(S_g) \to \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Mod}_g \to 1$

$\rightsquigarrow \mu \in H^2 \mathrm{Mod}_{g,1}$

<u>Thm</u> $e, \mu \in H^2 \mathrm{Mod}_{g,1} \simeq \mathbb{Z}^2$ linearly indep.

<u>$e \neq 0$</u>. $\quad \pi_1(S_g) \xrightarrow{p} \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Homeo}(S^1)$

induces
$$\begin{array}{ccc} S^1 \longrightarrow & E & = \dfrac{\mathbb{H}^2 \times S^1}{\pi_1(S_g)} \\ & \downarrow & \\ & S_g & \end{array} \qquad E \simeq T^1(S_g).$$

$\Rightarrow p^*(e) \in H^2(S_g)$ is Euler class of ~~$\pm$~~ TS_g (nonzero)

<u>Rmk</u> $p^*(\mu) = 0$.

<u>Next week</u> Surface bundles, Thurston norm.

Lecture 10

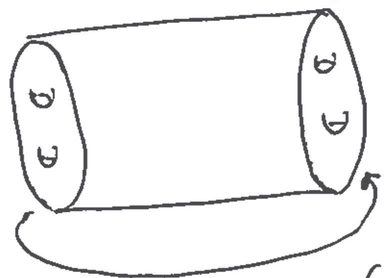
Last time: finished 1st part of course

Now turn to part ii which will be about surface bundles

Intro to I. Surface bundles and monodromy.

- surface bundle $S_g \to E$, $E \downarrow B$ — fiber bundle
  structure group $\mathrm{Homeo}(S_g)$, $\mathrm{Diff}(S_g)$

Example. Mapping torus.

$M_f =$ 

$M_f \downarrow S^1$ $\qquad f \in \mathrm{Homeo}(S_g)$

Example flat bundles $\quad \rho: \pi_1(B) \longrightarrow \mathrm{Homeo}(S_g)$. $\qquad b \in \tilde{B}$ $x \in S$ $\gamma \in \pi_1(B)$.

$S \longrightarrow E_\rho = \dfrac{\tilde{B} \times S}{\pi_1(B)}$ $\qquad (b, x) \sim (\gamma . b, \rho(\gamma) . x)$

$E_\rho \downarrow B$

Rmk ~~For~~ $B = S^1$ $\qquad E_\rho = M_f \qquad f = \rho(1)$.

Rmk $B = S_h$ $\qquad$ eg. $\pi_1(S_h) \longrightarrow F_n \longrightarrow \mathrm{Homeo}(S_g)$.

Some motivations ~~reasons to study~~

1) simplest nonlinear bundle theory
2) 3-mflds: almost every closed 3mfld is finitely covered by $S_g \to M_f \to S^1$.
3) 4-mflds: large class of symplectic mflds. (given concretely)

4) alg. geo : families of alg curves are topologically surface bundle

<u>Monodromy invariant</u> $\rho: \pi_1(B) \longrightarrow \mathrm{Mod}_g$. ~~for~~ $\begin{array}{c} S_g \to E \\ \downarrow \\ B \end{array}$

$[\gamma] \longmapsto [f_\gamma]$

$$\begin{tikzcd} M_{f_\gamma} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\ S^1 \arrow[r, "\gamma"] & B \end{tikzcd}$$

(or define w/ local trivialization)

Alternate POV: $\begin{array}{c} S_g \to E \\ \downarrow \\ B \end{array}$ $\rightsquigarrow$ $1 \to \pi_1(S_g) \to \pi_1(E) \to \pi_1(B) \to 1.$

$\mathrm{im}\left(\pi_2(B) \to \pi_1(S_g)\right) < Z(\pi_1(S_g)) =$

$\rightsquigarrow$ $\pi_1(B) \longrightarrow \mathrm{Out}(\pi_1(S_g)) \simeq \mathrm{Mod}(S_g)$ (Dehn-Nielsen-Baer)

<u>Rmk</u> For flat bundle ρ ~~[illegible]~~ is. $\pi_1(B) \longrightarrow \mathrm{Homeo}(S_g) \longrightarrow \mathrm{Mod}_g$.

<u>Thm</u> (Monodromy as complete invariant) For fixed B.

$$\left\{ \begin{array}{c} \text{bundles} \\ S_g \to E \\ \downarrow \\ B \end{array} \right\} \Big/ \text{iso} \quad \xleftrightarrow{1-1} \quad \left\{ \begin{array}{c} \text{homomorphisms} \\ \pi_1(B) \to \mathrm{Mod}_g \end{array} \right\} \Big/ \text{conj}$$

<u>Ex</u>. $B = S^1$. conj. comes b/c don't have canonical identification of fiber.

<u>Organizing problems</u>

1) classification for fixed base up to bundle iso, homeo/diffeo, hol/ Symplecto, biholo.

we'll talk about homeo prob for 3mflds fibering over S^1.

2) topology-monodromy dictionary eg. (Thurston) $M_f \to S^1_*$ hyperbolic

$\leftrightarrow$ f pseudo-Anosov

II. Surface bundles over S^1 ~~(examples)~~

<u>Knot complements</u> $S^3 \setminus K$.

(1) $K = \bigcirc$ trivial knot

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = D^2 \times S^1 \underset{S^1 \times S^1}{\cup} S^1 \times D^2$$

$\hookleftarrow K = S^1 \times 0.$

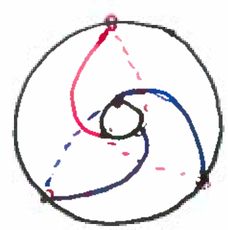
$$\Rightarrow S^3 \setminus K \simeq \begin{array}{c} D^2 \times S^1 \\ \downarrow \\ S^1 \end{array}$$

(2) $K =$  trefoil ~~knot~~

<u>Prop</u> There is a fibering $(T^2 \setminus pt) \longrightarrow \begin{array}{c} S^3 \setminus K \\ \downarrow \\ S^1 \end{array}$

<u>Pf</u> (family of Siefert surfaces) Construct foliation of $S^3 \setminus K$ w/ leaves $T^2 \setminus pt$ and leaf space S^1.

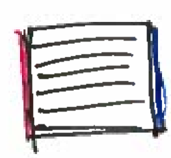
- K is a <u>torus knot</u>.



(construct foliation inside solid torus and then outside solid torus)

- interior solid torus.

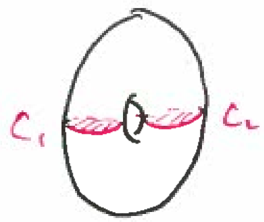
one leaf F_0



other leaves F_θ obtained by translation w/ rotation (screw motion)

Note $\partial F_0 = K \cup 2$ circles on boundary of solid torus.

- exterior solid torus

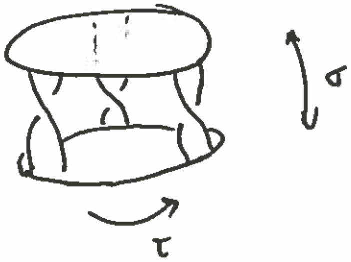


meridians on int. solid torus are longitudes on ext. solid torus $\Longrightarrow$ c_1, c_2 bound disks. $\square$

- altogether each leaf is a siefert surface for K (oriented surface S w/ $\partial S = K$)

$\chi(S) = -1 \implies S =$ Torus w/ 1 ~~planetare~~ ~~group~~ ∂comp. $\square$

<u>Monodromy</u>



$\mathrm{Mod}_{1,1} \ni \sigma \circ \tau \; : \; H_1(F_0) \simeq \mathbb{Z}^2 \circlearrowleft$

$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ order 6.

<u>Rmk</u> (Milnor) $K = S^3 \cap \{(z,w) \in \mathbb{C}^2 \mid z^2 + w^3 = 0\}$

$$\begin{aligned} S^3 \setminus K &\xrightarrow{\pi} S^1 \\ (z,w) &\longmapsto \frac{z^2+w^3}{|z^2+w^3|} \end{aligned}$$ fibration.

General procedure for producing fibered knots (not all fibered knots arise this way)

<u>Criteria for fibering</u>

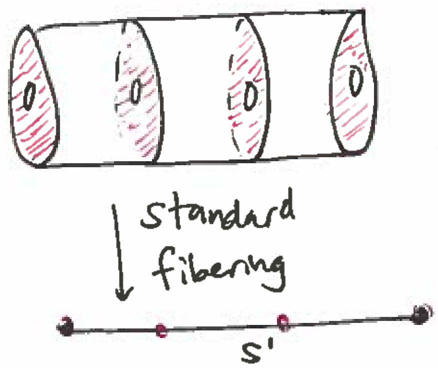
(1) <u>Thm</u> (Stallings) $K \subset S^3$ knot $\quad \pi := \pi_1(S^3 \setminus K)$

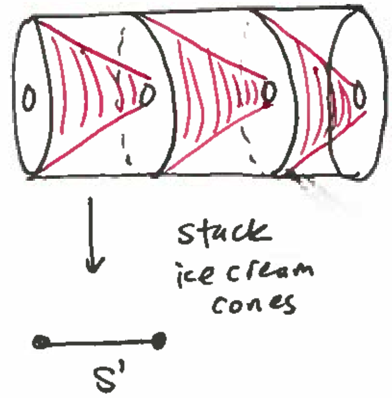
$S^3 \setminus K$ fibers $\iff \pi' \equiv [\pi, \pi]$ f.g.

Lecture 11

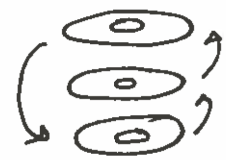
<u>Warmup</u> The trivial bundle $S_g \times S^1$ fibers $S_h \to S_g \times S^1$ in infinitely many ways.
$$\downarrow$$
$$S^1$$

<u>Observation</u> $A \times S^1$ fibers in many ways



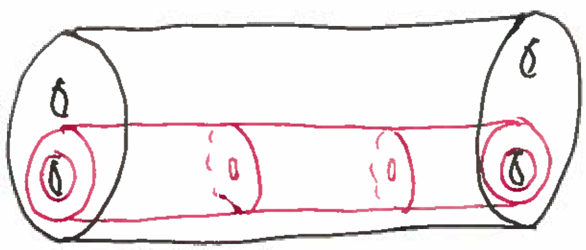


has fiber $A \sqcup A \sqcup A$ w/ monodromy

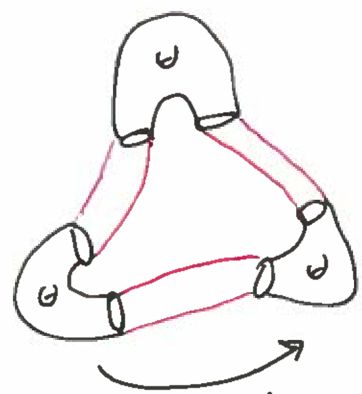


(can't be trivial monodromy b/c total sp. connected)

• insert into $S_g \times S^1$ to get new fibering



w/ fiber



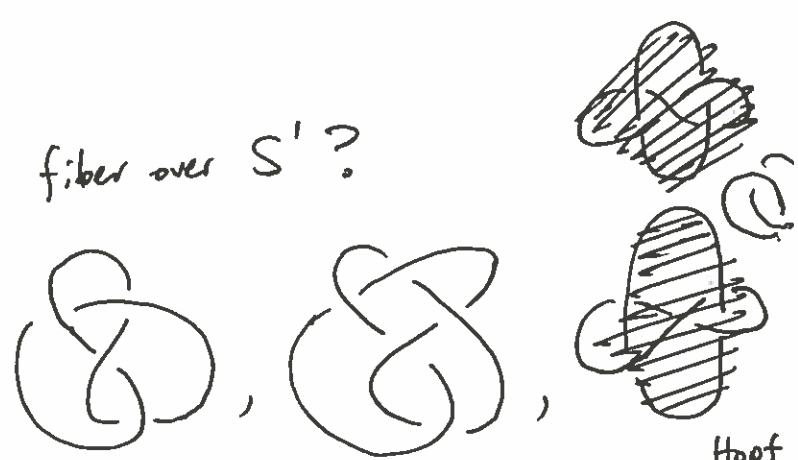
<u>Rmk</u> This gives iso $\pi_1(S_g) \times \mathbb{Z} \simeq \pi_1(S_h) \rtimes \mathbb{Z}$

where $\mathbb{Z}$ acts on $\pi_1(S_h)$ ~~by~~ via deck action.

<u>Questions</u>

(1) For 3 manifold M, does M fiber over S^1?

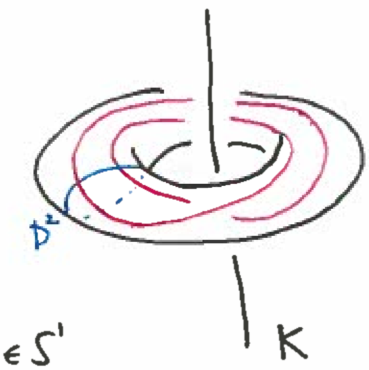
e.g. $M = S^3 \setminus K$ $K =$



(2) If $M \to S^1$ fibers, what are all the diff ways?

I. More knots that fiber

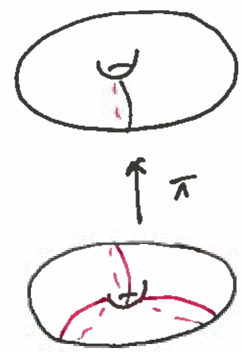
<u>Goldsmith branched cover construction</u>



- $K \subset S^3$ trivial knot
- $J \subset D^2 \times S^1 \subset S^3 \setminus K$ also trivial
  w/ $\#(J \cap D^2 \times \{\theta\})$ constant for $\theta \in S^1$
- branched cover of S^3, branched along J.

  Concretely $J \subset N \simeq J \times D^2$ tubular nbhd.

  – take $\mathbb{Z}/m$ cover of $S^3 \setminus N =$



  – glue back N to get
  $$X \cup N \approx S^3$$

  – π extends to $S^3 \xrightarrow{\pi} S^3$

  on $\pi|_N : \underset{J \times \mathbb{C}}{\underset{\wr}{N}} \longrightarrow \underset{J \times \mathbb{C}}{\underset{\wr}{N}}$

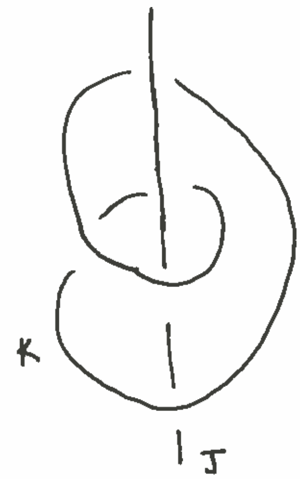
  $$(\theta, z) \longmapsto (\theta, z^m).$$

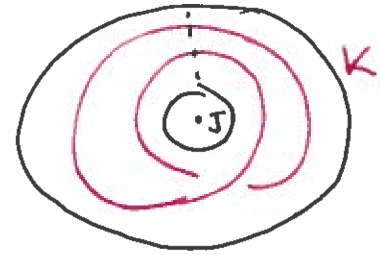
- define $K' = \pi^{-1}(K)$ $\quad S^3 \setminus K' \xrightarrow[\text{branched cover}]{} S^3 \setminus K \xrightarrow[\text{fibration}]{} S^1$

  defines fibration
  with fiber a branched cover of D^2 branched along $\#(J \cap D^2 \times \theta)$ points.

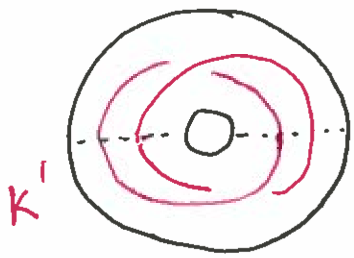
Examples

①



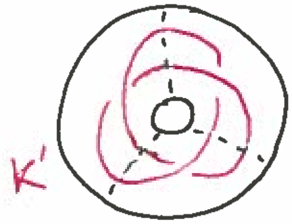


$m=2$



Hopf link – fibered.

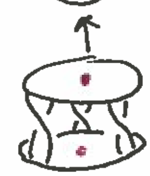
$m=3$



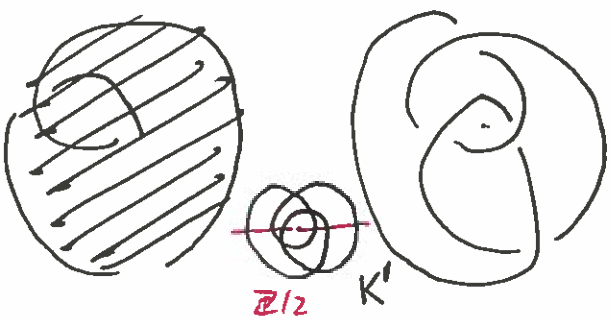
trefoil – fibered

Fiber is $\mathbb{Z}/3$ branched cover of

which is

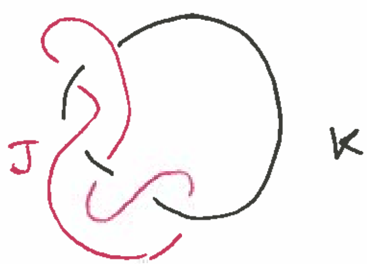


② Figure-8 knot : in general this method works well for knots w/ some symmetry. ($\mathbb{Z}/m$ rotational)



quotient $\rightsquigarrow$ knot

$\rightsquigarrow$ Goldsmith construction for $m=2$



and $m=2$ shows fig 8 knot complement is fibered.

<u>Rmk</u> Same J, K $m=3$ gives

Borromean rings fibered.

II. Criteria for fibering.

(1) <u>Thm</u> (Stallings) $K \subset S^3$ knot $\pi := \pi_1(S^3 \setminus K)$

$$S^3 \setminus K \text{ fibers} \iff \pi' := [\pi, \pi] \text{ f.g.}$$

<u>Rmk</u> ($\Rightarrow$ easy) Alexander duality $H_1(S^3 \setminus K) \cong H^1(K) \cong \mathbb{Z}$.

$\Rightarrow \pi^{ab} = \pi / \pi' \cong \mathbb{Z}$.

For fibration $S^3 \setminus K \longrightarrow S^1$ induced map $\pi_1(S^3 \setminus K) \xrightarrow{\phi} \mathbb{Z}$ is the abelianization homomorphism

$$\left(\begin{array}{ccc} \pi & \xrightarrow{\phi} & \mathbb{Z} \\ \downarrow & \nearrow \cong & \\ \mathbb{Z} = \pi^{ab} & & \end{array} \right)$$

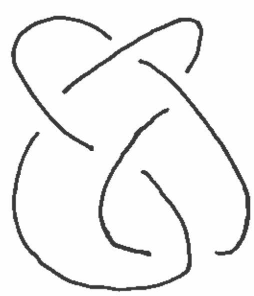
$\Rightarrow$ ~~$\pi_1(\text{fiber}) = \ker(\pi \to \pi^{ab}) \cong \pi'$ f.g.~~

$\pi' = \ker(\pi \longrightarrow \pi^{ab}) = \pi_1(\text{fiber})$ f.g.

<u>Rmk</u> $\pi'/\pi'' = H_1(\pi')$ is a $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ module.

"Alexander module" - knot invariant

<u>Ex</u> $K = 5_2$



$$\pi'/\pi'' \cong \frac{\mathbb{Z}[t, t^{-1}]}{\Delta(t)}$$

$\Delta(t) = 2 - 3t + 2t^2$ Alexander poly.

$\Rightarrow \pi'/\pi''$ not f.g. over $\mathbb{Z}$. ($t^2, t^3, \ldots$ not expressed in terms of lower deg terms)

$\Rightarrow S^3 \setminus K$ not fibered.

(2) <u>Thm</u> (Tischler) M^n closed mfld

(a) if $\exists$ $\alpha \in \Omega^1(M)$ closed 1-form s.t. $0 \neq \alpha_x : T_xM \to \mathbb{R}$ $\forall$ $x \in M$, then M fibers over S^1.

(b) $\mathcal{C} = \{ a \in H^1(M;\mathbb{R}) : a = [\alpha] \ \overset{\alpha}{\text{nonsingular}} \} \subset H^1(M;\mathbb{R})$

is open cone $\left(\text{nonempty} \overset{}{\iff} M \text{ fibers} \right)$.

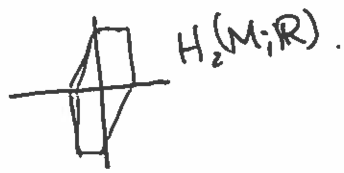
Theory of Thurston norm

- (semi) norm $\|\cdot\|_T$ on $H_2(M;\mathbb{R})$ $\left(\cong H^1(M;\mathbb{R})\right)$

  $\|x\|_T$ measures complexity of $\Sigma \hookrightarrow M$ w/ $[\Sigma] = x$.

  (norm eg. for M hyperbolic)

- (combinatorial str) $B_T := \{ x \in H_2 : \|x\| = 1 \}$ finite-sided

  rational polyhedron. 

  Cones on faces gives decomp of $H_2(M;\mathbb{R})$

  for $S_g \to M \downarrow S^1$ fiber $[S_g] \in H_2 M$ (where does it live?)

- <u>Thm</u> (Thurston)

  (i) $[S_g]$ lives in interior of cone on a face.

  (ii) If C such a cone, every $x \in C \cap H_2(M;\mathbb{Z})/\text{tor}$ is class of a fiber $S_h \to M \to S^1$.

Lecture 12

I. Thurston norm.

Q1. What are all the ways M^3 fibers over S^1?

Q2. Given $a \in H_2(M;\mathbb{Z})$ what is smallest genus for embedded surface representing a?

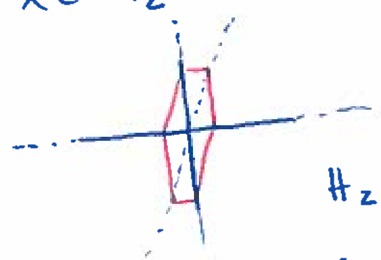
(Thurston) M 3mfld

- (semi) norm $\|\cdot\|_T$ on $H_2(M;\mathbb{R})$

$\|x\|_T$ measures complexity of $\Sigma \hookrightarrow M$ w/ $[\Sigma] = x$.

(norm eg. for M hyperbolic)

- (combinatorial str) $B_T = \{ x \in H_2 : \|x\| = 1\}$ finite-sided rational polyhedron.



H_2 (case $b_2 = 2$)

~~Cones on faces give decomp of $H_2(M;\mathbb{R})$.~~

for $S_g \to M \to S^1$ get $[S_g] \in H_2 M$ (where does it live?)

- Thm (Thurston)

(i) $[S_g]$ lives in interior of cone on a face of B_T.

(ii) If C such a cone, every $x \in C \cap H_2(M;\mathbb{Z})/\text{tor}$ is class of fiber $S_h \to M \to S^1$.

II. Defining the Thurston norm.

M oriented 3mfld. (compact usually closed)

Lemma (embedded representatives exist)

(i) $a \in H_2(M;\mathbb{Z})$ $\exists$ embedded, oriented $S \overset{f}{\hookrightarrow} M$

st. ~~a=f[S]~~ $a = f_*[S]$.

(ii) if $a = kb$ then any $S \overset{f}{\hookrightarrow} M$ w/ $[S] = a$ has the form

$S = S_1 \sqcup \cdots \sqcup S_k$ where $[S_i] = b$ $i = 1, \ldots, k$

Proof.

(i) $$\begin{array}{ccccc} H_2(M;\mathbb{Z}) & \simeq & H^1(M;\mathbb{Z}) & \simeq & [M, \underset{\overset{\textstyle S^1}{\simeq}}{K(\mathbb{Z},1)}] \\ \Psi & & & & \Psi \\ a & & & & \phi_a : M \longrightarrow S^1 \end{array}$$

$\phi_a : M \to S^1$ wlog smooth

$$\begin{array}{ccc} M & \longrightarrow & S^1 \\ \cup & & \Psi \\ S := \phi_a^{-1}(\theta) & & \theta \end{array}$$

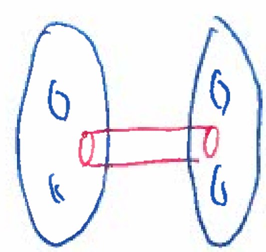
θ regular value

(ii) $$\begin{array}{ccc} M & \xrightarrow{\phi_b} & S^1 \\ & \searrow^{\phi_a} & \downarrow \theta \overset{p}{\longmapsto} k\theta \\ & & S^1 \end{array}$$

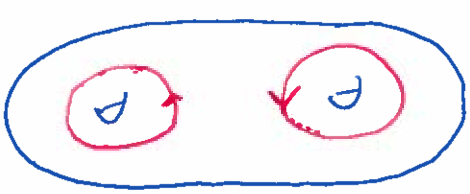
$$\phi_a^{-1}(\theta) = \phi_b^{-1}(\theta) \cup \phi_b^{-1}\left(\theta + \tfrac{2\pi}{k}\right) \cup \cdots \cup \phi_b^{-1}(\theta + \ldots$$
$$= S$$

$\square$

Rmk. Tubing is incompatible w/ orientation.



example one dim. down.



Connected

no embedded reps for these homology classes.

<u>Defn</u> • for S connected

$$\chi_-(S) := \max\{0, -\chi(S)\} \quad \text{eg} \quad \chi_-(S_g) = \begin{cases} 2g-2 & g \geq 2 \\ 0 & g = 0,1 \end{cases}$$

• $S = S_1 \sqcup \cdots \sqcup S_k$ $\quad S_i$ connected

$$\chi_-(S) := \sum \chi_-(S_i).$$

• Thurston norm on integer points $a \in H_2(M;\mathbb{Z})$.

$$\|a\| = \inf\{\chi_-(S) : S \hookrightarrow M \text{ embedded}, [S] = a\}.$$

<u>Key Properties</u>

(1) linear on rays $\quad \|ka\| = k\|a\| \quad k \in \mathbb{N}$.

(2) Δ-ineq. $\quad \|a+b\| \leq \|a\| + \|b\|$.

<u>Formal consequences:</u>

$\|\cdot\|$ admits ! cts extension $\|\cdot\| : H_2(M;\mathbb{R}) \longrightarrow \mathbb{R}_+$.

pseudonorm on $\mathbb{R}$-span of $\{a \in H_2(M;\mathbb{Z}) : \|a\| = 0\}$.

<u>Cor</u> If M is irreducible (no essential S^2's) and atoroidal (no essential T^2's) (eg. M hyperbolic), then $\|\cdot\|$ is a norm.

<u>Pf of (1)</u>. - any rep $[S] = a$ gives rep $[S \cup \cdots \cup S] = ka$

$$\Rightarrow \|ka\| \leq k\|a\|$$

- Lemma $\Rightarrow$ every rep of ka has form $S = S_1 \sqcup \cdots \sqcup S_k$

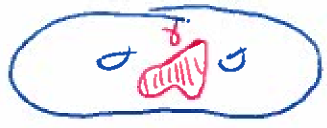
$$\Rightarrow \|ka\| \geq k\|a\|.$$

4

Pf of (2)

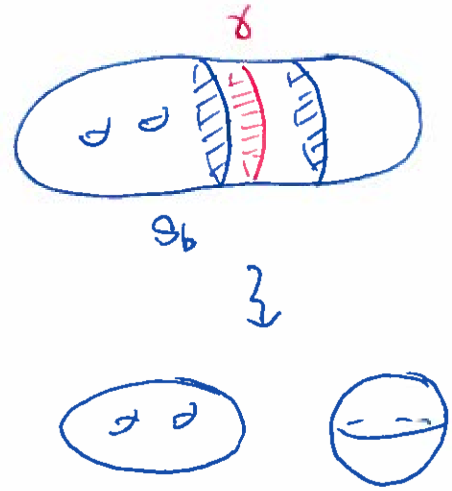
Cases

S_a, S_b norm minimizing, $S_a \pitchfork S_b$.
$S_a \cap S_b = \gamma_1, \ldots, \gamma_r$ closed curves.



(i) γ bounds disk on S_a (or S_b)

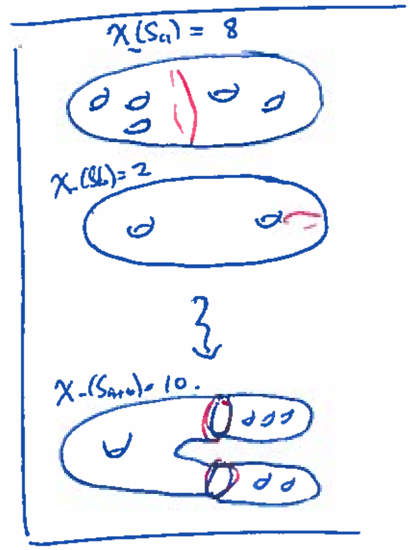
~~contracts~~ $\rightsquigarrow$ surger S_b along γ



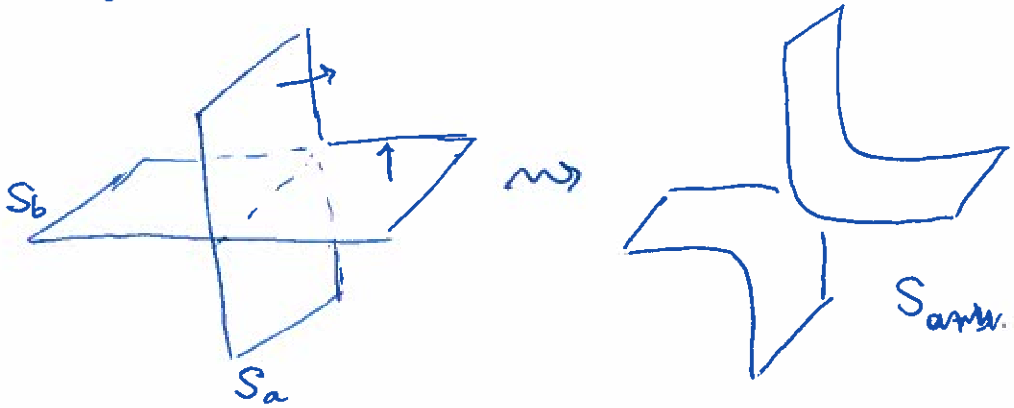
- doesn't change homology class
- doesn't change $\chi_-(S_b)$.
- reduces $S_a \cap S_b$.

Rmk May have nested γ on S_a that bound disks — have to start w/ innermost.

Note: γ must be ess. on S_b — o/w S_b is not norm minimizing



(ii) γ essential on both S_a and S_b



$\exists!$ way to resolve singularity in or. pres. way.

$$\chi_-(S_{a+b}) = \chi_-(S_a) + \chi_-(S_b)$$

Continue in this way until S_{a+b} embedded.

$$\Rightarrow \quad \|a+b\| \leq \|a\| + \|b\|.$$

Prop (Thurston)

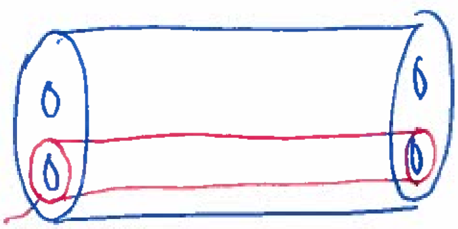
$V \simeq \mathbb{R}^d$ $\quad N : \mathbb{R}^d \to \mathbb{R}$ norm st. $N|_\Lambda : \Lambda \to \mathbb{Z}$

$\cup$

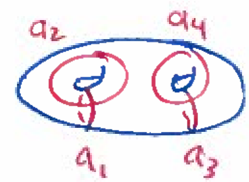
$\Lambda \simeq \mathbb{Z}^d$ lattice. $\quad$ Then $B_N = \{ x \in \mathbb{R}^d \mid N(x) \le 1 \}$ is compact finite-sided rational polyhedron.

(specified by finitely many linear inequalities)

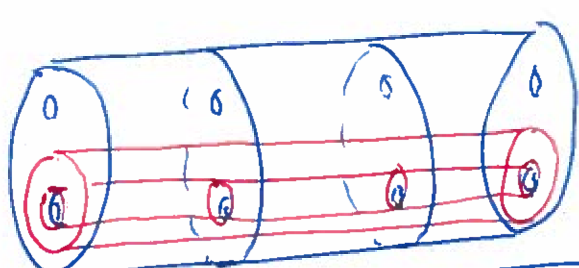
Ex. $M = S_2 \times S^1$



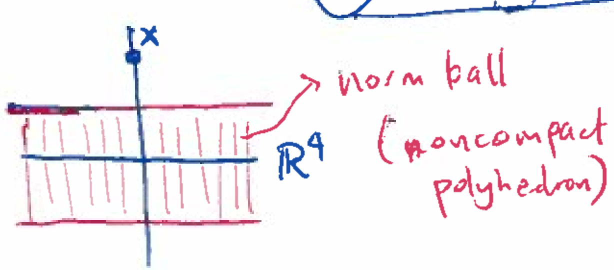
$$H_2(M;\mathbb{Z}) \simeq \underset{\mathbb{Z}\{x\}}{\underset{\|}{H_2(S_2)}} \oplus \underset{\mathbb{Z}\{a_i \otimes y\}}{\underset{\|}{H_1(S_2) \otimes H_1(S^1)}} \simeq \mathbb{Z} \oplus \mathbb{Z}^4.$$



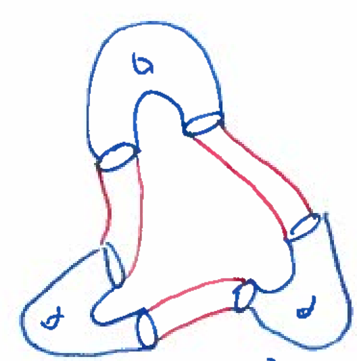
- $\|x\| = \chi_-(S_2) = 2$ (ie S_2 is norm minimizing)
- $\|a_i \otimes y\| = 0$ (have tori reps.)
- eg. $\|3x + 2(a_2 \otimes y)\| = 3\|x\| = 6.$ rep'd by surface.



resolve singularities to get surface.



Next time:
- more interesting ex.
- relate to fibers of fibrations.



(from last time)

Lecture 13

I. Thurston norm examples

<u>Last time</u> M cpt or 3mfld.

- Thurston semi-norm $|\cdot| : H_2(M;\mathbb{R}) \longrightarrow \mathbb{R}_+$

  (norm if M irreducible, atoroidal), e.g. M hyperbolic)

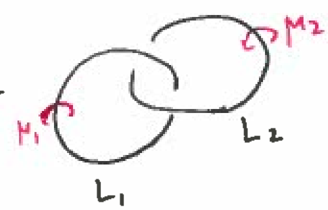
- Norm ball $B = \{ a \in H_2 : |a| \le 1 \}$ finite sided polyhedron

  (cpt if $|\cdot|$ is a norm)

<u>Cor</u> $|\cdot|$ is a norm $\Rightarrow$ $\mathrm{Diff}(M) \longrightarrow \mathrm{Aut}(H_2 M)$ finite image

(in fact $\mathrm{Diff}\, M$ has finitely many components – $\mathrm{Mod}(M)$ not so interesting)

<u>Examples</u>

(1) (Warmup) $M = S^3 \setminus N(L)$ $L =$  Hopf link. $N \supset L$ tubular nbhd

- <u>Note</u> ~~Alexander duality~~ $H_2(S^3 \setminus L) \simeq \tilde{H}^0(L) = \mathbb{R}$ (Alexander duality)

- <u>Rmk</u> When $\partial M \neq \phi$ can define $|\cdot|$ on $H_2(M, \partial M; \mathbb{R})$.

- $H_2(M, \partial M) \simeq H_2(S^3, N) \simeq H_1(N) \simeq \mathbb{R} L_1 + \mathbb{R} L_2$

~~$N \supset L$ tubular nbhd.~~

(How many times do wrap around L_1)

iso given explicitly by $S \longmapsto (\partial S \cdot \mu_1) L_1 + (\partial S \cdot \mu_2) L_2$

- $\|L_1\| = 0$ since rep'd by $S =$ (disk) $\chi_-(S) = 0$.

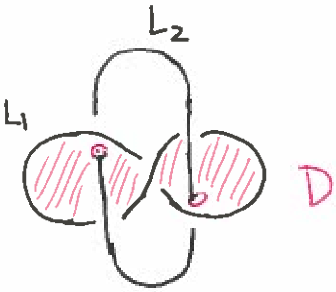
- Similarly $\|L_2\| = 0$. $\Rightarrow |aL_1 + bL_2| \le a|L_1| + b|L_2| = 0$.

Rmk. Previously showed $\begin{array}{c} A \to M \\ \downarrow \\ S^1 \end{array}$ (Goldsmith construction)

$A = \mathbb{Z}/2$ branched cover of 

In general, if $\begin{array}{c} F \to M \\ \downarrow \\ S^1 \end{array}$ and $\chi(F) \geq 0$ then

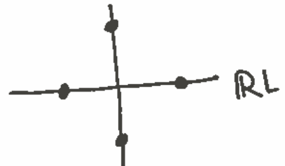
$|\cdot| \equiv 0$.

(2) $M = S^3 \setminus N(L)$ $\quad L =$ 

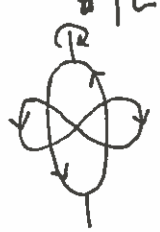
- As before $H_2(M, \partial M) \simeq \mathbb{R}L_1 + \mathbb{R}L_2$.
- Claim $|L_1| = 1$ $\quad \chi_-(D) = 1. \Rightarrow |L_1| \leq 1.$

$|L_1| = 0 \iff L_1$ rep'd by  or 

not this: L is nontrivial link $\qquad$ not this: $Lk(L_1, L_2) = 0$.

- similarly $|L_2| = 1$. 

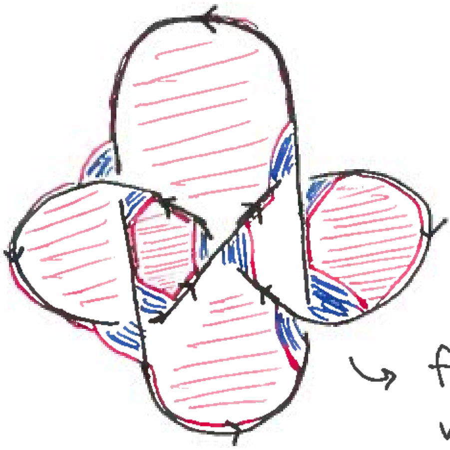
- Observe $|L_1 + L_2| = |L_1 - L_2|$ by symmetry of L



preserves or on L_1 switches L_2

- Claim $|L_1 + L_2| = 2$.

• $|L_1+L_2| \le 2$ by finding ~~Seifert~~ Seifert surface (connected, or surface S, $\partial S = L$)



$\chi(S) = \#\text{disks} - \#\text{strips} = -2$

(torus w/ 2 boundary comps)

↳ follow along line making jumps at crossings compatible w/ or. Gives disks that connect w/ strips at ~~corresp.~~ to crossing

• OTOH if $|L_1+L_2| < 2$ then $|L_1+L_2| = 0$ b/c

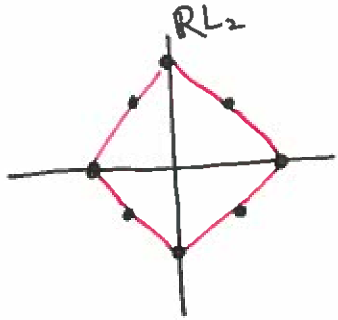
$$\chi(S) \equiv \#\begin{pmatrix}\text{boundary}\\ \text{comp of } S\end{pmatrix} \bmod 2$$

$|L_1+L_2| = 0$ violates convexity:

$$1 = |L_1| \le \tfrac{1}{2}|L_1-L_2| + \tfrac{1}{2}|L_1+L_2|$$

So $|L_1+L_2| \ge 2$.

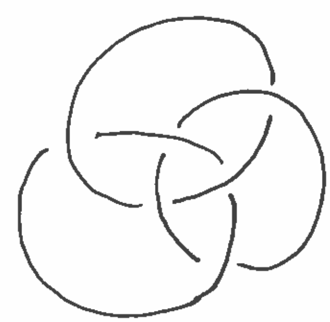
Norm ball det by these pts



$\Rightarrow |\cdot|_T = \|\ \|_{L^1}$

Cor Minimal rep of $17L_1 + 45L_2$ has $\chi_-(S) = 62$.

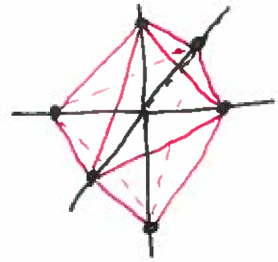
(3) Exercise Compute norm ball for $L =$



Borromean rings

$|L_i| = 1$, $|\pm L_1 \pm L_2 \pm L_3| = 3$ constant

(constant by sym, convexity $\Rightarrow \ge 3$, Seifert surface $\Rightarrow \le 3$)

Norm ball  octahedron.

II. Thurston norm & fiberings.

Prop $\begin{array}{c} F \longrightarrow M \\ \downarrow \\ S^1 \end{array}$ $\Rightarrow$ $[F] \in H_2(M)$ is norm-minimizing.
$|F| = \chi_-(F)$

Pf Take $S \hookrightarrow M$ $[S] = [F]$ norm minimizing.

(i) $\pi_1(S) \longrightarrow \pi_1(M)$ injective (equiv S incompressible
$\nexists\ D^2 \hookrightarrow M$ st. $D \cap S = \partial D$,
$\partial D \subset S$ essential)

S incompressible clearly b/c norm minimizing.

(equivalence is some nontrivial 3mfld topology) — loop theorem.

(ii) S lifts to $M_F \simeq F \times \mathbb{R}$ cover corresp to $\pi_1(F) < \pi_1(M)$.
$\downarrow$
M

(iii) Consider $f: S \longrightarrow M_F \simeq F \times \mathbb{R} \longrightarrow F$

- degree 1 b/c $[S] = [F]$ $\qquad |F| = \chi(S) = \chi(F)$ $\Uparrow$ $\square$
- π_1-injective

$\rightarrow$ f is h.e. (surfaces gps don't have injections endos that aren't isos)

Pf of (ii) Need to show $\pi_1(S) < \mathrm{Ker}\left(\pi_1(M) \xrightarrow{p} \mathbb{Z}\right) = \pi_1(F)$

(lifting criterion from covering sp. theory)

Since $\mathbb{Z}$ abelian, suffices to show

$$H_1(S) \xrightarrow{j} H_1(M) \xrightarrow{p} \mathbb{Z} \quad \text{is zero.}$$

- ~~Recall~~ Note: $[F] \in H_2(M;\mathbb{Z})$ for $F \to M \xrightarrow{p} S^1$

has $PD(F) \in H^1(M;\mathbb{Z}) = [M, S^1]$ rep'd by p.

- Alternatively, in de Rham coho $PD(F) = p^*(d\theta)$, $d\theta \in \Omega^1(S^1)$.

- Alternatively, $PD(F)$ is function $H_1(M) \longrightarrow \mathbb{Z}$, $[\gamma] \longmapsto \gamma \cdot F$

(intersection of oriented submflds)

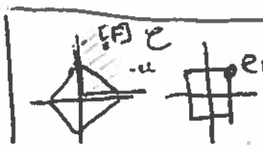
- $[S] = [F] \Rightarrow$ for $[\gamma] \in H_1(S)$ $\quad j(\gamma)\cdot F = j(\gamma)\cdot S = 0$

(push γ off S). $\square$

Next time

Thm (Thurston) For $F \to M \to S^1$

- $e_F \in H^2(M)$ vertical Euler class.
- $[F] \in H_2(M)$ lies in interior of cone on top-dim'l face of norm ball B
- $|e| = 1$ (dual norm) and e is a vertex of B^*
- $\langle e, F\rangle = |F|$ and if $[S] \in H_2(M)$ in same cone, $\langle e, S\rangle = |S|$.



$\langle e_F, a\rangle = |a|$ $\forall a \in \mathcal{C}$

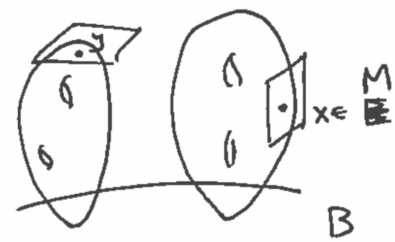
I. Thurston norm and fiberings

Vertical Euler class

$S_g \to M \xrightarrow{\pi} B$ smooth oriented surface bundle

- $\mathbb{R}^2 \to T_\pi M := \ker(d\pi : TM \to TB)$ fiber over $x \in M$ is tangent space in the "fiber direction"

$$\begin{array}{c} T_\pi M \\ \downarrow \\ M \end{array}$$



- Euler class $e_\pi := e(T_\pi M) \in H^2(M)$

obstruction to nonvanishing section of $T_\pi M \to M$

as a function $e_\pi : H_2(M) \longrightarrow \mathbb{Z}$.

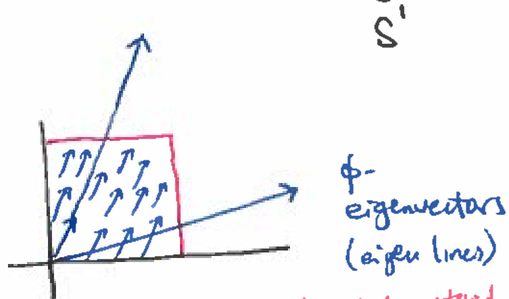
$[S \xrightarrow{f} M] \longmapsto$ (self intersection # of 0-section in $f^* T_\pi M \to S$).

Example. $T^2 \longrightarrow M \xrightarrow{\pi} S^1$

Claim $e_\pi = 0 \in H^2(M)$.

Pf $\phi \in \mathrm{Mod}(T^2) \simeq SL_2\mathbb{Z}$ monodromy.

A ϕ-inv. vector field on T^2 extends to v.f. on $M = \dfrac{T^2 \times [0,1]}{(x,0) \sim (\phi(x),1)}$



(what if negative evals?) rotate instead of straight line htpy

Note May have to interpolate on $[0,1]$ direction since $d\phi(v) = \lambda v$. $\lambda \neq 1$.

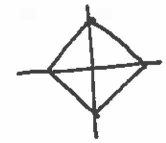
Example $S_g \to M \xrightarrow{\pi} S^1$ $g \neq 1 \Rightarrow e_\pi \neq 0 \in H^2(M)$ (always - ie for every monodromy)

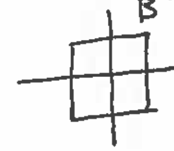
Pf. $\begin{array}{ccc} TS_g & \longrightarrow & T_\pi M \\ \downarrow & & \downarrow \\ S_g & \xhookrightarrow{i} & M \end{array}$ $\quad \begin{array}{c} H^2(S_g) \\ \Psi \\ i^*(e_\pi) = e(TS_g) \neq 0 \end{array}$

Dual norm M compact, oriented 3-mfld., $\|\cdot\| : H_2(M;\mathbb{R}) \to \mathbb{R}_+$ Thurston norm.

$\phi \in H^2(M;\mathbb{R})$ $\qquad \|\phi\|^* := \sup\limits_{\substack{a \in H_2(M) \\ \|a\| \le 1}} \phi(a)$

Ex. $M = S^3 \setminus L$ $\quad L =$  Whitehead link

$B \subset H_2$  $\|\cdot\| = \|\cdot\|_{L^1}$

$B^* \subset H^2$  $\|\cdot\|^* = \|\cdot\|_{L^\infty}$

Basic fact. $u, v \in B$, $\phi \in B^*$

- If $\phi(u) = \phi(v) = 1$. then u, v share face, ϕ on dual face.
- If $u_1, \dots, u_k \in B$ basis for H_2 & $\exists$ $\phi \in B^*$ s.t. $\phi(u_i) = 1$ $\forall i$ and ϕ vertex of B^*. then u_i contained in top dim'l face

Thm (Thurston) $S_g \to M \xrightarrow{\pi} S^1$ $\qquad \chi(S_g) < 0$.

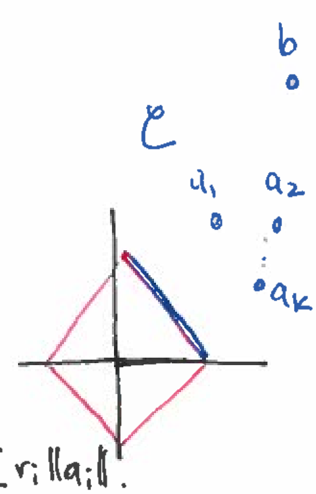
$F \subset B$.

(i) $\dfrac{[S_g]}{\|S_g\|} \in H_2$ lies on interior of top dim'l face ~~B~~.

$e_\pi \in H^2$ is vertex of B^*.

(ii) $a \in H_2(M;\mathbb{Z})$ and $\dfrac{a}{\|a\|} \in F \Rightarrow a$ corresponds to fibering $S' \to M \xrightarrow{\pi'} S^1$

$e_\pi = e_{\pi'}$, and $\|a\| = |\langle e_\pi, a \rangle|$.



Corollaries

(1) (Q of Yusheng) On a fibered face, norm det by basis.

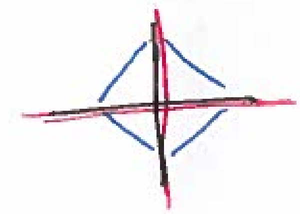
$b = \sum r_i a_i \quad r_i \geq 0$

$\|b\| = |\langle e_\pi, b \rangle| = |\sum r_i \langle e_\pi, a_i \rangle| = \sum r_i |\langle e_\pi, a_i \rangle| = \sum r_i \|a_i\|$.

(2) If $\|\cdot\|$ is norm and $\dim H_2 \geq 2$

(a) $\exists$ surface not fiber of fibration

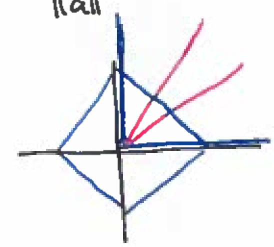
(b) M fibers in ∞ly many ways.



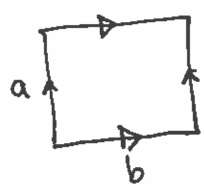
Will prove weaker statement:

(ii) $\exists$ nbhd ~~of~~ N of $\frac{S}{\|S\|} \in B$ s.t. statement true whenever $\frac{a}{\|a\|} = N$.

(pf will be easy after...)



II. 1-forms and fiber bundles.

Example. $M = T^2 =$ 

$\omega = f\,dx + g\,dy$ closed 1-form (e.g. $f, g \in \mathbb{R}$)

- periods $(A, B) := \left(\int_a \omega, \int_b \omega \right)$

- If ω nonsingular ($\omega_x : T_x M \longrightarrow \mathbb{R}$ nonzero $\forall x \in M$)

then ω defines foliation $\mathcal{F}$ / distribution $H_x = \{ v \in T_x M \mid \omega(v) = 0 \}$.

- If ω rational ($B = \frac{p}{q} A$) then $\mathcal{F}$ has compact leaves

~~and there is no fibration~~ which are fibers of fibration.

$T^2 \longrightarrow \mathbb{R} / \lambda\mathbb{Z}$ $\qquad 0 \in T^2$ base pt.

$x \longmapsto \int_0^x \omega$ ~~[illegible]~~ $\qquad \langle \lambda \rangle = \langle A, B \rangle \subset \mathbb{R}$

(Ehresmann: M, N cpt $f : M \longrightarrow N$ surjective submersion is a ~~fibration~~ loc. triv. fibration)

Here submersion b/c ω nonsingular, surj trivial: ~~[illegible]~~ $[\omega] \neq 0$

More generally M closed n-mfld.

fiber bundle $M \to S^1$ $\leftrightsquigarrow$ closed, nonsingular, rational 1-forms ω

$\hookrightarrow$ $\forall\, \gamma, \eta \in H_1(M;\mathbb{Z})$ $\exists\, p/q$ s.t. $\int_\gamma \omega = \frac{p}{q} \int_\eta \omega$

$$[\pi : M \to S^1] \longmapsto \pi^*(d\theta) \quad \text{nonsingular b/c } \pi \text{ submersion.}$$

$$\left[x \mapsto \int_0^x \omega \mod \mathbb{Z}\right] \longleftarrow \omega$$

Poincare duality: $H^1(M) \simeq H_2(M)$, $\pi^*(d\theta) \in H^1(M)$, $[S] \in H_2(M)$, $S \to M$, $M \xrightarrow{\pi} S^1$

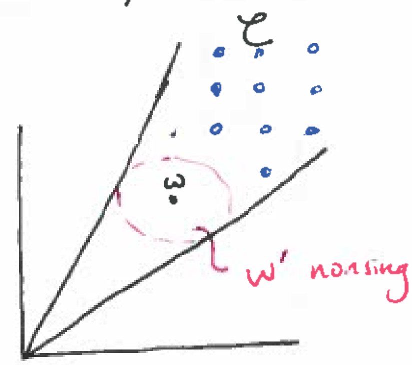
Perturbing a fibered class

$\omega = \pi^*(d\theta)$.

$\omega' := \omega + \sum \varepsilon_i \omega_i$ $\quad$ $\omega_1, \dots, \omega_d \in \Omega^1(M)$ reps for basis for $H^1(M;\mathbb{R})$, $\varepsilon_i \in \mathbb{R}$

For $\varepsilon_i << 1$, ω' nonsingular

$\omega' \in H^1(M;\mathbb{Q}) \Rightarrow \omega'$ corresp to fibration $S' \to M$, $M \xrightarrow{\pi'} S^1$



- $e_\pi = e_{\pi'}$ because foliations / distributions differ by small deformation $T_\pi M = \bigcup H_x(\omega)$ $\quad$ $T_{\pi'} M = \bigcup H_x(\omega')$.

- $|\langle e_\pi, s \rangle| = |\chi(s)| = \|S\|$ (fibers norm minimizing)

- $\|e_\pi\| = 1$ since $|\langle e_\pi, x \rangle| = \|x\|$ on open cone $\mathcal{C}$ (in particular on basis)

this also $\Rightarrow$ $e_\pi \in B^*$ is vertex (by basic fact earlier)

Application:
Homeo prob for $\begin{array}{c} S \to M \\ \downarrow \\ S^1 \end{array}$

Recall. $\left\{ \begin{array}{c} S \to M \\ \downarrow \\ S^1 \end{array} \right\} /$ bundle iso $\longleftrightarrow \{ \phi \in \mathrm{Mod}(S) \} / \mathrm{conj}.$

(haven't proved this, will later)

$\|$

$\left\{ \begin{array}{c} S \to M \\ \downarrow \\ S^1 \end{array} \right\} /$ fiberwise homeo

Problem: A given 3-mfld may fiber in many ways w/ diff fibergenus

Question Given $\phi \in \mathrm{Mod}(S_g)$, $\psi \in \mathrm{Mod}(S_h)$, when are $M_\phi \simeq M_\psi$ homeo ??

Algorithm

(1) Compute norm balls for M_ϕ, M_ψ.

if $M_\phi \simeq M_\psi$ then $H_2 M_\phi \simeq H_2 M_\psi$ as normed spaces.

Find ~~maxim~~ $\min\{ g \mid M$ fibers over S^1 w/ fiber $S_g \}$.

(2) List minimal genus fiberings $\phi_1, \dots, \phi_r$ $\left(\begin{array}{c} M_\phi \simeq M_\psi \Rightarrow \\ r = s \end{array} \right)$.

$\psi_1, \dots, \psi_s$

(3) $M_\phi \simeq M_\psi \iff \phi_i$ conjugate to ψ_j some i, j.

- conjugacy problem in Mod_g solvable.

(Hemion, Masur-Minsky, Hamenstädt)

Next week surface bundles over surfaces.

Lecture 15

I. Characteristic classes of surface bundles

Defn M mfld. A characteristic class c of M bundles is an assignme

$$(M \to E \to B) \longmapsto c(E) \in H^*(B)$$

natural wrt bundle pullbacks

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

$$c(f^*E) = f^*c(E).$$

(equiv. $c : \mathrm{Bund}_M(-) \longrightarrow H^*(-)$ natural transformation)

Defn G top. ~~space~~ group. A principal G-bundle is a fiber bundle $P \to B$ w/ action

$$\begin{tikzcd} P \times G \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & B & \end{tikzcd}$$

free & ~~simple~~ transitive on fibers

$$\begin{tikzcd} \pi^{-1}(b) \times G \arrow[rr] \arrow[dr] & & \pi^{-1}(b) \arrow[dl] \\ & b & \end{tikzcd}$$

A pGb $EG \to BG$ is universal if EG contractible.
BG called a classifying space (unique up to htpy).
note $\pi_i(BG) \cong \pi_{i-1}(G)$

Key property. Spse $G \curvearrowright M$. For nice B

$$\left\{\begin{matrix} M \text{ bundles} \\ E \to B \text{ w/} \\ \text{str. gp. } G \end{matrix}\right\}\Big/\text{iso} \longleftrightarrow \left\{\begin{matrix} \text{principal} \\ G \text{ bundles} \\ P \to B \end{matrix}\right\}\Big/\text{iso} \longleftrightarrow \left\{\begin{matrix} \text{cts maps} \\ B \to BG \end{matrix}\right\}\Big/\text{htpy}.$$

$$\begin{matrix} M \to E = \dfrac{P \times M}{G} \\ \downarrow \\ B \end{matrix} \quad \longleftarrow\!\shortmid \quad \begin{matrix} P = f^*EG \\ \downarrow \\ B. \end{matrix} \quad \longleftarrow\!\shortmid \quad f : B \to BG.$$

$\Rightarrow$ characteristic classes are elts of $H^\bullet(BG)$:

for $c \in H^\bullet(BG)$ $\quad c\left(\begin{matrix} E & \longrightarrow & \frac{EG\times M}{G} \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{matrix}\right) = f^*(c)$.

Examples

(1) G discrete. (principle bundle is covering sp. w/ deck group G) $\quad \begin{matrix}\widetilde{K(G,1)} \\ \downarrow \\ K(G,1)\end{matrix}$ universal $\Rightarrow$ $BG \sim K(G,1)$

(2) vector bundles. $G = GL_n\mathbb{R}$.

EG = n-frames in $\mathbb{R}^\infty$ (Stiefel manifold) (contractible, G acts in obvious way on n frames...)

BG = n-planes in $\mathbb{R}^\infty \equiv Gr_n\mathbb{R}^\infty$ (Grassmannian)

$H^\bullet(BGL_n\mathbb{R}) \simeq H^\bullet(BO(n))$ ~~is~~ generated by $p_i \in H^{4i}(\ ;\mathbb{Q})$, $w_i \in H^{i}(\ ;\mathbb{Z}/2)$.

Pontryagin classes / SW classes.

(3) $G = \mathrm{Diff}(M)$. [Rmk: BG always exists, but explicit model frequently useful to prove ~~do~~ anything. Where to find model? View $E(GL_n\mathbb{R}) = \mathrm{Hom}_{inj}(\mathbb{R}^n, \mathbb{R}^\infty)$.

$EG = \mathrm{Emb}(M, \mathbb{R}^\infty) \circlearrowleft \mathrm{Diff}(M)$ freely precomposition.

$\downarrow$

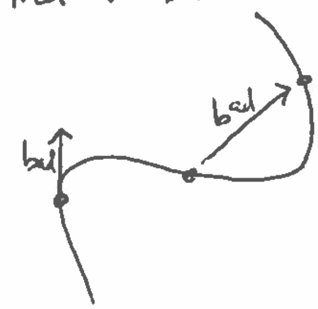
BG = submflds of $\mathbb{R}^\infty$ diffeo to M.

Prop EG is (weakly) contractible. (parameterized whitney embedding)

(Weak Whitney embedding) $\exists\, M^n \hookrightarrow \mathbb{R}^{2n+1}$

Pf idea: Given $f: M \longrightarrow \mathbb{R}^N$ if $N > 2n+1$, find $v \in \mathbb{R}^N$

s.t. $\pi_v \circ f$ still embedding.



bad directions $S^k \times \overbrace{(M \times M)}^{2n} \longrightarrow S^{N-1}$

$$(x,y) \longmapsto \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$$

$$S^k \times \overbrace{(T'M)}^{2n-1} \longrightarrow S^{N-1}$$

Sard: $N-1 > 2n \Longrightarrow \exists$ good direction.

Parameterized version: $f_t : M^n \longrightarrow \mathbb{R}^N \quad t \in S^k$

$N > 2n+1+k \Longrightarrow \exists$ good direction.

$\Longrightarrow \exists \quad \pi : \mathbb{R}^N \longrightarrow \mathbb{R}^{2n+1+k}$ s.t. $\pi \circ f_t$ emb. $\forall t$.

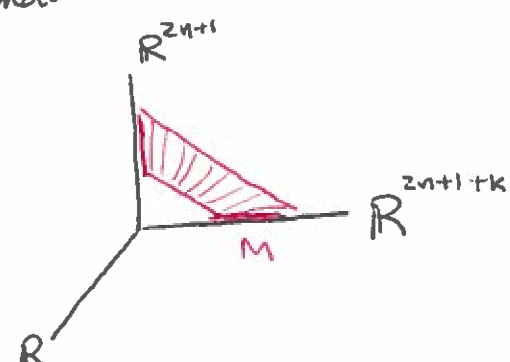
Application: for $N \gg n, k$, $f : S^k \longrightarrow \mathrm{Emb}(M, \mathbb{R}^N)$ extends over D^{k+1}.

Fix any $g : M \longrightarrow \mathbb{R}^{2n+1}$. Define

$\hat{f}_{s,t} : M \longrightarrow \mathbb{R} \times \mathbb{R}^{2n+1+k} \times \mathbb{R}^{2n+1}$ embeddings,

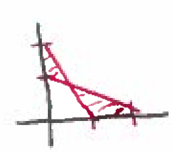
$\hat{f}_{s,t} = (s, (1-s) f_t, s g)$.

$\hat{f}_{0,t} = f_t$. $\quad \hat{f}_{1,t} = (1, 0, g)$ constant.



Rmk. use $\mathbb{R}$ direction to ensure embedding

to prevent



$\square$

Problem Compute $H^\bullet(B\mathrm{Diff}(S))$.

II. Miller–Morita–Mumford classes.

Recall. $S \to E^{d+2} \xrightarrow{\pi} B^d$ oriented. $\quad \mathbb{R}^2 \to T_\pi E = \ker(d\pi : TE \to TB)$, $T_\pi E \to E$.

$e \in H^2(E)$ vertical Euler class (cc's should be in $H^\cdot(B)$).

– Gysin homomorphism $\pi_! : H^k(E) \longrightarrow H^{k-2}(B)$

Defn MMM classes $e_k(E) = \pi_!(e^{k+1}) \in H^{2k}(B)$.

Defining $\pi_!$ (case ev'tg mfld) (many options)

① Poincaré duality.

$$\pi_! : H^k(E) \simeq H_{d+2-k}(E) \xrightarrow{\pi_*} H_{d+2-k}(B) \simeq H^{k-2}(B).$$

② de Rham coho $\quad \pi_! : \Omega^k(E) \longrightarrow \Omega^{k-2}(B)$ "integration-along fiber"

$$\omega \longmapsto \int_S \omega$$

③ Evaluate in total space.

$$e_k : H_{2k}(B) \longrightarrow \mathbb{Q}$$
$$[N^{2k} \xrightarrow{f} B] \longmapsto \langle e^{k+1}, [f^*E] \rangle.$$

$$\begin{array}{ccc} f^*E & \to & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & B \end{array}$$

Problem. Show $e_k \in H^{2k}(B\mathrm{Diff}(S))$ nonzero / linear indep.

(will give one geometric / one homotopy theoretic pf)

Rmk. $\pi_!$ not ring hom $\quad \pi_!(e^{k+1}) \neq (\pi_!(e))^{k+1}$

(so no obvious relation btwn the e_k's.)

Rmk. Defn works for S replaced by M^n.

$$\begin{array}{c} M \to E \\ \downarrow \pi \\ B \end{array} \rightsquigarrow \text{ classes } \pi_!(c(T_\pi E)) \text{ for } c \in H^\cdot(BO(n)).$$

Example $\quad e_0 = \pi_!(e(T_\pi)) \in H^0(B)$

$e_0 : H_0(B) \longrightarrow \mathbb{Q}$. multiplication by $\chi(S)$.

Example/Lemma. For $\begin{array}{c} S_g \to E \\ \downarrow \\ S_h \end{array}$ $\quad \langle e_1(E), S_h \rangle = 3\,\mathrm{sig}(E)$.

Recall. $\mathrm{sig}(M^4)$.

$$H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \xrightarrow{B} \mathbb{R}$$
$$(a,b) \longmapsto \langle a \cup b, [M] \rangle.$$

nondeg. symmetric bilinear form.

For some basis $\quad (B(e_i,e_j)) = \begin{pmatrix} Id_p & \\ & -Id_q \end{pmatrix}$. $\quad \mathrm{sig}(M) = p-q$

homotopy inv.

Hirzebruch signature Thm $\quad \mathrm{sig}(M) = \langle \tfrac{1}{3} p_1(TM), [M] \rangle$.

Pf of Lemma

<u>Pf of Lemma</u>. Note $TE \simeq T_\pi E \oplus \pi^*(TB)$, $\quad p_1 = e^2$

$3\,\mathrm{sig}(E) = \langle p_1(TE), [E] \rangle$

$= \langle p_1(T_\pi E \oplus \pi^*(TB)), [E] \rangle = \langle e^2(T_\pi E), E \rangle + \langle \pi^*(\overset{0}{\cancel{p_1(TB)}}), E \rangle$

$= \langle \pi_!(e^2(T_\pi E)), [S_n] \rangle$

$= \langle e_1(E), [S_n] \rangle$. $\quad \square$.

<u>Next</u>. $e_1 \neq 0$. in $H^2(B\mathrm{Diff}(S))$.

Lecture 16

Last time

- $BDiff(S) = EDiff(S)/Diff(S)$ classifying space.
- MMM construction
  - M^n oriented mfld $c \in H^k(BSO(n)) \rightsquigarrow \kappa_c \in H^\cdot(BDiff\, M)$

$$\kappa_c\left(\begin{matrix} M^n \to E \\ \downarrow\pi \\ B\end{matrix}\right) = \pi_!\left(c\left(\begin{matrix}\mathbb{R}^n_i \to T_\pi E \\ \downarrow \\ E\end{matrix}\right)\right) \in H^{k-n}(B).$$

$\pi_! : H^k(E) \longrightarrow H^{k-n}(B)$ Gysin hom $\left(\begin{matrix}\text{IOU in} \\ \text{general case}\end{matrix}\right)$

Rmk $\kappa_c = \kappa_c\left(\begin{matrix} M \to \frac{EDiff M \times M}{Diff M} \\ \downarrow \\ BDiff(M)\end{matrix}\right) \in H^\cdot(BDiff\, M)$

  - $(n=2)$ $H^\cdot(BSO(2);\mathbb{Z}) = H^\cdot(\mathbb{CP}^\infty;\mathbb{Z}) \simeq \mathbb{Z}[e]$ $\begin{matrix} e \in H^2 \\ \text{Euler class}\end{matrix}$

$e_i := \kappa_{e^{i+1}}$ ith MMM class $\in H^{2i}(BDiff(S))$.

  - $e_0\left(\begin{matrix} S \to E \\ \downarrow\pi \\ B\end{matrix}\right) \in H^0(B) \simeq \mathbb{Z}$ (say B connected)

$\langle e_0(E), pt\rangle = \langle e(T_\pi E), \pi^{-1}(pt)\rangle = \langle e(TS), [S]\rangle = \chi(S)$

$\Rightarrow e_0(E) = \chi(S)$ (so have cc that knows sg about top. of fiber)

I. ~~Interpretting~~ Interpreting $e_1 \in H^2(BDiff(S))$.

Lemma For $\begin{matrix} S_g \to E \\ \downarrow \\ S_h\end{matrix}$ $\langle e_1(E), [S_h]\rangle = 3 \cdot sig(E)$.

Recall $\mathrm{sig}(M^4)$ $\quad B: H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \longrightarrow \mathbb{R}$.

$$(a,b) \longmapsto \langle a \smile b, [M] \rangle$$

nondeg. sym. bilinear form.

For some basis $(B(e_i,e_j)) = \begin{pmatrix} Id_p & \\ & -Id_q \end{pmatrix}$ $\quad \mathrm{sig}(M) = p-q$ htpy invt.

Thm (Hirzebruch signature) $\quad \mathrm{sig}(M^4) = \frac{1}{3}\langle p_1(TM), [M] \rangle$.

$p_1\left(\begin{matrix}\mathbb{R}^n \to V \\ \downarrow \\ B\end{matrix}\right) = c_2\left(\begin{matrix}\mathbb{C}^n \to V\otimes\mathbb{C} \\ \downarrow \\ B\end{matrix}\right) \in H^4$

Cor: $p_1^{\#}$ is htpy invt. and $3 \mid p_1^{\#}$.

(also follows from interpretation $p_1^{\#} = 3 \cdot \#\left(\begin{matrix}\text{triple int.} \\ \text{of } M^4 \hookrightarrow \mathbb{R}^6\end{matrix}\right)$.)

Pf of lemma (choose a connection)

- $TE \simeq T_\pi E \oplus \pi^*(TB)$
- For 2-plane bundle $p_1 = e^2$

$$3\,\mathrm{sig}(E) = \langle p_1(TE), [E] \rangle = \langle p_1(T_\pi E \oplus \pi^* TS_h), [E] \rangle.$$

$$= \langle p_1(T_\pi E) + \underbrace{\pi^* p_1(TS_h)}_{=0}, [E] \rangle = \langle e^2(T_\pi E), [E] \rangle$$

$$= \langle e_1(E), [S_h] \rangle \qquad \square.$$

Recall W/ $\mathbb{Q}$ coeff

$$H^2(B\mathrm{Diff}(S_g)) \simeq H^2(\mathrm{Mod}_g) \simeq H^2(\mathcal{M}_g) \simeq \mathbb{Q}\{\mu\}.$$

$$H^2(B\mathrm{Diff}(S_{g,*})) \simeq H^2(\mathrm{Mod}_{g,*}) \simeq H^2(\mathcal{M}_{g,*}) \simeq \mathbb{Q}\{\mu, e\}.$$

$\mathcal{M}_g$ = moduli space of genus g Riem. surfaces. $\quad \mathcal{M}_{g,*}$ = surfaces w/ marked pt.

$\boxed{e}$ $\quad e \in H^2(\mathrm{Mod}_{g,*})$ Euler class of

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Homeo}(S^1) \arrow[r] & \mathrm{Homeo}(S^1) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \mathrm{Mod}_g^1 \arrow[r] \arrow[u] & \mathrm{Mod}_{g,1} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

(a) $$\begin{array}{ccc} S_g & \to & \mathcal{M}_{g,*} \\ & & \downarrow \pi \\ & & \mathcal{M}_g \end{array}$$

induced map on (orbifold) π_1 is Birman seq.

$$1 \to \pi_1(S_g) \xrightarrow{P} \mathrm{Mod}_{g,*} \longrightarrow \mathrm{Mod}_g \longrightarrow 1.$$

$$e = e(T_\pi \mathcal{M}_{g,*}) \in H^2(\mathcal{M}_{g,*};\mathbb{Q}) \simeq H^2(\mathrm{Mod}_{g,*};\mathbb{Q})$$

$e \neq 0$ since evaluates nontrivially on fiber ($g \geq 2$).

(b) $\bar{e} \in H^2(B\mathrm{Diff}(S_g,*))$

- $B\mathrm{Diff}(S_g,*)$ classifies surface bundles w/ section

$$\begin{array}{ccc} S_g & \longrightarrow & E \\ & & \pi \downarrow \uparrow \sigma \\ & & B \end{array}$$

Define $\bar{e}(E) = \sigma^* e(T_\pi E) \in H^2(B)$

defines cc $\bar{e} \in H^2(B\mathrm{Diff}(S,*))$.

For $$\begin{array}{ccc} S_g & \to & E \\ & & \downarrow \uparrow \\ & & S_h \end{array}$$

$$\langle \bar{e}(E), [S_h] \rangle = \langle \sigma^* e(T_\pi E), S_h \rangle = \langle e(T_\pi E), \sigma(S_h) \rangle$$

$$= \langle \text{Euler class of normal bundle to } \sigma(S_h), \sigma(S_h) \rangle = \sigma(S_h) \cdot \sigma(S_h)$$

self intersection #.

<u>Example</u> Pointpushing subgroup $\pi_1(S_g) \xrightarrow{P} \mathrm{Mod}_{g,*}$ is monodromy

of trivial bundle w/ diagonal section

$$\begin{array}{c} E = S_g \times S_g \\ \downarrow \uparrow \Delta \\ S_g \end{array}$$

$$T_\pi|_\Delta \simeq TS_g \quad \Rightarrow \quad \bar{e}(E) = e(TS_g) \in H^2(S_g) \simeq \mathbb{Z}.$$

$\boxed{\mu}$ $\quad \mu \in H^2(\mathrm{Mod}_g)$ Euler class of $0 \to \mathbb{Z} \to \widetilde{Sp_{2g}\mathbb{R}} \to Sp_{2g}\mathbb{R} \to 1$

$$\begin{array}{c} \uparrow \\ \mathrm{Mod}_g \end{array}$$

as a charactistic class:

$$\begin{array}{ccc} S_g \to E & \rightsquigarrow & \mathbb{R}^{2g} \simeq H_1(S_g) \to V \\ \downarrow & & \downarrow \\ B & & B \end{array}$$

classified by $f: B \to BDiff(S_g) \xrightarrow{\sim} B\mathrm{Mod}_g \longrightarrow BSp_{2g}\mathbb{R} \sim BU(g)$.

$H^2(BU(g)) \simeq \mathbb{Q}\{c_1\}$ $\qquad \mu(E) = f^*(c_1) = c_1(V) \in H^2(B)$.

<u>Rmk</u> (connection to e_1) $\quad \begin{array}{c} \mathbb{C}^g \to V \\ \downarrow \\ B \end{array}$ $\quad$ defines $V \in K(B)$ (Grothendieck group of v.b. over B)

$V - \overline{V} = \mathrm{ind}(D) \in K(B)$ $\qquad D = \{D_b\}_{b \in B}$ family of elliptic ops on S_g.

<u>Atiyah-Singer index thm</u> $\qquad$ a-ind $=$ t-ind $\equiv \pi_!(\sigma(D))$

$$\text{a-ind} \equiv \mathrm{ind}(D)$$

$\sigma(D) \in K(E)$ symbol class

$\pi_!: K(E) \to K(B)$ Gysin.

$$\Rightarrow \quad ch(V - \overline{V}) = \pi_!\left(\frac{x}{\tanh(x/2)}\right) \qquad x = e(T_\pi E).$$

deg 2 term: $\quad 2c_1(V) = \frac{1}{6}\pi_!(e(T_\pi E)^2) = \frac{1}{6} e_1(E)$

Note $\frac{x}{\tanh(x/2)} \approx 2 + \frac{x^2}{6} + O(x^4)$ $\qquad \Rightarrow \quad e_1 = 12\mu$.

<u>Rmk</u> even w/ all this, not clear $e_1 \in H^2(BDiff(S))$ nonzero.

<u>Problem</u>: Show $\exists$ $\begin{array}{c} S_g \to E \\ \downarrow \\ S_h \end{array}$ w/ $sig(E) \neq 0$.

Ex $\mathrm{sig}(S_g \times S_h) = 0$.

Note if $M^4 = \partial W^5$ $\quad \mathrm{sig}(M) = \langle p_1(TM), [M] \rangle$
$= \langle i^* p_1(TW), [M] \rangle = \langle p_1(TW), \underbrace{i_*[M]}_{=0} \rangle$

$S_g \times S_h = H_g \times S_h$ $\qquad H_g =$ 

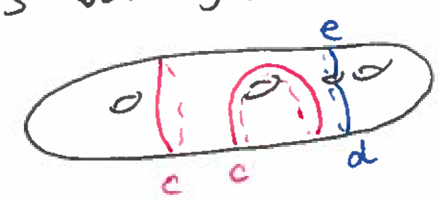
Ex by discussion above if $\begin{matrix} S_g \to E \\ \quad\ \downarrow \\ \quad\ B^2 \end{matrix}$ st.

$\pi_1(B) \curvearrowright H_1(S_g)$ trivial (ie $f: \pi_1(B) \to \mathrm{Mod}_g \to Sp_{2g}(\mathbb{Z})$ trivial)

then $\mathrm{sig}(E) = 4 f^*(c_1) = 0$.

Rmk $\ker(\mathrm{Mod}_g \longrightarrow Sp_{2g}\mathbb{Z}) =: \mathcal{I}_g$ Torelli group.

(Birman–Powell) $\mathcal{I}_g = \left\langle T_c, T_d T_e \;\middle|\; \begin{matrix} c \subset S \text{ separating} \\ d, e \subset S \text{ bounding pair} \end{matrix} \right\rangle$



Ex. $\forall \begin{matrix} S_g \to E \\ \quad\ \downarrow \\ \quad\ T^2 \end{matrix}$ $\qquad \mathrm{sig}(E) = 0$.

For $k > 0$ consider
$$\begin{matrix} E_k & \xrightarrow{F_k} & E \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{f_k} & T^2 \end{matrix}$$
f_k deg k cover.
($\Rightarrow$ so is F_k)

OTOH $\quad \mathrm{sig}(E_k) = k\, \mathrm{sig}(E)$. (Pont #'s mult. under covers)

OTOH $\quad |\mathrm{sig}(E)| \leq \dim H_2(E) \leq \dim(H_2(S_g \times T^2)) = 4g + 2$.

$\Rightarrow \mathrm{sig}(E) = 0$.

Next time SBC w/ $\mathrm{sig} \neq 0$.

Lecture 17

I. Nontriviality of e_1

- MMM classes $\quad \mathbb{Z}[e_1, e_2, \ldots] \longrightarrow H^*(BDiff(S); \mathbb{Z}) \qquad e_i \in H^{2i}$
- interpretations of e_1

1) signature $\quad$ $\begin{array}{c} S_g \to E \\ \downarrow \\ S_h \end{array}$ $\quad \langle e_1(E), [S_h] \rangle = 3\,\mathrm{sig}(E)$

2) Chern class of Hodge bundle. $\quad$ $\begin{array}{c} S_g \to E \\ \downarrow \\ B \end{array} \rightsquigarrow \begin{array}{c} \mathbb{C}^g \to V \\ \downarrow \\ B \end{array}$ $\quad e_1(E) = 12 c_1(V)$.

<u>Thm.</u> (Atiyah, Kodaira) Construction of $\begin{array}{c} S_6 \to E \\ \downarrow \\ S_{129} \end{array}$

$\mathrm{sig}(E) \neq 0$.

<u>Cor.</u> $e_1 \neq 0 \in H^2(BDiff(S_g))$

$H^2(\mathrm{Mod}_g; \mathbb{Q}) = \mathbb{Q}\{\mu\}$. (previously only showed $\dim H^2 \leq 1$).

<u>Cor</u> (Morita) $\quad e_i \neq 0 \;\; \forall\, i \geq 1$.

<u>Warmup.</u> Naive constructions.

(1) For $\begin{array}{c} S_g \to E \\ \downarrow \\ S_h \end{array}$ if $\pi_1(S_h) \curvearrowright H_1(S_g)$ trivial, then $\mathrm{sig}(E) = 0$.

(since then $\begin{array}{c} \mathbb{C}^g \to V \\ \downarrow \\ S_h \end{array}$ trivial. $V = S_h \times \mathbb{C}^g$. $\Rightarrow c_1(V) = 0$)

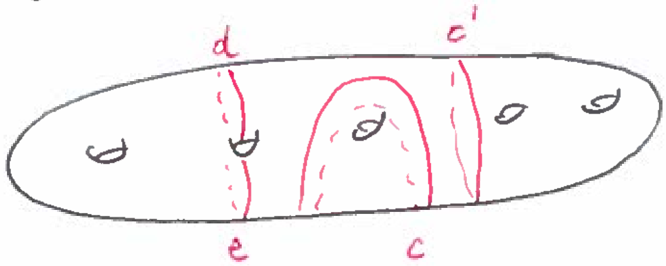
Equivalently, define $\mathcal{I}_g := \ker(\mathrm{Mod}_g \longrightarrow Sp_{2g}(\mathbb{Z}))$ Torelli group.

if monodromy factors

$$\begin{tikzcd} \pi_1(S_n) \arrow[r] \arrow[dr, dashed] & \mathrm{Mod}_g \\ & \mathcal{I}_g \arrow[u, phantom, "\cup"] \end{tikzcd}$$

then $\mathrm{sig}(E)=0$.

(Birman-Powell) $\mathcal{I}_g$ generated by separating twists T_c and bounding pairs $T_d T_e^{-1}$.



(2) ~~Pick~~ For $f_1,\dots,f_n \in \mathrm{Diff}(S_g)$, define $\underset{\langle a_1,b_1,\dots,a_n,b_n \mid \rangle}{\pi_1(S_n)} \longrightarrow \mathrm{Diff}(S_g)$

$$a_i \longmapsto f_i$$
$$b_i \longmapsto 1.$$

Defines flat bundle.

$$S_g \to E = \frac{\widetilde{S}_n \times S_g}{\pi_1(S_n)}$$
$$\downarrow$$
$$S_n$$

$\mathrm{sig}(E)=0$. since ~~its~~ classifying map of E factors

$$S_n \longrightarrow \bigvee_n S^1 \longrightarrow B\mathrm{Diff}(S_g)$$

and $H^2(\vee S^1)=0$.

(3) <u>Prop</u> $\forall$ $S_g \to E \downarrow T^2$ $\mathrm{sig}(E)=0$.

<u>Rmk.</u> Saw a version of this in proof ~~of~~ H^2 Mod cyclic using Hopf's formula. $S_g \to E \downarrow T^2 \leftrightarrow \mathbb{Z}^2 < M\dots$

<u>Pf.</u> Main observation: For such E $|\mathrm{sig}(E)| \leq 4g+2$.

By defn. $|\mathrm{sig}(E)| \leq \dim H_2(E)$.

sseq arg $\dim H_2 E \leq \dim H_2(S_g \times T^2) = 4g+2$.

if $sig(E) \neq 0$ consider

$$\begin{array}{ccc} E_k & \xrightarrow{\tilde{f}_k} & E \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{f_k} & T^2 \end{array}$$

$f_k : T^2 \to T^2$ deg k cover

($\Rightarrow$ $\tilde{f}_k$ also cover).

signature is a characteristic # ~~$sig(E) = \frac{1}{3}\langle p_1(TE), E \rangle$~~.

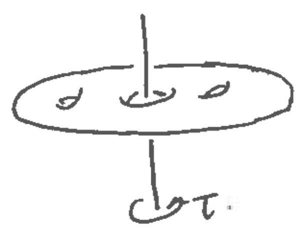
$\Rightarrow$ $sig(E_k) = k\, sig(E)$ since

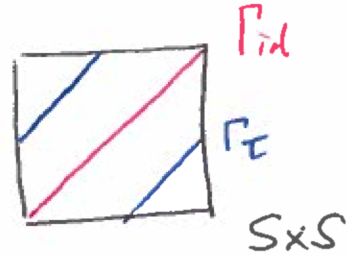
$$\begin{aligned} sig(E_k) = \tfrac{1}{3}\langle p_1(TE_k), E_k \rangle &= \tfrac{1}{3}\langle f_k^* p_1(TE), E_k \rangle \\ &= \tfrac{1}{3}\langle p_1(TE), f_k(E_k) \rangle \\ &= \tfrac{k}{3}\langle p_1(TE), E \rangle = k\, sig(E) \end{aligned}$$

$\square$.

<u>Idea of Atiyah-Kodaira construction</u>: branched covers

$G = \mathbb{Z}/2 \curvearrowright S$ free.





- want to take $\mathbb{Z}/2$ branched of $S \times S$ branched over $\Gamma_{id} \cup \Gamma_\tau$.
- result $\begin{array}{c} E \to S \times S \\ \downarrow \\ S \end{array}$ fibers over S w/ fiber over $x \in S$ $\mathbb{Z}/2$ branched cover of S branched at $x, \tau(x)$.

<u>Problem</u>. branched covers don't always exist.

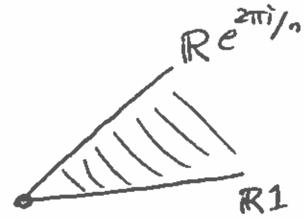
$$\begin{array}{ccccccc} S_6 \to E & \to & S_{129} \times S_3 & \to & S_3 \times S_3 \\ \downarrow & & \downarrow & & \downarrow \\ S_{129} & = & S_{129} & \to & S_3 \end{array}$$

- will pass to cover of base to ensure we can branch.

II. Hirzebruch criterion for branched covers.

$\mathbb{Z}/m$ branched covers

- prototype: $\mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^m$ — branched at 0. m fold covering on $\mathbb{C}\setminus\{0\}$.



alternate POV: $\mathbb{Z}/m \curvearrowright \mathbb{C}$ rotation of order m. $\pi : \mathbb{C} \to \mathbb{C}/(\mathbb{Z}/m) \simeq \mathbb{C}$

<u>Defn.</u>

$$\begin{array}{ccc} f: \hat{M} & \longrightarrow & M \\ \cup & & \cup \\ \hat{B} & \longrightarrow & B \end{array}$$

is a $\mathbb{Z}/m$ branched cover if B – codim 2 submfld. (B = <u>branched set</u>)

(1) $f|_{\hat{B}}$ diffeo.

(2) $f|_{\hat{M}\setminus\hat{B}}$ $\mathbb{Z}/m$ regular cover

(3) near $p \in \hat{B}$ f has form $U\times\mathbb{C} \longrightarrow U\times\mathbb{C}$, $(x,z) \longmapsto (x, z^m)$ for some $U \subset \hat{B}$.

In this case, $G = \mathbb{Z}/m = \langle \tau \rangle \curvearrowright \hat{M}$ w/ $\hat{B} = \{x \in \hat{M} : \tau x = x\}$ & $G \curvearrowright \hat{M}\setminus\hat{B}$ free.

<u>Thm</u> (Hirzebruch) $B^{n-2} \subset M^n$ oriented mflds. If $\exists$ $z \in H_{n-2}(M;\mathbb{Z})$, $m \in \mathbb{Z}$ s.t. $mz = [B]$, then $\exists$ $\mathbb{Z}/m$ branched cover $\hat{M} \longrightarrow M$ branched over B.

Poincare duality in codim 2

M^n closed or.

(Part of connection of Thurston ~~connected~~ norm on H_2 w/ fibrations over S^1)

Recall. $H_{n-1}(M;\mathbb{Z}) \simeq [M, S^1]$

$$H_{n-2}(M;\mathbb{Z}) \simeq H^2(M;\mathbb{Z}) \simeq [M, K(\mathbb{Z},2)] \simeq [M, BU(1)] \simeq \left\{ \begin{array}{c} \mathbb{C} \to V \\ \downarrow \\ M \end{array} \right\} / \text{iso}$$

$$B := \{\sigma = 0\} \longleftarrow\!\!\shortmid \left(\begin{array}{c} \mathbb{C} \to V \\ \downarrow \uparrow \sigma \\ M \end{array} \right)$$

$\sigma \pitchfork$ 0-section

Cor Every $z \in H_{n-2}(M;\mathbb{Z})$ rep'd by embedded submfld.

Conversely, given $B^{n-2} \subset M$ or. can construct $\begin{array}{c} \mathbb{C} \to V \\ \downarrow \uparrow \sigma \\ M \end{array}$

s.t. $B = \{\sigma = 0\}$. :

- $$\begin{tikzcd} & \mathbb{C} \arrow[d] \\ V_0 = \{(x,y) : \pi(x) = \pi(y)\} \arrow[r] \arrow[d, "\pi'"] & N(B) \arrow[d, "\pi"] \\ N(B) \arrow[r, "\pi"'] & B \end{tikzcd}$$

  $N(B)$ – tubular nbhd / normal bundle.

- **Observe** $\begin{array}{c} \mathbb{C} \to V_0 \\ \downarrow \\ N(B) \end{array}$ has section $\sigma(x) = (x,x)$ which vanishes $\iff x \in B$.

  In particular $V_0|_{N(B) \setminus B} \simeq (N(B) \setminus B) \times \mathbb{C}$ trivial since it has a nonvanishing section.

- To obtain $\begin{array}{c} \mathbb{C} \to V \\ \downarrow \\ M \end{array}$ glue $\begin{array}{c} M \setminus N(B) \times \mathbb{C} \\ \downarrow \\ M \setminus N(B) \end{array}$ to $\begin{array}{c} V_0 \\ \downarrow \\ N(B). \end{array}$

Lecture 18

No class Friday

Goal: Construction of $\begin{array}{c} S_6 \to E \\ \downarrow \\ S_{129} \end{array}$ w/ $\mathrm{sig}(E) \neq 0$.

Hirzebruch's criterion for branched covers

Recall • $\mathbb{Z}/m$ branched cover $f: \begin{array}{ccc} \hat{M}^n & \longrightarrow & M^n \\ \cup & & \cup \\ \hat{B} & \xrightarrow{\simeq} & B^{n-2} \end{array}$ $f|_{\hat{M} \setminus \hat{B}}$ $\mathbb{Z}/m$ cover in normal dir to $\hat{B}$ f looks like $\begin{array}{c} \mathbb{C} \to \mathbb{C} \\ z \mapsto z^m \end{array}$

• Thm (Hirzebruch) M^n closed oriented. $B^{n-2} \subset M$ or. submfld. If $m \mid [B]$ in $H_{n-2}(M)$ then $\exists$ $\hat{M} \to M$ branched over B.

• Key: there is a bijection $H_{n-2}(M;\mathbb{Z}) \simeq H^2(M;\mathbb{Z}) \simeq \left\{ \begin{array}{c} \mathbb{C} \to V \\ \downarrow \\ M \end{array} \right\} / \mathrm{iso.}$

- $\begin{array}{c} \mathbb{C} \to V \\ \downarrow \uparrow \sigma \\ M \end{array} \rightsquigarrow B = \{\sigma = 0\}$

- last time: ~~constructed~~ construction given $B \subset M$ of ~~bax~~ $\begin{array}{c} \mathbb{C} \to V_B \\ \downarrow \uparrow \sigma \\ M \end{array}$ w/ $B = \{\sigma = 0\}$.

- divisibility: $m \mid [B] \iff m \mid PD(B) = c_1(V)$

$\iff \exists$ $\begin{array}{c} \mathbb{C} \to W \\ \downarrow \\ M \end{array}$ st. $W^{\otimes m} \simeq V$.

Proof of Hirzebruch

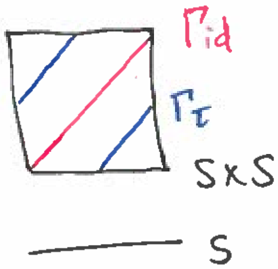
$$\begin{array}{ccc} W & \xrightarrow{f} & W^{\otimes m} \simeq V \\ \cup & & \downarrow \uparrow \sigma \\ \hat{M} := f^{-1}(\sigma(M)) & \longrightarrow & M \\ \cup & & \\ \hat{B} := f^{-1}(B) = & \hat{M} \cap \text{0-section.} & \end{array} \qquad B = \{\sigma = 0\}.$$

$f(u) = u \otimes \cdots \otimes u$

fiberwise is $\mathbb{C} \to \mathbb{C}$, $f|_{\hat{B}}$ diffeo since f diffeo on 0-section $\square$

Atiyah–Kodaira construction

Setup $G = \mathbb{Z}/2 = \langle \tau \rangle \curvearrowright S = S_g = S_3$ freely

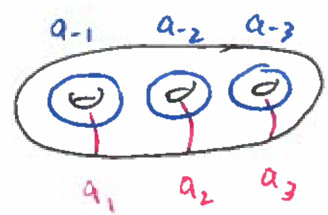


Step 1 Apply Hirzebruch. (Find right submfld to branch over.)

Want to take $\mathbb{Z}/2$ cover branched over $\Gamma_{id} \cup \Gamma_\tau \subset S \times S$.

Claim $D_1 = [\Gamma_{id}] + [\Gamma_\tau] \in H_2(S \times S)$ not even

- on a 4mfld $z \in H_2$ even $\iff$ $z \cdot u \in 2\mathbb{Z}$ $\forall u \in H_2$.
- $H_2(S \times S) \cong \underset{[S]_h}{H_2(S) \otimes 1} \oplus \underset{a_i \otimes a_j}{H_1(S) \otimes H_1(S)} \oplus \underset{[S]_v}{~~H~~ 1 \otimes H_2(S)}$.



* Note $D_1 \cdot [S]_h = 2 = D_1 \cdot [S]_v$.

OTOH for $u = a_1 \otimes a_{-1}$, u rep'd by torus $T \subset S \times S$ $\quad T = \{ (a_1(t), a_{-1}(s)) \}$

$$D_1 \cdot u = \Gamma_{id} \cdot T + \Gamma_\tau \cdot T$$
$$= \pm 1 + 0.$$

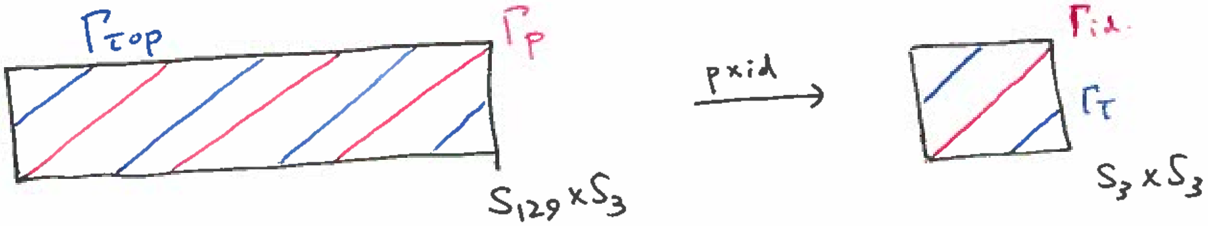
$(x,x) = (a_1(t), a_{-1}(s))$ for some x, t, s $\iff x \in a_1 \cap a_{-1}$

$(x, \tau x) = (a_1(t), a_{-1}(s))$ $\iff$ $x \in a_1 \cap \tau a_{-1}$ but $a_1 \cap \tau a_{-1} = \emptyset$.

//

• Consider cover $(\mathbb{Z}/2)^{2g} \simeq H_1(S;\mathbb{Z}/2) \longrightarrow \hat{S} \xrightarrow{p} S$

$\chi(\hat{S}) = 2^{2g}\,\chi(S) \Rightarrow \text{genus}(\hat{S}) = 2^{2g}(g-1)+1 = 129 \qquad (g=3)$



<u>Claim</u> $D = [\Gamma_p] + [\Gamma_{\tau p}]$ even

- Note $D \cdot [S_3] = 2 \qquad D \cdot [S_{129}] = 2 \cdot \deg(p)$ both still even.

- WTS $D \cdot (b_j \otimes a_j)$ even for $b_j \otimes a_j \in H_1(S_{129}) \otimes H_1(S_3)$.

• <u>Main point</u> if $b \in H_1(S_{129})$, then $p(b) \in 2 \cdot H_1(S_3;\mathbb{Z})$.

$$\begin{tikzcd} \pi_1(\hat{S}) \arrow[r] \arrow[dr] & \pi_1(S) \arrow[r] & H_1(S;\mathbb{Z}) \arrow[r] & H_1(S;\mathbb{Z}/2) \\ & H_1(\hat{S};\mathbb{Z}) \arrow[ur, dotted] & & \end{tikzcd}$$

zero.

• rep $b_j \otimes a_j$ by torus $T = \{(b_j(t), a_j(s))\}. \subset S_{129} \times S_3$.

$(x, px) = (b_j(t), a_j(s)) \iff p(x) \in p(b) \cap a$

$\Rightarrow \Gamma_p \cdot T = p(b) \cdot a$

• similarly $\Gamma_{\tau p} \cdot T = p(b) \cdot \tau a$

$\Rightarrow D \cdot (b \otimes a) = p(b) \cdot (a + \tau a)$ ~~even since~~ by the main point. //

Apply Hirzebruch

$$\begin{array}{ccc} X & \xrightarrow{f} & S_{129} \times S_3 \\ \cup & & \cup \\ \hat{D} & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} S_6 & \to & X \\ & & \downarrow \\ & & S_{129} \end{array}$$

<u>Step2</u> Compute $\mathrm{sig}(X) \neq 0$.

<u>Option1</u> Hirzebruch G-signature thm:

For $\mathbb{Z}/2$ branched cover $\begin{matrix} \hat{M} & \longrightarrow & M^4 \\ \cup & & \cup \\ \hat{B} & \longrightarrow & B \end{matrix}$ $\quad \mathrm{sig}(\hat{M}) = 2\,\mathrm{sig}(M) - \hat{B}\cdot\hat{B}$

(works more generally for any dim, $G = \mathbb{Z}/m$. More complicated formula)

$$\Longrightarrow \mathrm{sig}(X) = 2 \cdot \underbrace{\mathrm{sig}(S_{129} \times S_3)}_{=0} - \hat{D}\cdot\hat{D}$$

<u>Note/Claim</u> $\hat{D}\cdot\hat{D} = \frac{1}{2}\, D\cdot D$ $\quad$ (under branched cover euler class of normal bundle dec. by factor of $1/m$).

$$\Rightarrow \mathrm{sig}(X) = -\tfrac{1}{2}\left((\Gamma_p + \Gamma_\tau)\cdot(\Gamma_p + \Gamma_\tau)\right) = -\Gamma_p\cdot\Gamma_p = -\chi(S_{129})$$
$$= 256.$$

<u>Option2</u> Vertical vector fields. (completely elementary)

<u>Recall</u> $\mathrm{sig}(X) = \frac{1}{3}\langle p_1(TX), X\rangle = \frac{1}{3}\langle e(T_\pi X)^2, X\rangle$

Recipe for computing $\mathrm{sig}(X)$

1. Find vertical v.f. ξ on X w/ isolated zeros

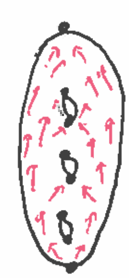
   (ie section of $\mathbb{R}^2 \to T_\pi X \to X$ transverse to 0-section)

   $N := \{\xi = 0\} = PD(e(T_\pi X))$.

2. by PD $\langle e(T_\pi X)^2, X\rangle = N\cdot N$.

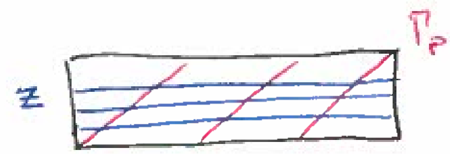
- Constructing ξ.

  - Define ξ_1 on $S_{129} \times S_3$ constant ~~[illegible]~~ vert v.f.



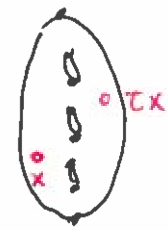
$Z = \{\xi_1 = 0\} \subset S_{129} \times S_3$ $\qquad$ $Z \simeq \coprod_8 S_{129} \times \{x_1, \dots, x_8\}$.

$[Z = PD(e(T_\pi(S_{129} \times S_3)))]$



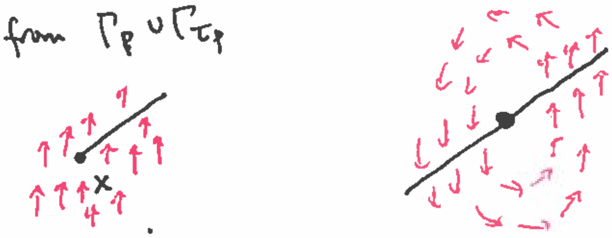
Define Lift to ξ on $X \setminus \hat{D}$

<u>Claim</u> ξ extends to X by zero on $\hat{D}$.



Case 1. e.g. in fiber where $\check{\xi}_1$ misses branch points

ie Z disjoint from $\Gamma_p \cup \Gamma_{\tau p}$



fiber in $S_{129} \times S_3$ $\qquad$ fiber in X.

Case 2. in fiber where ξ Z intersects $\Gamma_p \cup \Gamma_{\tau p}$

Omit.

$N := \{\xi = 0\}$. $\qquad$ $[N] = [\hat{Z}] + [\hat{\Gamma}_p] + [\hat{\Gamma}_{\tau p}]$

$\hat{Z} = f^{-1}(Z)$

$\hat{D} = \hat{\Gamma}_p \cup \hat{\Gamma}_{\tau p}$

$$N \cdot N = (\hat{Z} + \Gamma_p + \Gamma_{\tau p})^2 = \underbrace{2\hat{Z}\cdot\hat{\Gamma}_p + 2\hat{Z}\cdot\hat{\Gamma}_{\tau p}}_{4\hat{Z}\cdot\hat{\Gamma}_p} + \underbrace{\hat{\Gamma}_p\cdot\hat{\Gamma}_p + \hat{\Gamma}_{\tau p}\cdot\hat{\Gamma}_{\tau p}}_{\frac{1}{2}\Gamma_p\cdot\Gamma_p + \frac{1}{2}\Gamma_p\cdot\Gamma_p}$$

$=$

$= 4(2g-2)\cdot\deg(p)$ $\qquad$ $= \Gamma_p \cdot \Gamma_p$.

$= 4(2g-2)\cdot 2^{2g}$ $\qquad$ $= 2^{2g}(2-2g)$

$N \cdot N = (16-4)\,2^6$

$\Rightarrow \mathrm{sig}(X) = \frac{1}{3}\cdot 12 \cdot 2^6 = 2^8 = 256$

No class Friday

Lecture 19

I. Homeomorphism problem for surface bundles

<u>Problem</u> Fix d. Give computable invariants of $\begin{array}{c} S \to E \\ \downarrow \\ B^d \end{array}$

that can be used to determine if $\begin{array}{c} S \to E \\ \downarrow \\ B \end{array}$ & $\begin{array}{c} S' \to E' \\ \downarrow \\ B' \end{array}$

have $E \simeq E'$ homeo.

<u>Ex</u> (3-manifold case) $\left\{ \begin{array}{c} S_g \to M \\ \downarrow \\ S^1 \end{array} \right\} / \substack{\text{bundle} \\ \text{iso}} \longleftrightarrow \left\{ \phi \in \mathrm{Mod}_g \right\} / \mathrm{conj}$

$\Rightarrow$ Homeo prob reduces to: Given $\phi \in \mathrm{Mod}_g$ $\psi \in \mathrm{Mod}_h$,

determine if $M_\phi \simeq M_\psi$ homeo.

<u>Algorithm</u> (Farb):

(1) Compute Thurston norm balls for M_ϕ, M_ψ

($M_\phi \simeq M_\psi \Rightarrow H_2(M_\phi) \simeq H_2(M_\psi)$ as normed spaces)

$g_0^{(M)} := \min \left\{ g : \exists \begin{array}{c} S_g \to M \\ \downarrow \\ S^1 \end{array} \right\}$

(2) List minimal genus fiberings $\phi_1, \ldots, \phi_r$ for M_ϕ

$\psi_1, \ldots, \psi_s$ for M_ψ.

($M_\phi \simeq M_\psi \Rightarrow r = s$).

(3) $M_\phi \simeq M_\psi \iff \phi_i$ conj to ψ_j in Mod_{g_0} for some i, j.

Conjugacy Prob in Mod_g solvable (Hemion, Masur–Minsky, Hamenstädt)

4-manifold case: open.

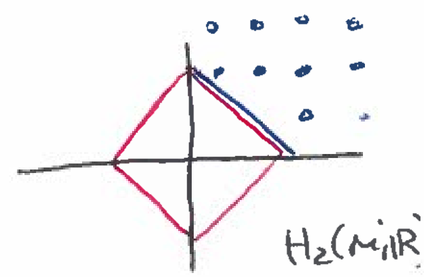
Basic question: Can 4-manifolds fiber as surface bundles over surfaces (SBS) in many ways?

(eg if every SBS fibers in ! way, monodromy is homeo invt.)

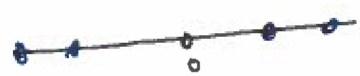
II. Multiple fiberings of SBS

Recall (Thurston) M^3 mfld. If $M \to S^1$ fibers & $b_2(M) \geq 2$ ($= \dim H_2(M;\mathbb{R})$)

then M fibers in ∞'ly many ways (w/ fiber genus $\to \infty$)



(OTOH if $b_2(M)=1$



multiples of fibered class corresp to fiberings w/ disconnected fiber)

Rmk $\exists$ $S_g \to E \to T^2$ that fiber in ∞'ly many ways.

Take $E = M \times S^1$ where $S_g \to M \to S^1$

Assume g,h $S_g \to E \to S_h$ $g,h \geq 2$.

Note $\chi(E) = \chi(S_g)\cdot\chi(S_h) = 4(g-1)(h-1)$

$\Rightarrow$ E can fiber w/ only finitely many fiber genera.

Defn $N(d) = \max\{ n \mid \exists\, E \ \chi(E) \leq 4d \text{ w/ } n \text{ distinct fiberings} \}$.

- upper bound

<u>Thm</u> (Johnson) $\begin{matrix} S_g \to E \\ \downarrow \\ S_h \end{matrix}$ $g, h \geq 2$, $\chi(E) = 4d$. Then E has at most $(d+1)^{2d+7}$ fiberings

ie $N(d) \leq (d+1)^{2d+7}$

- lower bound
  - $S_g \times S_h$ fibers in 2 ways.
  - AK examples fiber in 2 ways

$$4(6-1)(129-1) = \chi(X) = 4(3-1)(?-1)$$

$$\begin{tikzcd} S_6 \arrow[dr] & & S_{321} \arrow[dl] \\ & X \arrow[d] & \\ & S_{129} \times S_3 \arrow[dl] \arrow[dr] & \\ S_{129} & & S_3 \end{tikzcd}$$

$$\Rightarrow 2 \leq N(d) \leq (d+1)^{2d+7}$$

Evidence that fiberings may be rare.

Fix $\begin{matrix} S_g \to E \\ \downarrow \\ S_h \end{matrix}$ w/ monodromy $\rho : \pi_1(S_h) \to \mathrm{Mod}_g$.

<u>Thm</u> (Salter) If $H^1(S_g)^\rho = \{ v \in H^1 \mid \rho(\gamma) v = v \ \forall \gamma \in \pi_1(S_h) \} = 0$

Then E fibers in ! way.

<u>Rmk</u>. For $\begin{matrix} S_g \to M_\phi \\ \downarrow \\ S^1 \end{matrix}$ $H^1(M;\mathbb{R}) = H^1(S^1) \oplus H^1(S_g)^\phi$

so $b_1(M) = b_2(M) = 1 \Leftrightarrow H^1(S_g)^\phi = 0$.

(so Thm generalizes 3 mfld case — diff proof)

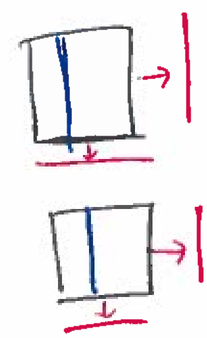
(Then might guess if $H^1(S_g)^\rho \neq 0$ could have many fiberings.)

Thm (Salter) If $\operatorname{im}(\rho) < K_g = \langle T_c \mid c \subset S \text{ separating} \rangle$
(Johnson Kernel) then either $E = S_g \times S_h$ or E fibers!

Note: In this case $H^1(S_g)^\rho = H^1(S_g)$ since separating twists act trivially on H^1. (so guess from above wrong)
nevertheless ...

Thm (Salter) $\exists$ $S_g \to E \to S_h$ that fiber in many ways.

II. Salter construction.

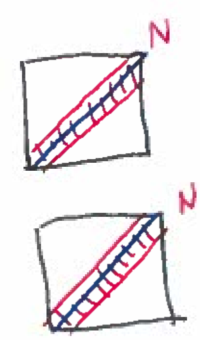


Observation. $E = S \times S \sqcup S \times S$ fibers in 4 ways

fiber $S \sqcup S$ disconnected.

WT alter to get connected total space. Use AK as guide.

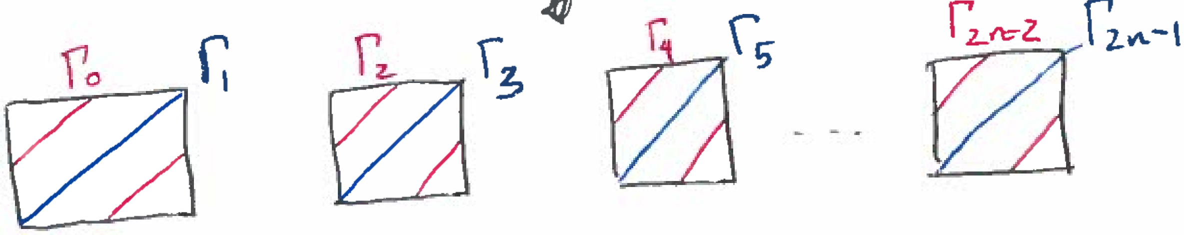
~~[illegible]~~ Section sum. $N \subset S \times S$ nbhd of diagonal



$$E := (S \times S \setminus N) \cup_{\partial N} (S \times S \setminus N)$$

4-different fiberings. $S \# S \to E \to S$.

2^n fiberings   Take $\mathbb{Z}/2 \curvearrowright S$ free



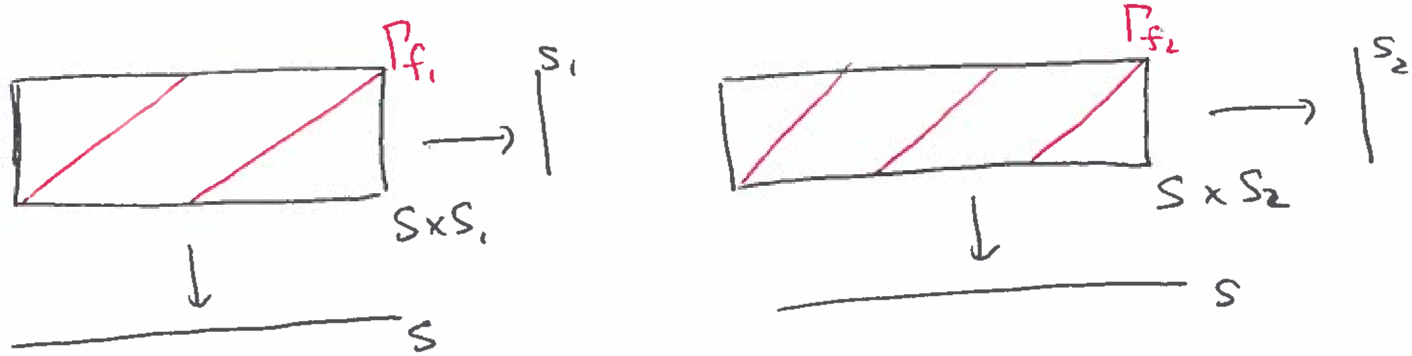
glue Γ_1 to Γ_2, Γ_3 to Γ_4, ..., Γ_{2n-1} to Γ_0.

issue: These bundles all have fiber $S^{\#n}$. In fact, they're all fiberwise diffeo. (although not "π_1-fiber diffeo")

Different fiber genera.

$$\begin{array}{ccc} & S & \\ {}^{f_1}\swarrow_{2:1} & & {}_{3:1}\searrow^{f_2} \\ S_1 & & S_2 \end{array}$$

eg. genes $g(S)=7$, $g(S_1)=4$, $g(S_2)=3$.

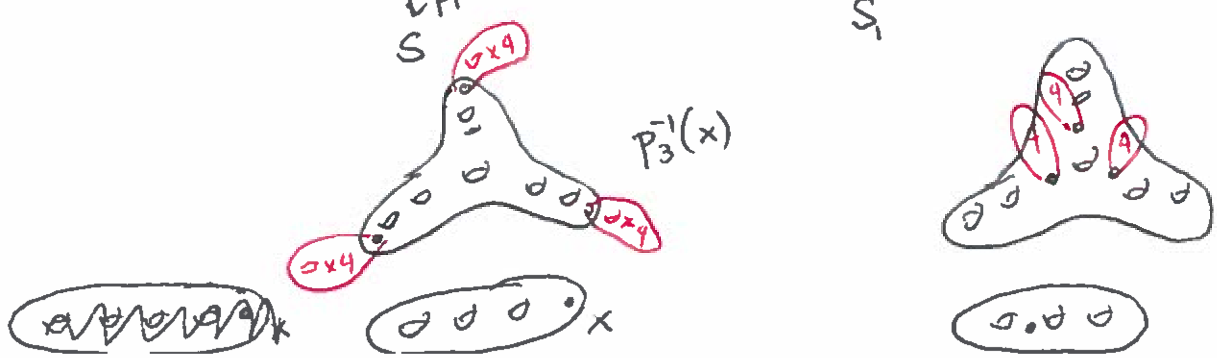


$$E = \left[S \times S_1 \setminus N(\Gamma_{f_1}) \right] \cup_\partial \left[S \times S_2 \setminus N(\Gamma_{f_2}) \right]$$

fiberings

$$\begin{array}{ccc} g=7 & g=13 & g=19 \\ S_1 \# S_2 \to E & S \# S_2 \# S_2 \to E & S \# S_1 \# S_1 \# S_1 \to E \\ \downarrow p_1 & \downarrow p_2 & \downarrow p_3 \\ S & S_1 & S_2 \end{array}$$

$\boxed{P_3}$



Cor
$2^{(d+2)/6} \leq N(d$

I. Mumford Conjecture   Lecture 20.   No class Friday   1

Previously

- Computed $H_1(\mathrm{Mod}_g) \equiv (\mathrm{Mod}_g)^{ab} = 0$ $g \geq 3$ gen set for Mod_{g} (lantern relation)

- Computed $H_2(\mathrm{Mod}_g) \simeq \mathbb{Z}$ $g \geq 6$

  (Hatcher-Thurston presentation, Hopf's formula, $\begin{matrix} S_g \to E \\ \downarrow \\ S_h \end{matrix}$ $\mathrm{sig}(E) \neq 0$)

- Defined MMM classes / tautological classes $e_i \in H^{2i}(\mathcal{M}_g) \simeq H^{2i}(\mathrm{Mod}_g; \mathbb{Q})$

$$\mathrm{Hyp}(S_g) = \left\{ \begin{matrix} \text{hyperbolic} \\ \text{metrics on } S_g \end{matrix} \right\} \circlearrowleft \mathrm{Diff}(S_g) \supset \mathrm{Diff}_0(S_g)$$ diffeos isotopic to id.

$$\mathrm{Teich}(S_g) := \mathrm{Hyp}/\mathrm{Diff}_0 \simeq \mathbb{R}^{6g-6} \circlearrowleft \mathrm{Diff}/\mathrm{Diff}_0 = \mathrm{Mod}_g$$

prop. disc., finite stabilizer ~~not free~~.
(stab $\leftrightarrow$ conn gp)

$$S_g \simeq \mathrm{Diff}(S)/\mathrm{Diff}(S,*) \longrightarrow \mathcal{M}_{g,1} := \mathrm{Hyp}(S_g)/\mathrm{Diff}(S_g,*)$$
$$\downarrow \pi$$
$$\mathcal{M}_g := \mathrm{Hyp}(S_g)/\mathrm{Diff}(S_g) \cong \mathrm{Teich}/\mathrm{Mod}_g.$$

$$e_i := \int_{S_g} e(T_\pi \mathcal{M}_{g,1})^{i+1} \in H^{2i}(\mathcal{M}_g).$$

<u>Conjecture</u> (Mumford, 1983) $H^i(\mathcal{M}_g; \mathbb{Q}) \simeq \mathbb{Q}[e_1, e_2, \ldots]_{\deg i}$

for $0 \leq i \leq \phi(g)$ where ~~for some~~ $\phi(g) \to \infty$ as $g \to \infty$.

<u>Cor to Conj</u>

(1) $H^1(\mathrm{Mod}_g; \mathbb{Q}) = 0$ for $g \gg 0$

$H^2(\mathrm{Mod}_g; \mathbb{Q}) \simeq \mathbb{Q}$ for $g \gg 0$.

(2) (Earle-Eells) $\chi(S) < 0 \Rightarrow \mathrm{Diff}(S) \to \mathrm{Mod}(S)$ h.e.

$\Rightarrow B\mathrm{Diff}(S) \sim B\mathrm{Mod}(S)$.

$\Rightarrow$ MMM classes $e_i \in H^{2i}(B\mathrm{Diff}(S_g))$ nontrivial $g \gg i$.

(~~nonexplicit~~ constructive proof - recall proof $e_1 \neq 0$ via AK.)

(3) Algebraic geometry?

<u>Rmk</u>. $\mathcal{M}_g$ finite dim'l (orbifold) $\Rightarrow$ eg $H^i(\mathcal{M}_g; \mathbb{Q}) = 0$ for $i > 6g-6$

so can't have iso in all deg for fixed g.

<u>Rmk</u>. $R_g^\circ := \langle e_1, e_2, \ldots \rangle \subset H^\circ(\mathcal{M}_g; \mathbb{Q})$ "Tautological ring"

Relations?

<u>Faber conjecture</u> ~~(relations)~~ $R_g = \langle e_1, \ldots, e_{\lfloor g/3 \rfloor} \rangle$

and R_g° is a PD ring dim $g-2$. $\left(\begin{array}{c} R_g^{g-2} \simeq \mathbb{Q} \\ R^k \times R^{g-2-k} \to \mathbb{Q} \\ \text{perfect pairing} \\ R^k \simeq (R^{g-k-2})^* \end{array} \right)$

parts known - perfect pairing open.

<u>II</u>. Resolution of Mumford conjecture

~~Improvement~~

Ingredients

1. Homological stability.

<u>Thm</u> (Harer stability, improved by Ivanov, Boldsen, Randal-Willia[ms]

$$H_i(\mathrm{Mod}_g^1;\mathbb{Z}) \longrightarrow H_i(\mathrm{Mod}_{g+1}^1;\mathbb{Z})$$

iso for $i \leq \frac{2}{3}(g-1)$

and $$H_i(\mathrm{Mod}_g^1;\mathbb{Z}) \longrightarrow H_i(\mathrm{Mod}_g;\mathbb{Z})$$

iso for $i \leq \frac{2}{3}g$.

<u>Defn</u> $\mathrm{Mod}_\infty := \mathrm{colim}\ \mathrm{Mod}_g^1$

Stability $\Longrightarrow$ $H_i(\mathrm{Mod}_g^1) \simeq H_i(\mathrm{Mod}_\infty)$ for $i \leq \frac{2}{3}(g-1)$

Mumford conj $\Longleftrightarrow$ $H^\bullet(\mathrm{Mod}_\infty;\mathbb{Q}) \simeq \mathbb{Q}[e_1, e_2, \ldots]$

2. Stable homotopy theory (Pontryagin–Thom scanning method. Group completion thm)

McDuff–Segal

<u>Defn</u> $\mathrm{Diff}_\infty = \mathrm{colim}\ \mathrm{Diff}(S_g^1)$

<u>Thm</u> $H^\bullet(B\mathrm{Diff}_\infty;\mathbb{Z}) \simeq H^\bullet(\Omega_0^\infty MTSO(2);\mathbb{Z})$

- $MTSO(2)$ Madsen–Tillmann spectrum
- $\Omega_0^\infty(\ \ )$ ~~Component~~ component of assoc. ∞-loop space.
- easy computation $H^\bullet(\Omega_0^\infty MTSO(2);\mathbb{Q}) \simeq \mathbb{Q}[e_1, e_2, \ldots]$

3. Earle - Eells $\Rightarrow$

$$H^\bullet(\mathrm{Mod}_g^{(1)}) \underset{\uparrow \text{in range}}{\simeq} H^\bullet(\mathrm{Mod}_\infty) \simeq H^\bullet(B\mathrm{Diff}_\infty) \simeq \mathbb{Q}[e_1, e_2, \dots]$$

//.

<u>Precursors to Mumford Conj</u>

1. Symmetric groups S_k.
   - Homological stability (Nakaoka) $H_\bullet(S_k) \to H_\bullet(S_{k+1})$ for $\bullet < k/2$
   - Stable homology (Barratt, Priddy, Quillen, Segal)

     $H_\bullet(S_\infty) \simeq H_\bullet(\Omega_0^\infty \mathbb{S})$ $\quad\mathbb{S}$ sphere spectrum

2. Braid groups B_k.
   - Stability (Arnold)
   - (Segal) $H_\bullet(B_\infty) \simeq H_\bullet(\Omega_0^2 S^2)$ — component of based maps $S^2 \to S^2$.

<u>Other applications</u> $\mathrm{Out}(F_n)$, $B\mathrm{Diff}(\#_g S^n \times S^n)$, ...

III. Homological Stability.

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$$

<u>Strategy for stability</u> to show $H_i(G_n) \longrightarrow H_i(G_{n+1})$ $i \ll n$.

Find simplicial complexes X_n $n \geq 1$ with

- $G_n \curvearrowright X_n$ simplicially.
- Transitivity $G_n \curvearrowright X_n(p) = \{p\text{-simplices}\}$ transitively.
- Stabilizers $\mathrm{Stab}(\sigma_p) \simeq G_{n-p-1}$ for $\sigma_p \in X_n(p)$.
- Connectivity X_n highly connected, ie. $\pi_i(X_n) = 0$ for $i \ll n$.

spectral seq arg gives stability.

Next time. Explain for symmetric group S_k.

Example $G_n = S_n$ symmetric group.

Note $S_n \curvearrowright X_n = \Delta^{n-1}$ (n-1) simplex

- Connectivity: Δ^{n-1} contractible. ✓
- transitivity on $\{p\text{-simplices}\} \Leftrightarrow \{(p+1)\text{ elt subsets of } \{1,\dots,n\}\}$ ✓
- Stabilizers $\sigma_p = \{1,\dots,p+1\}$.

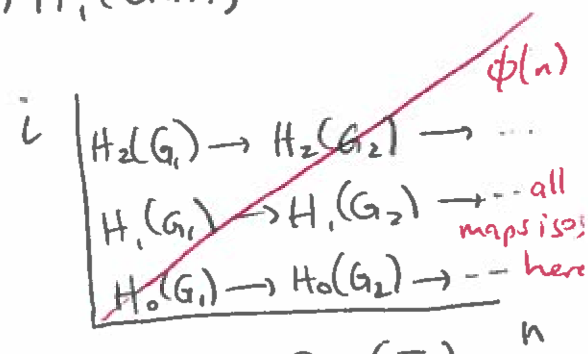
$$\mathrm{Stab}(\sigma_p) = \mathrm{Sym}\{1,\dots,p+1\} \times \mathrm{Sym}\{p+2,\dots,n\}$$
$$\simeq S_{p+1} \times S_{n-p-1} \qquad \times$$

No class friday

Lecture 21

I. Homological Stability

Defn A family of groups $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \cdots$ is <u>homologically stable</u> if $\exists$ $\phi(n) \to \infty$ as $n \to \infty$ s.t. the induced maps $H_i(G_n) \longrightarrow H_i(G_{n+1})$ are isomorphisms for $i \le \phi(n)$.



<u>Groups satisfying homological stability</u>

symmetric groups, braid groups, $GL_n\mathbb{Z}$, Mod_g, $\mathrm{Out}(F_n)$ ...

<u>Strategy of proof</u> Find ~~complexes~~ G_n-complex X_n s.t.

(1) Stabilizers: Denote $X_n(p) = \{p\text{-simplices}\}$.

For any $\sigma_p \in X_n(p)$ $\quad \mathrm{Stab}(\sigma_p) \simeq G_{n-p-1} < G_n$.

(up to conj in G_n).

(2) Transitivity: $G_n \curvearrowright X_n(p)$ transitive $\forall p$.

(3) Connectivity: X_n is highly connected

i.e. $\pi_i(X_n) = 0$ for $i < n$.

<u>Prop</u> (G_n, X_n) as above. If X_n is $(n-2)$-connected, then

$H_i(G_n) \to H_i(G_{n+1})$ iso for $i < \frac{1}{2}(n-1)$.

<u>Proof</u>. (Quillen) Equivariant homology + spectral seq. argument.

(Next week)

RMK. Flexible technique ~~depending~~ don't need specific case or transitivity...

II. Example: Symmetric groups.

Want $X_n \circlearrowleft S_n$ w/ above properties.

<u>Attempt 1</u>. $S_n \curvearrowright X_n = \Delta^{n-1}$

$X_n(0) = \{1, \ldots, n\} =: [n]$

$X_n(p) = \{(p+1)\text{-elt subset of } \cancel{X_n(0)}\ [n]\}$.

<u>Properties</u>.

- Connectivity: X_n contractible ✓.
- transitive $S_n \curvearrowright \{p+1 \text{ elt subsets of } \cancel{\{1,\ldots,n\}}\ [n]\}$ transitive ✓.
- Stabilizer $\sigma_p = \{1, \ldots, p+1\}$.

$$\mathrm{Stab}(\sigma_p) = \mathrm{Sym}\{1, \ldots, p+1\} \times \mathrm{Sym}\{p+2, \ldots, n\}.$$

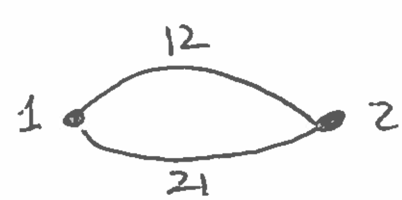
$$\cong S_{p+1} \times S_{n-p-1}$$ ✗.

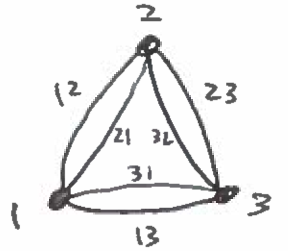
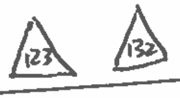
<u>Attempt 2</u>. Complex of ordered simplices

$X_n(0) = \{1, \ldots, n\}$.

$X_n(p) = \{\text{ordered } (p+1) \text{ elt subset of } \cancel{X_n(0)}\ [n]\}$.

<u>Examples</u>. $X_1 = \bullet\ 1$.

$X_2 =$ 

$X_3 =$  $\cup$ 6 faces  ...

Note. X_n not simplicial complex but it is a Δ-complex / geometric realization of simplicial set.

(not simplicial since simplex isn't det. by its simplices. ~~[illegible]~~ Δ complex is space made of simplices... ~~[illegible]~~)

- Stabilizer $\quad \sigma_p = (1, \dots, p+1) \qquad \mathrm{Stab}(\sigma_p) = \mathrm{Sym}\{p+2, \dots, n\}$
$\simeq S_{n-p-1}$.

- Transitivity $\quad$ 1 orbit of p-simplex $\quad 0 \le p \le n-1$.

- Connectivity.

<u>Claim</u> X_n is $n-2$ connected $\qquad$ Cor Stability for S_n.

By Hurewicz, suffices to show $\tilde{H}_i(X) = 0 \quad i \le n-2$

<u>Closer look at $X = X_n$</u> (how is X_n built from X_k $\;k<n$?)

- consider subcomplex of simplices w/ 1st vertex $i =: Y_i$

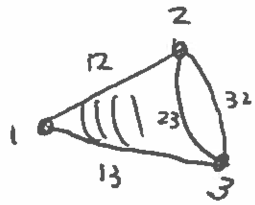
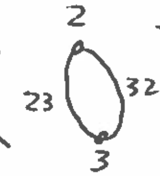
<u>Note</u> $X = Y_1 \cup \cdots \cup Y_n$.

Can compute $\tilde{H}_i(X)$ if we understand (inductively)

$\tilde{H}_\bullet(Y_i)$, $\tilde{H}_\bullet(Y_i \cap Y_j)$, ... $\qquad$ (Mayer-Victoris)

– $Y_i \supset Y_{\hat{i}} := \bigcup$ simplices that don't contain i.

Note $Y_{\hat{i}} \simeq X_{n-1}$ and $Y_i = \mathrm{Cone}(Y_{\hat{i}})$.

E.g. in X_3 $Y_1 =$  $=$ Cone()

faces (123), (132)

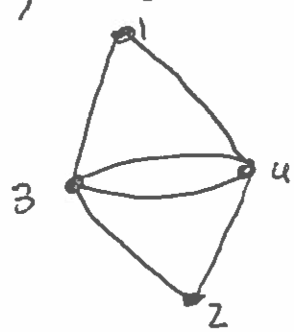
(Given simplex τ in $Y_{\hat{i}}$, $\exists !$ simplex containing i & τ.)

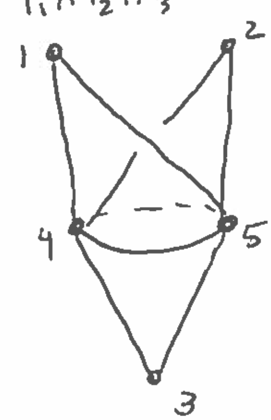
$\Rightarrow$ $\tilde{H}_0(Y_i) \simeq 0$.

– Similarly $Y_i \cap Y_j \supset Y_{\hat{i}} \cap Y_{\hat{j}} = \bigcup$ simplices containing neither i nor j.

$Y_{\hat{i}} \cap Y_{\hat{j}} \simeq X_{n-2}$

$Y_i \cap Y_j = (Y_{\hat{i}} \cap Y_{\hat{j}}) * \{i, j\}$.

in X_4 $Y_1 \cap Y_2 =$ 

in X_5 $Y_1 \cap Y_2 \cap Y_3 =$ $(Y_{\hat{1}} \cap Y_{\hat{2}} \cap Y_{\hat{3}}) * \{1,2$ 

Rmk F finite, Z k-conn $\Rightarrow$ $Z * F$ $(k+1)$ connected.

5

<u>Summary</u>. If we assume (by induction) that X_i is $(i-2)$ connected for $1 \le i \le n-1$, then.

- $X = Y_1 \cup \dots \cup Y_n$ where $Y_i = \mathrm{Cone}(X_{n-1}) \sim *$
- $Y_i \cap Y_j \cong X_{n-2} * \{i,j\}$ $(n-3)$-connected.
- $Y_{i_1} \cap \dots \cap Y_{i_k} = X_{n-k} * \{i_1, \dots, i_k\}$ $(n-k-1)$-conn.

<u>Lemma</u>. If $Y = Y_1 \cup \dots \cup Y_m$ and Y_i is r-conn. and $Y_{i_1} \cap \dots \cap Y_{i_k}$ is $(r-k+1)$ conn, then Y is r-conn.

<u>Cor</u>. X_n is $(n-2)$ connected.

<u>Pf of Lemma</u>. Induct on m. Base $m=1$ trivial.

$m \geq 2$ $\quad Y = \underbrace{Y_1}_{A} \cup \underbrace{(Y_2 \cup \dots \cup Y_m)}_{B}$

$A \cap B = (Y_1 \cap Y_2) \cup \dots \cup (Y_1 \cap Y_m)$.

IH applies to $A, B, A \cap B$ $\Rightarrow$ A, B r-conn.
$A \cap B$ $(r-1)$ conn.

Mayer-Vietoris.

$$H_i(A) \oplus H_i(B) \longrightarrow H_i(Y) \longrightarrow H_{i-1}(A \cap B)$$

$\Rightarrow \tilde{H}_i(Y) = 0$ for $i \le r$. $\square$

no class friday. Monday equiv. hom. & stab

Lecture 22

I. ~~Spectral~~ Homological stability

$G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ seq of groups.

<u>Prop.</u> X_n G_n-complex s.t.

(1) $\mathrm{Stab}(\sigma_p) \cong G_{n-p-1}$ $\forall\, \sigma_p \in X_n(p)$ p-simplex.

(2) $G_n \curvearrowright X_n(p)$ transitive $\forall p$.

(3) X_n is $(n-2)$-connected

$\Rightarrow$ $H_i(G_n) \to H_i(G_{n+1})$ iso for $i < \frac{n}{2}$.

<u>Application</u> $G_n = S_n$. $X_n(p) = \left\{ \begin{array}{c} \text{injections} \\ [p] \to [n] \end{array} \right\}$

~~Yester~~ last time: X_n is $(n-2)$-connected

<u>Goal</u>: Explain prop. Rough idea: X_n is guide for building $K(G_n,1)$ from $K(G_m,1)$ $m<n$.

main tool:

II. Equivariant homology.

- G discrete group, X simplicial complex, $G \curvearrowright X$ simplicially.
  (so ~~$G\backslash X$~~ X/G also simplicial)

- EG contractible w/ proper free G action
  $BG := EG/G$ is $K(G,1)$ $\left(\begin{array}{c} G \to EG \to BG \text{ universal} \\ \text{principal } G\text{-bundle} \end{array} \right)$.

Defn. The <u>equivariant homology</u> of G-space X is

$$H_\bullet^G(X) := H_\bullet\left(\frac{EG\times X}{G}\right).$$

Examples/remarks.

(1) $H_\bullet^G(pt) = H_\bullet(BG) \equiv H_\bullet(G)$.

(2) $G \curvearrowright EG$ free $\Rightarrow$ $X \longrightarrow \frac{EG\times X}{G} \longrightarrow BG$ is fibration.

serre spectral sequence $E^2_{pq} = H_p(BG; H_q(X)) \Rightarrow H_\bullet^G(X)$.

(if don't know ssegs it's okay. Computational tool – here can for theory compute $H_\bullet^G(X)$ given ...)

- $G \curvearrowright X$ trivial $\Rightarrow$ $\frac{EG\times X}{G} \simeq BG \times X$

$\Rightarrow H_\bullet^G(X) \simeq H_\bullet(X) \otimes H_\bullet(BG)$

(3) if $G \curvearrowright X$ freely then $EG \longrightarrow \frac{EG\times X}{G} \longrightarrow X/G$ fibration.

$EG \sim * \Rightarrow H_\bullet^G(X) \simeq H_\bullet(X/G)$.

(4) if X acyclic (ie $\tilde{H}_\bullet(X)=0$) then $H_\bullet^G(X) = H_\bullet(G)$.

expand on (3-4): show $H_\bullet^G(X)$ useful for computing ~~this~~ $H_\bullet(G)$.

Lemma. X k-connected ($\pi_i(X)=0$ $0\le i\le k$)

$\Rightarrow H_i(G) \simeq H_i^G(X)$ for $i \le k+1$.

Rmk: Could prove w/ sseq. $\pi_i(X)=0 \Rightarrow H_i(X)=0 \Rightarrow \dots$ We'll give more basic proof not req. sseq

WTS. $H_i(X_G) \simeq H_i(BG)$ $i \le k-1$

<u>Proof</u>. Set $X_G = \frac{EG \times X}{G}$. Consider $X \to X_G \xrightarrow{f} BG$ ~~fibration~~

$\pi_i(X) = 0$ $i \le k$ $\Longrightarrow$ $\pi_i(X_G) \simeq \pi_i(BG)$ $i \le k$.

$\Rightarrow$ $\pi_i(M_f, X_G) = 0$ $i \le k$ $M_f = X_G \times [0,1] \cup_f BG$ mapping cyl.

$\Rightarrow$ (Hurewicz) $H_i(M_f, X_G) = 0$ $i \le k$.

$$H_{i+1}(M_f, X_G) \longrightarrow H_i(X_G) \longrightarrow \underset{\substack{\simeq \\ BG.}}{H_i(M_f)} \longrightarrow H_i(M_f, X_G)$$

iso for $i+1 \le k$. $\square$

<u>Rmk</u>. Alternatively $X_G \xrightarrow{f} BG$ k-connected $\Longrightarrow$ can build $BG = K(G,1)$ from X_G by attaching cells of dim $\geq k+1$.

<u>Takeaway</u> Can compute $H_\cdot(G)$ by computing $H_\cdot^G(X)$.

<u>Computing $H_\cdot^G(X)$</u>.

- Assume for each simplex $\sigma \subset X$ G_σ acts trivially on σ (always can achieve by barycentric subdivision)

<u>Note</u> $E^2_{p,q} = H_p(G; H_q(X)) \Rightarrow H^G_{p+q}(X)$ not useful for learning about $H_\cdot(G)$ from $H_\cdot^G(X)$.

instead consider $\pi: \frac{EG \times X}{G} \longrightarrow X/G$.

... [illegible] ... $G \curvearrowright X$ not free but ...

<u>Thm</u> (Borel) There is a sseq. $E^2_{p,q} = H_p(X/G; \mathcal{H}_q) \Rightarrow H_\cdot^G(X)$

<u>Rmk</u>. This will be main tool for computing $H_\cdot^G(X)$ and applying to $H_\cdot(G)$.

<u>Defining</u> $H_p(X/G; \mathcal{H}_q)$.

- Fix $\sigma \subset X/G$ simplex and lift $\tilde{\sigma} \subset X$.

$$\begin{array}{ccc} X & \longrightarrow & X/G \\ \cup & & \cup \\ G/G_\sigma \times \mathrm{int}(\tilde\sigma) & & \mathrm{int}(\sigma) \end{array} \qquad \frac{EG \times X}{G} \xrightarrow{\pi} X/G.$$

- $$\pi^{-1}(\mathrm{int}\,\sigma) = \frac{EG \times G/G_\sigma \times \mathrm{int}(\tilde\sigma)}{G} \simeq \frac{EG \times G/G_\sigma}{G} \times \mathrm{int}(\tilde\sigma).$$

$$\simeq \frac{EG}{G_\sigma} \times \mathrm{int}(\tilde\sigma) \simeq BG_\sigma \times \mathrm{int}(\tilde\sigma).$$

Rmk have $G/G_\sigma \to \frac{EG \times G/G_\sigma}{G} \to B$

also have $G/G_\sigma \to \frac{EG}{G_\sigma} \to \frac{EG}{G}$

<u>Claim</u>. over $\mathrm{int}(\tilde\sigma)$ π is locally trivial w/ fiber $K(G_\sigma, 1)$.

<u>Defn</u> Y simplicial complex, viewed as category (obj = simplices, morphisms = face inclusions) (poset).

A <u>coeff system</u> on Y is a functor $\mathcal{H}: Y \longrightarrow \mathrm{AbGp}$.

<u>Ex</u>. (1) Constant functor $\mathcal{H}(\sigma) = \mathbb{Z}$ $\forall \sigma$.

(2) on X/G define $\mathcal{H}_q(\sigma) = H_q(G_\sigma)$.

Note $\tau \subset \sigma$ face $\Rightarrow$ $G_\sigma \subset G_\tau$ $\rightsquigarrow$ $H_q(G_\sigma) \to H_q(G_\tau)$.

<u>Defn</u>. Given coeff syst. $\mathcal{H}$ on Y, define

$H_\cdot(Y;\mathcal{H})$ via chain cplx $C_k(Y,\mathcal{H}) := \left\{ \sum c_i \sigma_i \,\middle|\, \begin{array}{l} \sigma_i \in Y(k) \\ c_i \in \mathcal{H}(\sigma_i \end{array} \right.$

<u>Rmk</u>. From group POV.

$$H_\cdot^G(X) = H_\cdot\left(P_\cdot \otimes_{\mathbb{Z}G} C_\cdot(X) \right) \quad \text{where}$$

$P_\cdot \to \mathbb{Z}$ proj. resolution of $\mathbb{Z}$ by $\mathbb{Z}G$ modules.
(eg $P_\cdot = C_\cdot(EG) \ldots$)

$C_\cdot(X)$ cellular chains.

The two ssqs
$$E^2_{p,q} = H_p(G; H_q(X)) \Rightarrow H_\cdot^G(X)$$
$$E^2_{p,q} = H_p(X/G; \mathcal{H}_q) \nearrow$$

Corresp. to sseqs assoc. to horz/vert filtrations of double complex $P_\cdot \otimes_{\mathbb{Z}G} C_\cdot(X)$.

<u>Next time</u>. Apply to prove Homological Stability Prop.

Lecture 23

I Equivariant homology

Last time

- Defined equivariant homology $H_\bullet^G(X)$.
- Lemma. X k-connected $\Rightarrow$ $H_i^G(X) = H_i(G)$ for $i \leq k-1$
- Thm There is a spectral seq that computes $H_\bullet^G(X)$ with $E^2_{pq} = H_p(X/G; \mathcal{H}_q)$.

Warm-up computations

$G_n = S_n$ $\quad X_n$ complex w/ p-simplices

$$X_n(p) = \left\{ \begin{matrix} \text{ordered } (p+1) \text{ elt} \\ \text{subset of } [n] \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{injections} \\ [p+1] \to [n] \end{matrix} \right\}.$$

face maps $\quad \partial_i : X_n(p) \longrightarrow X_n(p-1)$

$$\left(f : [p+1] \to [n]\right) \longmapsto f|_{\{1,\dots,\hat{i},\dots,p+1\}}.$$

① $H_\bullet(X_n/G_n)$

Cellular chains $\quad C_p(X_n/G_n) \simeq \mathbb{Z} \qquad 0 \leq p \leq n-1$

since $\quad G_n \curvearrowright X_n(p)$ transitively.

boundary : $\quad$ eg. $\quad C_1(X_n) \xrightarrow{\partial = \partial_0 - \partial_1} C_0(X_n)$

$$(ij) \longmapsto (j) - (i).$$

$\Rightarrow \partial : C_1(X_n/G_n) \longrightarrow C_0(X_n/G_n)$ is zero.

OTOH $\quad \partial : C_2(X_n) \longrightarrow C_1(X_n)$

$(ijk) \longmapsto (jk) - (ik) + (ij)$

$\Rightarrow \partial : C_2(X_n/G_n) \longrightarrow C_1(X_n/G_n)$ is iso.

$$\leadsto \quad 0 \longrightarrow \underset{\mathbb{Z}}{C_{n-1}(X_n/G_n)} \longrightarrow \cdots \longrightarrow \underset{\mathbb{Z}}{C_2(X_n/G_n)} \xrightarrow{\simeq} \underset{\mathbb{Z}}{C_1(X_n/G_n)} \xrightarrow{0} \underset{\mathbb{Z}}{C_0(X_n/G_n)}$$

$$\Rightarrow \quad H_i(X_n/S_n) = \begin{cases} \mathbb{Z} & i = 0. \\ 0 & 1 \le i \le n-2. \end{cases}$$

Ex. Check. $X_2/S_2 \simeq S^1$, $X_3/S_3 \simeq D^2$, $X_4/S_4 \simeq D^2 \vee S^3$

② $H_\cdot(X_n/G_n ; \mathcal{H}_q)$.

Recall. $\mathcal{H}_q : \coprod_p X_n/G_n^{(p)} \longrightarrow \mathrm{AbGp}.$ $\quad \tilde{\sigma} \in X_n(p)$. lift.

$\sigma \longmapsto H_q(G_{\tilde{\sigma}})$.

$$C_p(X_n/G_n ; \mathcal{H}_q) := \left\{ \sum c_i \sigma_i \;\middle|\; \begin{array}{l} \sigma_i \in X_n/G_n(p) \\ c_i \in \mathcal{H}_q(\sigma) \end{array} \right\} \simeq H_q(G_{\sigma_p}) \simeq H_q(G_{n-p-1})$$

$\Big\downarrow \partial.$

$$C_{p-1}(X_n/G_n ; \mathcal{H}_q) \simeq H_q(G_{\sigma_{p-1}}) \simeq H_q(G_{n-p}).$$

As above $\partial = \left(\sum_{i=0}^{p} (-1)^i\right) H_q(j) = \begin{cases} 0 & p \text{ odd} \\ H_q(j) & p \text{ even.} \end{cases}$

where $H_q(j)$ induced by ~~[illegible]~~ $j : G_{n-p-1} \to G_{n-p}$

<u>Rmk</u>. These computations only used.

(i) $G_n \curvearrowright X_n(p)$ transitive

(ii) For $\sigma_p \in X_n(p)$ $G_{\sigma_p} \simeq G_{n-p-1}$.

(and not fact that $G_n = S_n$ $X_n = \cdots$)

II. Homological stability

<u>Thm</u> $G_n \curvearrowright X_n$. $n \geq 1$.

(i) $G_n \curvearrowright X_n(p)$ transitive (ii) $G_{\sigma_p} \simeq G_{n-p-1}$.

(iii) X_n is $(n-2)$-connected.

$\Rightarrow H_i(G_n) \longrightarrow H_i(G_{n+1})$ for $i \leq \frac{1}{2}(n-1)$.

<u>Proof</u>.

- Note X_n $(n-2)$-conn $\Rightarrow H_i(G_n) \simeq H_i^{G_n}(X_n)$ $i \leq n-3$.
- Assume know thm for $G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_{n-1}$
- WT Show : ~~[illegible]~~ $G_{n-1} \longrightarrow G_n$ induces H_*-iso in low deg.
- use sseq $E^2_{p,q} = H_0(X_n/G_n; \mathcal{H}_q) \Rightarrow H_{p+q}^{G}(X_n)$.

$$E^1_{p,q} = C_p(X_n/G_n, \mathcal{H}_q) \simeq H_q(G_{n-p-1}).$$

$$\begin{array}{c|llllll}
q & & & & & & \\
1 & H_1(G_{n-1}) & \xleftarrow{0} & H_1(G_{n-2}) & \xleftarrow{\cong} & H_1(G_{n-3}) & \xleftarrow{0} \cdots \\
0 & H_0(G_{n-1}) & \xleftarrow{0} & H_0(G_{n-2}) & \xleftarrow{\cong} & H_0(G_{n-3}) & \xleftarrow{0} \cdots \\
\hline
 & & & & & & p
\end{array}$$

(use ~~...~~ IH to get $0, \simeq$)

$E^2_{p,q}$.

$$\begin{array}{c|l|l}
k & H_k(G_{n-1}) & \\
\vdots & & \\
2 & H_2(G_{n-1}) & E^2_{p,q} = 0 \\
1 & H_1(G_{n-1}) & \text{by induction} \\
0 & H_0(G_{n-1}) & \\
\hline
 & 0 & 1 \quad 2
\end{array}$$

$$\Rightarrow H_i^{G_n}(X_n) \simeq H_i(G_{n-1}) \quad \text{for} \quad i \leq \tfrac{1}{2}(n-2).$$

$$\begin{array}{c} \wr\wr \\ H_i(G_n) \quad \text{for} \quad i \leq n-3 \end{array}$$

$$\Rightarrow H_i(G_n) \simeq H_i(G_{n+1}) \quad \text{for} \quad i \leq \tfrac{1}{2}(n-1). \qquad \square.$$

III. Application $\mathcal{M}_g$ vs $K(\mathrm{Mod}_g, 1)$.

- $\mathcal{M}_g = \mathrm{Teich}(S_g)/\mathrm{Mod}_g$. Moduli space.
- $\mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6} \circlearrowleft \mathrm{Mod}_g$ proper disc.

Prop. $H_\bullet(\mathcal{M}_g;\mathbb{Q}) \simeq H_\bullet(\mathrm{Mod}_g;\mathbb{Q})$.

Key: for $\Sigma \in \mathrm{Teich}(S_g)$ $\quad \mathrm{Stab}(\Sigma) < \mathrm{Mod}_g$ finite.

$$\Rightarrow \quad H_i(\mathrm{Stab}(\Sigma);\mathbb{Q}) = 0 \qquad i>0.$$

Proof $\mathrm{Teich}(S_g) \sim * \quad \Rightarrow \quad H_\bullet(\mathrm{Mod}_g;\mathbb{Q}) \simeq H_\bullet^{\mathrm{Mod}_g}(\mathrm{Teich};\mathbb{Q})$

sseq $\quad E^2_{p,q} = H_p(\mathcal{M}_g;\mathcal{H}_q)$

$$\mathcal{H}_q(\sigma) = H_q(G_\sigma;\mathbb{Q}) = 0 \quad \text{for } q>0 \quad \text{since } G_\sigma \text{ finite.}$$

$$\Rightarrow \quad E^2_{p,q} = \begin{array}{|lll} & 0 & \\ H_0(\mathcal{M}_g) & H_1(\mathcal{M}_g) & \cdots \\ \hline \end{array}$$

$$\Rightarrow \quad H_\bullet^{\mathrm{Mod}_g}(\mathrm{Teich};\mathbb{Q}) \simeq H_\bullet(\mathcal{M}_g;\mathbb{Q}) \qquad \square.$$

Next time: homological stability for Mod_g^1.

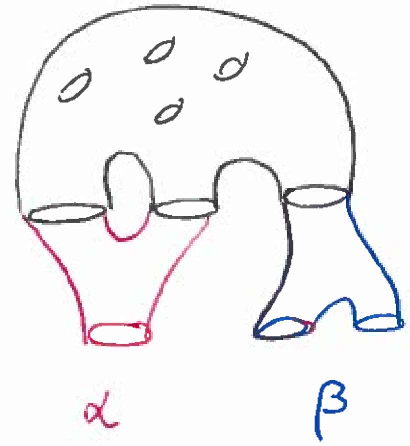
Lecture 24

I. Homological Stability for Mod_g

Stabilization maps $\mathrm{Mod}_{g,b}$ genus g, b boundary comp.

For $b \geq 1$ have

- $\alpha_{g,b} : \mathrm{Mod}_{g,b+1} \longrightarrow \mathrm{Mod}_{g+1,b}$
- $\beta_{g,b} : \mathrm{Mod}_{g,b} \longrightarrow \mathrm{Mod}_{g,b+1}$



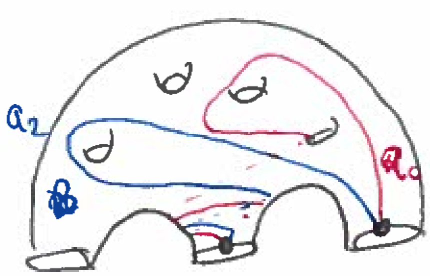
Note $\alpha \circ \beta : \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Mod}_{g+1,1}$

Thm. For $g \geq 0$, $b \geq 1$,

- $H_i(\alpha_{g,b})$ iso for $i \leq \frac{2}{3}(g-1)$.
- $H_i(\beta_{g,b})$ iso for $i \leq \frac{2}{3} g$.

Arc complexes for $\mathrm{Mod}(S)$ Assume $\partial S \neq \phi$. Fix $z_0, z_1 \in \partial S$.

$X(S, z_0, z_1)$: vertices = isotopy classes of nonseparating embedded arcs w/ $\partial = \{z_0, z_1\}$



p-simplices = collection $\{a_0, \dots, a_p\}$ nonsep. disjointly embeddable away from z_0, z_1.

$\{a_0, a_1\}$ = 1 simplex

Two cases

- $X^1_{g,b} := X(S_{g,b}, z_0, z_1)$ w/ z_0, z_1 on same ∂ comp.
- $X^2_{g,b} := X(S_{g,b}, z_0, z_1)$ w/ z_0, z_1 on diff ∂ comps.

Prop. (a) $\mathrm{Mod}_{g,b} \curvearrowright X^i_{g,b}(p)$ transitive $\forall p$ $i=0,1$.

(b) $\sigma \in X^1_{g,b}(p) \Rightarrow \mathrm{Stab}(\sigma) \simeq \mathrm{Mod}_{g-p-1,\, b+p+1}$.

$\sigma \in X^2_{g,b}(p) \Rightarrow \mathrm{Stab}(\sigma) \simeq \mathrm{Mod}_{g-p,\, b+p-1}$.

Thm $X^i_{g,b}$ $i=1,2$ is $(g-2)$-connected.

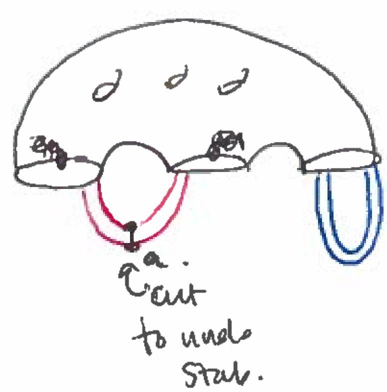
Rmk How to choose X?

Roughly X parameterizes the ways to undo the stabilization maps.

- Ex. For $S_n = \mathrm{Aut}([n])$. Stabilization $[n] \longrightarrow [n+1]$.

$X_n(p) = \{ [p+1] \hookrightarrow [n] \}$ ways to forget $(p+1)$-points.

- For $\mathrm{Mod}_{g,b}$ Stabilization

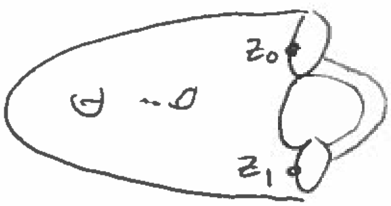


undo stabilization by cutting along arcs.

<u>Rmk</u>. Need both $X^1_{g,b}$ & $X^2_{g,b}$.

stabilization gives $\alpha : X_{g,b} \longrightarrow X_{g+1,b-1}$

$\circlearrowleft$ $\mathrm{Mod}_{g,b}$ $\qquad$ $\circlearrowleft$ $\mathrm{Mod}_{g+1,b-1}$.



II. ~~Facts from~~ Isotopy extension (fact from diff top that we'll need)

<u>Thm</u>. (Palais–Cerf fibering thm)

M, N mflds $\quad V \subset M$ cpt submfld.

then the Restriction map $\quad \mathrm{Emb}(M,N) \longrightarrow \mathrm{Emb}(V,N)$

is locally trivial fibration.

<u>Eg</u> $M = N \quad \rightsquigarrow \quad$ ~~Diff~~ $\quad \mathrm{Diff}(M,V) \longrightarrow \mathrm{Diff}(M) \longrightarrow \mathrm{Emb}(V,M)$

eg. $V = pt \qquad \mathrm{Diff}(M,*) \longrightarrow \mathrm{Diff}(M) \longrightarrow M$.

(explained proof in this case) $\rightsquigarrow$ BES $\quad 1 \to \pi_1(S_g) \to \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g \to 1$

<u>Cor</u>. (isotopy extension) For $i : V \longrightarrow M$ as above &

$\gamma : [0,1] \longrightarrow \mathrm{Emb}(V,M) \quad \exists \quad \hat{\gamma} : [0,1] \longrightarrow$ ~~Emb~~ $\mathrm{Diff}(M)$

st $\quad \hat{\gamma}(t)|_V = \gamma(t)$. $\qquad$ Pf: isotopy ext. is a path lifting

III. Properties of Arc Complexes.

<u>Prop</u> (a) (Transitivity) Classification of surfaces.

<u>Prob</u> (b). Focus on $X = X'_{g,b}$.

Fix $\sigma = \{a_0, \ldots, a_p\} \in X(p)$

<u>WTS</u> $\mathrm{Stab}(\sigma) \simeq \mathrm{Mod}_{g-p,\, r+p-1}$.

i.e. $\mathrm{Stab}(\sigma) \simeq \mathrm{Mod}(S \setminus \sigma)$ where $S \setminus \sigma$ is compact surface (obtained by cutting σ)

<u>Note</u>. ~~Theorem is~~ Have $\mathrm{Mod}(S \setminus \sigma) \xrightarrow{\phi} \mathrm{Stab}(\sigma) < \mathrm{Mod}(S)$

induced by $\mathrm{Diff}(S \setminus \sigma) \longrightarrow \mathrm{Diff}(S)$ (assume diffeos are id near ∂ ...)

<u>Claim</u> ϕ is an isomorphism

<u>Surjective</u>. If $[f] \in \mathrm{Stab}(\sigma)$ and $f(a_i) = a_i$.

cut ~~tangent~~ along $\sigma = \bigcup a_i$ to get $[f'] \in \mathrm{Mod}(S \setminus \sigma)$

– in general $[\varphi] \in \mathrm{Stab}(\sigma)$ only means $\varphi(a_i) \sim a_i$ isotopic. But by isotopy extension, if $\varphi(a_i) \sim a_i$

then ~~from~~ $\varphi \sim f$ st $f(a_i) = a_i$

Rmk. $[f] \in \mathrm{Stab}(\sigma)$ can't permute $\{a_0, \dots, a_p\}$.
b/c f fixes ∂S pointwise ($f \sim$ diffeo fixing nbhd of ∂S)

Injective Case $\sigma = \{a\}$ vertex (p-simplex case similar...)

Must show following is impossible:

$\exists$ diffeo $f : S \longrightarrow S$ s.t. (i) $f(a) = a$.

(ii) $f \sim id$ isotopic, but not isotopic through diffeos fixing a.

for such f and isotopy f_t to id, $f_t(a) \in \mathrm{Emb}(I, S)$ loop.

Ex. $\mathrm{Mod}(S, *) \longrightarrow \mathrm{Mod}(S)$ not injective. When $f(*) = *$ isotopic to id. isotopy gives elt of $\pi_1(S)$.

Thm. $\mathrm{Emb}_a(I, S) = \{$arcs from z_0 to z_1 isotopic to $a\}$

is contractible.

Proof of injectivity. fiber seq. $\mathrm{Diff}(S, a) \longrightarrow \mathrm{Diff}(S) \to \mathrm{Emb}_a(I, S)$

$\rightsquigarrow 0 = \pi_1 \mathrm{Emb}_a(I, S) \longrightarrow \mathrm{Mod}(S \setminus a) \xrightarrow{\phi} \mathrm{Mod}(S)$.

$\square$

$\Rightarrow \phi$ injective.

For more arcs, eg $\sigma = \{a_0, a_1\}$ consider $S_1 = S \setminus a_1$

$$\mathrm{Mod}(S_1 \setminus a_0) \longrightarrow \mathrm{Stab}(a_0) < \mathrm{Mod}(S_1 = S \setminus a_1) \longrightarrow \mathrm{Stab}(a_1)$$

$$\downarrow$$

$$\mathrm{Mod}(S)$$

image in $\mathrm{Stab}(a_0 \cup a_1)$.

or $\mathrm{Stab}(a_0) \cap \mathrm{Stab}(a_1)$.

Lecture 25

I. Connectivity of arc complexes

Last time: – defined $\mathrm{Mod}_{g,b}$ complexes $X^1_{g,b}$, $X^2_{g,b}$

– proved transitivity & stabilizer props for $\mathrm{Mod}_{g,b}$ action

Today: $X^i_{g,b}$ highly connected. $(\pi_k = 0 \quad k \leq g-2)$

Warmup Hatcher flow on arc complexes.

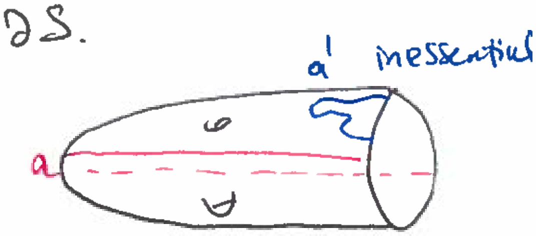
S compact surface $\partial S \neq \emptyset$.

Defn $A(S)$ vertices: isotopy class of essential embedded arc a

p-simplex: $\sigma = \{a_0, \dots, a_p\}$ a_i ~~disjoint~~ distinct disjointly embeddable

Essential means can't homotope a to ∂S.

Note. arcs allowed to separate.



Thm (Hatcher) $A(S)$ contractible.

Proof Fix vertex $v = \{a\}$ in $A(S)$

Will define deformation retract $R_t : A(S) \searrow \mathrm{Star}(v)$

$\mathrm{Star}(v) \equiv \bigcup$ simplices containing v. (always $\sim *$)

Note. $u = \{b\}$ vertex of $\mathrm{Star}(v) \Leftrightarrow$ a, b disjoint (up to isotopy).

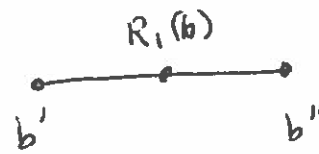
ie $i(a,b) = 0$.

Define R_t on each simplex $\sigma = \{b_0, \ldots, b_p\}$.

Case 1. $\sigma = \{b\}$ vertex.

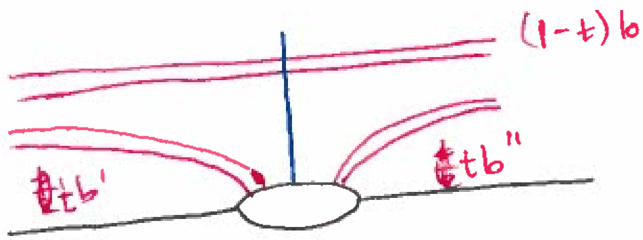
– isotope b so a, b in minimal position

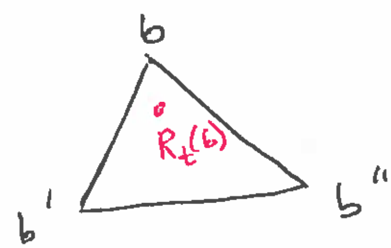
idea:



– thicken b to strip of width 1.

$R_t(b)$:



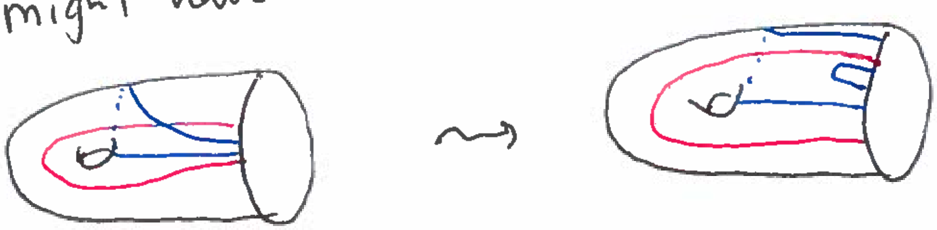


$\shortparallel$

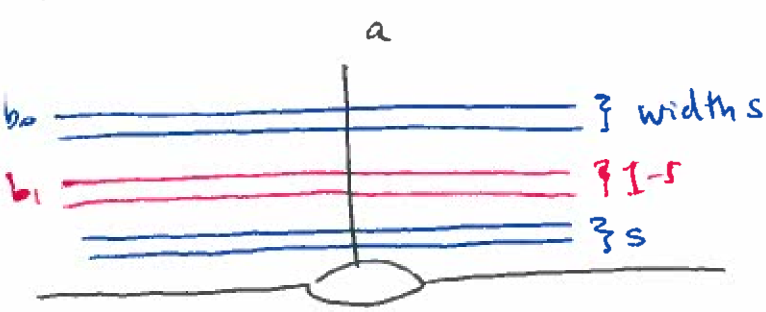
~~$(1-t)b + \frac{t}{2}b' + \frac{t}{2}b''$~~ barycentric coords in

$\frac{(1-t)}{1+t}b + \frac{t}{1+t}b' + \frac{t}{1+t}b''$ in

Rmk. might have b' or b'' inessential but not both inessential



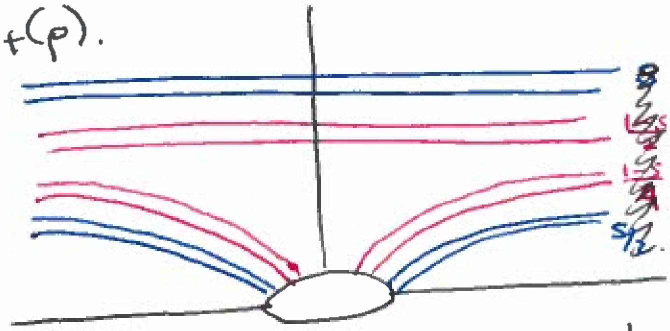
Case 2. $\sigma = \{b_0, b_1\}$. $p = s\,b_0 + (1-s)\,b_1 \in \sigma$.



represent $p \in \sigma$ as strips of varying thickness on S.

$R_t(p)$: push $t \cdot \left(\begin{matrix}\text{total width}\\ \text{of strips}\end{matrix}\right)$ to ∂.

$\rightsquigarrow$ barycentric coords for $R_t(p)$.



similarly for p-simplices.

$t = \frac{1}{2}$.

The only choice made is direction along a, so defn of R_t well-defined. ~~consistent~~ $\square$.

<u>Rmk</u>. $A(S)$ not suited for homological stability b/c. action not transitive & stabilizers are wrong.

II. High connectivity of $X^i_{g,b}$.

<u>Auxiliary complexes</u> $\quad \Delta \subset \partial S$ finite nonempty $\quad \Delta = \Delta_0 \sqcup \Delta_1$

$$X^i_{g,b} \hookrightarrow B_0(S; \Delta_0, \Delta_1) \hookrightarrow B(S; \Delta_0, \Delta_1) \hookrightarrow A(S, \Delta)$$

a nonseparating. $\quad \partial_0 a \in \Delta_0$, $\partial_1 a \in \Delta_1$. $\quad$ vertices: arcs a s.t. $\partial a \subset \Delta$.

$$X^i_{g,b} \simeq B_0(S; \{z_0\}, \{z_1\})$$

<u>WTS</u> spaces/maps highly connected. (this is a lot of work ... only sketch)

3 main types of arguments.

(i) Show cplx contractible (Hatcher flow)

(2) express complex as suspension of simpler complex.

(3) Wahl's "inductive deduction" from connectivity of larger complex.

E.g. $A(S, \Delta)$ contractible by args of type (1), (2).

((2) is like how we showed X_n for S_n was highly conn.)

<u>Prop.</u> $\mathcal{B} = \mathcal{B}(S; \Delta_0, \Delta_1)$ highly conn.

$\Rightarrow \mathcal{B}_0 = \mathcal{B}_0(S; \Delta_0, \Delta_1)$ is too.

(argument of type (3))

<u>Rough sketch</u>.

$$\begin{tikzcd}
S^k \arrow[r, "f"] \arrow[d, hook] & \mathcal{B}_0 \arrow[d, phantom, "\cap"] \\
D^{k+1} \arrow[r, "\hat{f}"] \arrow[ur, dashed, "\hat{\hat{f}}"] & \mathcal{B}
\end{tikzcd}$$

- Given $S^k \xrightarrow{f} \mathcal{B}_0$ — WTS f null homotopic.
- extend $D^{k+1} \xrightarrow{\hat f} \mathcal{B}$ — WT ~~homotope~~ use $\hat f$ ~~into $\mathcal{B}_0$~~ to find $\hat{\hat f}$ ~~(relative)~~.

- Wlog $\hat f$ simplicial.
- main problem: may have $\sigma \subset D^{k+1}$ all of whose vertices land in $\mathcal{B} \setminus \mathcal{B}_0$. ie $\sigma = \{x_0, \ldots, x_p\}$ where each $\hat f(x_i) \subset S$ separating.

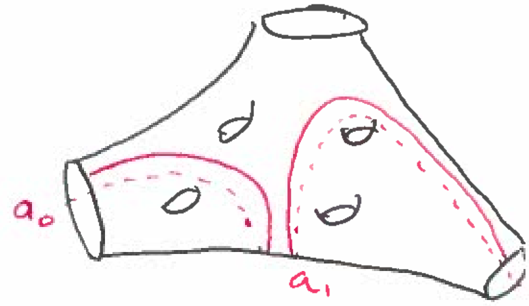
<u>Defn</u> $\sigma \subset D^{k+1}$ is <u>bad</u> if $\hat f(x_i)$ separates S $i = 0, \ldots, p$

$\sigma = \{x_0, \ldots, x_p\}$.

σ <u>maximally bad</u> if σ not contained in bad $(p+1)$-simplex.

- Fix bad σ. Write $S \setminus \hat{f}(\sigma) = S_1 \sqcup \cdots \sqcup S_\ell$.

- Key. if σ maximally bad, then.

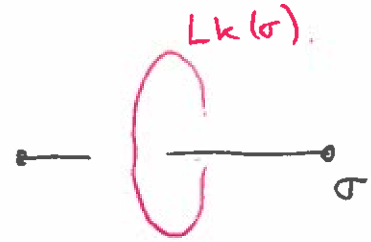


$$\hat{f}\big|_{Lk(\sigma)} : \underset{\simeq S^{k-p}}{Lk(\sigma)} \longrightarrow J_\sigma \subset B$$

where $J_\sigma \simeq \underbrace{B_0(S_1, \Delta_0^1, \Delta_1^1) * \cdots * B_0(S_\ell, \Delta_0^\ell, \Delta_1^\ell)}_{\text{highly conn. by induction.}}$ $\sigma = (x_0, x_1)$

$a_i = f(x_i)$.

-
$$\begin{tikzcd} S^{k-p} \simeq Lk(\sigma) \arrow[r, "\hat{f}"] \arrow[d, phantom, "\cap"] & J_\sigma \\ D^{k-p+1} \simeq K \arrow[ur, dotted] & F \end{tikzcd}$$



- replace $\hat{f}\,|\,\mathrm{star}(\sigma) \simeq \partial\sigma * K$ with (agreeing

$$\hat{f} * F : \partial\sigma * K \longrightarrow B.$$

(by defn map remains unchanged on $\partial(\partial\sigma * K) = \partial\sigma * Lk(\sigma)$.)
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\quad \partial\,\mathrm{star}(\sigma)$.

Check. Any bad simplex $\tau * \tau' \subset \partial\sigma * K$ has dim $\leq p-1$.
(must be face of σ...) $\Rightarrow$ can inductively simplify $\hat{f}$
re simplex of $\partial\sigma$. so no bad simplices. $\square$

Mumford conj: Homological stability ✓

Earle-Eells thm: $\mathrm{Diff}_0(S) \sim *$. next time.

Madsen-Weiss.

Lecture 26

I. Diffeomorphism groups of spheres

$\mathrm{Diff}(S^n)$ orientation-preserving diffeos.

Question / Problem. Determine homotopy type of $\mathrm{Diff}(S^n)$.

e.g. compute $\pi_i \mathrm{Diff}(S^n)$.

$\pi_0 \mathrm{Diff}(S^n)$ & exotic spheres. There are ~~maps~~ homomorphisms

$$\boxed{\pi_0(\mathrm{Diff}_\partial D^n) \xrightarrow{\phi_1} \pi_0(\mathrm{Diff}\, S^n) \xrightarrow{\phi_2} \mu(\mathrm{Diff}\, S^n) \xrightarrow{\phi_3} \Theta_{n+1}}$$

where – $\mathrm{Diff}_\partial D^n$ = diffeos identity near ∂.

– $\mu(\mathrm{Diff}\, S^n)$ "pseudo-isotopy mapping class group"

Defn. $f_0, f_1 : M \to M$ pseudo-isotopic if $\exists$ diffeo.

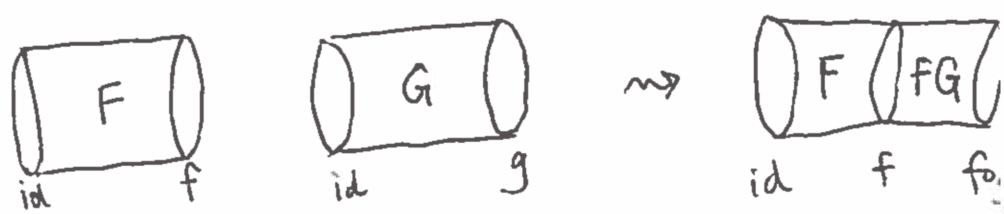
$\exists$ $f : M \times [0,1] \to M \times [0,1]$ s.t. $f|_{M \times \{i\}} = f_i$

$i = 0, 1$.

(isotopic $\Rightarrow$ pseudo isotopic)

Note: $N = \{ f \in \mathrm{Diff}\, M \mid f$ p-isotopic to id$\} < \mathrm{Diff}\, M$ normal subgroup.

Subgroup:



$$\mu(\mathrm{Diff}\, S^n) = \mathrm{Diff}(S^n) / \text{pseudo isotopy} = \mathrm{Diff}(S^n)/N.$$

- Θ_{n+1} group of (oriented) exotic $(n+1)$-spheres

$$= \left\{ f : N \xrightarrow[\text{h.e.}]{} S^n \right\} / \sim \quad \text{where } (N,f) \sim (N',f')$$

if $\exists$
$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "d"'] & S^n \\ N' \arrow[ur, "f'"'] & \end{tikzcd}$$

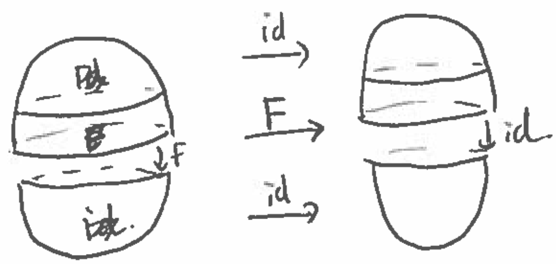
$f \sim f' \circ d$ homotop ~~isotopic~~

group under connected sum.

- ϕ_1 is "extend by identity.
- ϕ_2 obvious quotient map.
- ϕ_3 : $[f] \longmapsto D^{n+1} \cup_f D^{n+1}$

[Aside: can define $D^{n+1} \cup_f D^{n+1} \longrightarrow D^{n+1} \cup_{id} D^{n+1}$ by id on bottom & f on top, levelwise

Note: $f \underset{\text{p-isotopic}}{\sim} id \implies \phi_3(f) = S^{n+1}$.



gives diffeo.

Facts about ϕ_i

- ϕ_3 iso (Smale's h-cobordism thm, $n \neq 4$).
- Cerf pseudo-isotopy thm: $\pi_0(\mathrm{Diff}\, M) \cong \mu(\mathrm{Diff}\, M)$ if $\dim M \geq 5$ & $\pi_1 M = 0$, then ^ i.e. ϕ_2 iso.
- For ϕ_1 consider fibration

$$\mathrm{Diff}(S^n, T_pS^n) \longrightarrow \mathrm{Diff}(S^n) \xrightarrow{\eta} \mathrm{Fr}(S^n)$$

$f : S^n \to S^n$ st. $f(p) = p$, $df_p = id$.

frame bundle

Note $\mathrm{Fr}(S^n) \simeq \mathrm{Isom}(S^n) \simeq SO(n+1)$.

η splits $\Rightarrow$ $\mathrm{Diff}(S^n) \simeq SO(n+1) \times \mathrm{Diff}(S^n, T_pS^n)$ topologically.

Prop 1 $\mathrm{Diff}(S^n, T_pS^n) \sim \mathrm{Diff}_\partial D^n$ homotopy equiv.

(Easier) Prop 2. $\mathrm{Emb}((\mathbb{R}^n, 0), (\mathbb{R}^n, 0)) \xrightarrow{\sim} GL_n\mathbb{R}$

$$f \longmapsto (df)_0.$$

Proof. deformation retract

$$R_t(f)(x) = \begin{cases} f(tx)/t & t \neq 0 \\ (df)_0(x) & t = 0. \end{cases}$$

$R_1 = id$ $\quad R_0 = (df)_0$ $\square$

Rmk. For $f \in \mathrm{Diff}(S^n, T_pS^n)$ restricting to $N_\varepsilon(p) \simeq (\mathbb{R}^n, 0)$ gives embedding $(\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$.

Exercise Use Prop 2 to show $\mathrm{Diff}_\partial D^n \hookrightarrow \mathrm{Diff}(S^n, T_pS^n)$ weak h.e. (ie π_i-iso $\forall i$) ~~[illegible]~~

$\mathrm{Diff}(\ \ )$ has homotopy type of CW $\xRightarrow{\text{Whitehead}}$ Prop 1.

[Claim: ~~For~~ Given compact $K \subset \mathrm{Diff}(S^n, T_pS^n)$ $\exists$ $\varepsilon > 0$ and $q \in S^n$ st. for $f \in K$ image of $\eta_\varepsilon(f)$ does not contain q.
$\eta_\varepsilon : \mathrm{Diff}(S^n, T_pS^n) \longrightarrow \mathrm{Emb}(N_\varepsilon(p), S^n)$

<u>Cor</u> $\mathrm{Diff}(S^n) \sim SO(n+1) \times \mathrm{Diff}_\partial D^n$

and $\pi_0 \mathrm{Diff}(S^n) \simeq \pi_0 \mathrm{Diff}_\partial D^n \Rightarrow \phi_1$ iso.

<u>Thus</u> $\pi_0(\mathrm{Diff}_\partial D^n) \simeq$ ~~$\pi_0 \mathrm{Diff}$~~ $\pi_0(\mathrm{Diff}\, S^n) \simeq \Theta_{n+1}$.

- (Milnor–Kervaire) typically $\Theta_{n+1} \neq 0$ for $n \geq 5$.

  eg. $\Theta_7 \simeq \mathbb{Z}/28$

- Θ_4 ~~cases with~~ unknown. (since $\pi_0 \mathrm{Diff}_\partial D^3 = 0$ if smooth 4D Poincaré conj. false, the examples aren't obtained by twisting.)

- $\pi_0(\mathrm{Diff}_\partial D^n) = 0$ $n \leq 3$. (Munkres $n=2$, Cerf $n=3$).

II. Smale conjecture.

<u>Conjecture</u> (Smale) $\mathrm{Isom}(S^n) \longrightarrow \mathrm{Diff}(S^n)$ is h.e. for $n \leq 3$.

Equivalently $\mathrm{Diff}_\partial D^n$ contractible.

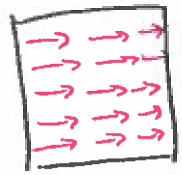
- $n=2$ (Smale 1958)
- $n=3$ (Hatcher 1983)
- $n=1$ exercise : $\mathrm{Diff}_\partial D^1 \simeq id$ by straight-line homotopy.

<u>Cor</u> $B\mathrm{Diff}(S^n) \sim BSO(n+1)$ for $n \leq 3$.

ie. ~~all smooth~~ smooth sphere bundles all come from v. bundles.

<u>Rmk</u>. Will use $\mathrm{Diff}_\partial D^2 \sim *$ to prove $\mathrm{Diff}_0(S) \sim *$ when $\chi(S) < 0$ (Earle-Eells thm).

Proof sketch (following Thurston)

Key observations. Let $Z =$ 

(A) For v.f. X on D^2 s.t. $X = Z$ near ∂D^2

can define canonical path X_t from X to Z:

$$\begin{tikzcd} & \widetilde{\mathbb{R}^2 \setminus 0} \arrow[d] \\ D^2 \arrow[r, "X"] & \mathbb{R}^2 \setminus 0 \end{tikzcd}$$

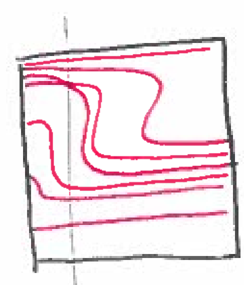
View $X : D^2 \to \mathbb{R}^2 \setminus 0$. Def ret. $\widetilde{\mathbb{R}^2 \setminus 0} \searrow 0$ gives ~~canonical path~~ X_t.

(B) A nonvanishing v.f. on $\mathbb{R}^2$ has no closed orbits.
(false in higher dims)

Sketch. Define $R_t : \mathrm{Diff}_\partial D^2 \rightsquigarrow \{id\}$ on single $f \in \mathrm{Diff}_\partial D^2$ (in cts fashion)

(i) $X := f_*(Z)$. $\overset{(A)}{\rightsquigarrow}$ X_t path X to Z.

(ii) flow of X_t $\rightsquigarrow$ diffeo $h_t : D^2 \to D^2$ (not nec. $= id$ near ∂)



(iii) (B) $\Rightarrow$ ~~Bo~~ hitting function diffeo $\phi_t : [0,1] \to [0,1]$.

(iv) take straight line ~~isotopy~~ isotopy of ϕ_t^{-1} to id.
Compose h_t levelwise w/ isotopy for ϕ_t^{-1} to get diffeo $R_t(f)$.

(on $S \times [0,1]$ do time s of isotopy for ϕ_t.)

Lecture 27

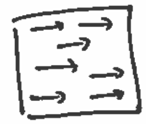
I. ~~Proof of~~ Smale's theorem

$D = [0,1] \times [0,1]$. $\mathrm{Diff}_\partial(D)$ diffeos identity near ∂D.

Thm $\mathrm{Diff}_\partial D \sim *$ contractible.

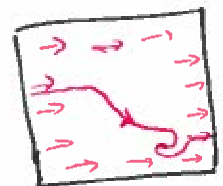
Diffeomorphisms & vector fields on D

- ~~Basic fact~~ Consider $Z \equiv e_1$



- **Basic fact**: X nonvanishing v.f. s.t. $X = Z$ near ∂D

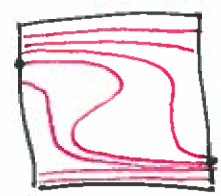
  $\Rightarrow$ every trajectory of X hits $\{1\} \times [0,1]$



  (follows from Poincaré – Bendixson Thm – basic fact about dynamics in plane)

  Rmk False in higher dims (∃ nonvanishing v.f. const. on ∂, w/ traj that's trapped in spiral).

  Cor. Given X get ~~diffeo~~ $\phi \in \mathrm{Diff}_\partial [0,1]$.

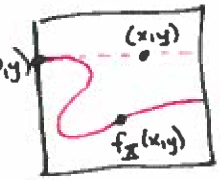


- Correspondence btwn diffeos & v.f's.

  (i) $f \in \mathrm{Diff}_\partial D \rightsquigarrow f_*(Z)$, $f_* Z = Z$ near ∂D.

  (ii) Conversely suppose X v.f. $X = Z$ near ∂D.

  define $h_X : (x,y) \longmapsto$ follow trajectory starting at $(0,y)$ for time x.



Note: h_X typically only embedding $D \longrightarrow \mathbb{R}^2$.

Can scale X by (unique) fxn δ constant on each traj. so $h_{\delta X}$ diffeo.

E.g. Given $f \in \mathrm{Diff}_\partial D$, $\quad X := f_*(Z) \qquad u := \frac{X}{\|X\|}$.

For some fxn $\delta \qquad h_{\delta u} : D^* \to D$ diffeo

by isotopy preserving trajectories of X

Note $f \sim h_{\delta u}$ isotopic (~~slide along trajectories~~)

Proof

Sketch of Thm: Fix $f \in \mathrm{Diff}_\partial D$. Will define isotopy f_t to id s.t. isotopy depends continuously on f. This will give deformation retract $\mathrm{Diff}_\partial D \rightsquigarrow \{id\}$.

~~Then gives~~

– choose def. retract $R_t : \mathbb{R}^2 \setminus \{0\}$. Fix $Z \equiv e_1$ as above.

(1) $X := f_*(Z) \qquad u := \frac{X}{\|X\|} : D \to S^1$

$$\begin{tikzcd} D \arrow[r] \arrow[dr, dashed] & S^1 \\ & \mathbb{R} \arrow[u] \end{tikzcd}$$

u_t homotopy to Z defined using R_t

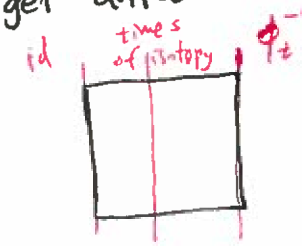
(2) u_t defines diffeo $h_t := h_{\delta_t u_t}$

Note. h_t not nec. identity near ∂D.

$\rightsquigarrow \phi_t : [0,1] \to [0,1]$ diffeo.

Choose (straight-line) isotopy of ϕ_t^{-1} to id.

(3) Compose h_t levelwise w/ isotopy of ϕ_t^{-1} to get diffeo $f_t \in \mathrm{Diff}_\partial D$.



Check f_0 canonically isotopic to f. $\quad f_1 = id$ since $u_1 = Z$. ▨.

II Generalized Smale conjecture.

M compact mfld.

Prob. Homotopy type of $\mathrm{Diff}\, M$.

Rmk. $1 \to \mathrm{Diff}_0(M) \to \mathrm{Diff}\, M \to \pi_0 \mathrm{Diff}\, M \to 1$.

(i) understand $\pi_0 \mathrm{Diff}\, M$ (ii) homotopy of ~~understand~~ $\mathrm{Diff}_0 M$

topologically $\mathrm{Diff}\, M = \coprod_{\pi_0 \mathrm{Diff}\, M} \mathrm{Diff}_0 M$.

2-dimensions

- $M = S^2$ (Smale) $\pi_0 \mathrm{Diff}(S^2) = 1$ $SO(3) \hookrightarrow \mathrm{Diff}_0(S^2)$ h.e.
- $M = T^2$ $\pi_0 \mathrm{Diff}(T^2) \simeq \mathrm{Out}(\mathbb{Z}^2) \cong SL_2\mathbb{Z}$ $T^2 \hookrightarrow \mathrm{Diff}_0(T^2)$ h.e.
- $M = S_g$ $g \geq 2$ $\pi_0 \mathrm{Diff}(S_g) \simeq \mathrm{Out}(\pi_1 S_g) \cong \mathrm{Mod}(S_g)$ $* \hookrightarrow \mathrm{Diff}_0(S_g)$ h.e.
  (Earle-Eells)

Naive general guess

(1) $\mathrm{Diff}\, M \curvearrowright \pi_1 M$ Guess $\pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Out}(\pi_1 M)$ iso.

(2) If g Riem. w/ "maximal symmetry"
Guess $\mathrm{Isom}(M,g)^0 \hookrightarrow \mathrm{Diff}_0 M$ h.e.

Counterexamples

(1) Milnor-Kervaire $\pi_0 \mathrm{Diff}(S^n) \simeq \pi_0 \mathrm{Diff}_\partial(D^n) \simeq \Theta_{n+1}$ typically nonzero.

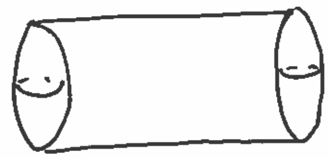
(2) (Hatcher) For $n \geq 5$

$$1 \to (\mathbb{Z}/2)^{\infty} \oplus \bigoplus_{i=0}^{n} \Theta_{i+1}^{\binom{n}{i}} \longrightarrow \pi_0 \mathrm{Diff}(T^n) \longrightarrow SL_n\mathbb{Z} \longrightarrow 1.$$

(3) even in low dim:

(Hatcher) $\mathrm{Diff}(S^1 \times S^2) \sim SO(2) \times SO(3) \times \Omega SO(3)$.

not unexpected: this is group of bundle auts.



$S^1 \xrightarrow{f} SO(3)$ defines diffeo

$$S^1 \times S^2 \longrightarrow S^1 \times S^2$$
$$(x,y) \longmapsto (x, f(x)(y)).$$

<u>Conjecture</u> (Generalized Smale) Naive guess correct for constant curvature 3-dim geometries (elliptic, Euclidean, hyperbolic)

<u>Known for</u> • S^3 (Hatcher) • Lens spaces (McCullough ...)

• hyperbolic 3 mflds (Gabai)

still open for certain elliptic 3-mflds (eg $\mathbb{RP}^3$).

<u>III.</u> Earle-Eells thm $\mathrm{Diff}(S)$ diffeos s.t. $f|_{\partial S} = \mathrm{id}$.

<u>Thm</u> S compact surface $\chi(S) < 0$ $\Rightarrow \mathrm{Diff}_0(S) \sim *$.

<u>Rmk</u> Enough to show $\pi_i \mathrm{Diff}(S) = 0$ $i \geq 0$.

- $\mathrm{Diff}\, M$ metrizable Banach mfld.
- (Palais) metrizable Banach mflds have homotopy type of CW cplx.

- Whitehead $f: X \to Y$ map of CW inducing π_i-iso $i \geq 0 \Rightarrow f$ h.e.

Strategy (following Hatcher)

Use Evaluation fibrations as bootstrapping tool.

Step 1. Reduction to case w/ ∂.

Prop 1. $S = S_g$ closed $g \geq 2$. Then $\mathrm{Diff}(S,*) \hookrightarrow \mathrm{Diff}(S)$ induces iso on π_k $k \geq 1$.

Pf. Fibration $\mathrm{Diff}(S,*) \longrightarrow \mathrm{Diff}(S) \longrightarrow S$

- $\pi_k(S) = 0$ $k \geq 1$ $\Longrightarrow$ Prop for $k \geq 2$ ~~$\pi_k \mathrm{Diff}(S,*) \cong \pi_k \mathrm{Diff}(S)$ for $k \geq 2$.~~

- For $k=1$ $\quad 0 \to \pi_1 \mathrm{Diff}(S,*) \to \pi_1 \mathrm{Diff}(S) \to \pi_1(S) \xrightarrow{\delta} \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g \to$

δ injective (c.f. proof of BES). $\square$.

Prop 2 Fix $(D,0) \hookrightarrow (S,*)$ embedded disk.

$\mathrm{Diff}(S,D) := \{ \text{diffeos} : f|_D = id \}$. Then $\mathrm{Diff}(S,D) \hookrightarrow \mathrm{Diff}(S,*)$ iso on π_k $k \geq 1$.

Pf. Fibration $\mathrm{Diff}(S,D) \longrightarrow \mathrm{Diff}(S,*) \longrightarrow \mathrm{Emb}_+((D,0),(S,*))$

- **Exercise:** $\mathrm{Emb}_+((D,0),(S,*)) \sim SL_2\mathbb{R} \sim SO(2)$. $\Rightarrow$ ~~the~~ Prop true for $k \geq 2$.

- For $k=1$ $\quad 0 \to \pi_1 \mathrm{Diff}(S,D) \to \pi_1 \mathrm{Diff}(S,*) \to \underset{\mathbb{Z}.}{\underset{\parallel}{\pi_1 \mathrm{Emb}}} \xrightarrow{\delta} \mathrm{Mod}_g' \to \mathrm{Mod} \to 1$

$\delta(1)$ Dehn twist about ∂. $\square$.

Lecture 28

I. Diffeomorphism groups of surfaces

<u>Thm.</u> S_g closed genus $g \geq 2$. Then $\pi_k \mathrm{Diff}(S_g) = 0$ for $k \geq 1$.

<u>Cor.</u>
- $\mathrm{Diff}(S_g) \longrightarrow \mathrm{Mod}_g$ h.e.
- $B\mathrm{Diff}(S_g) \longrightarrow B\mathrm{Mod}_g = K(\mathrm{Mod}_g, 1)$ h.e.
- $\left\{ \begin{matrix} S_g \to E \\ \downarrow \\ B \end{matrix} \right\}_{/\mathrm{iso}} \cong [B, B\mathrm{Diff}(S_g)] \cong [B, K(\mathrm{Mod}_g, 1)] \cong \left\{ \pi_1(B) \to \mathrm{Mod}_g \right\}_{/\mathrm{conj}}$

<u>Strategy</u> (following Hatcher) Use evaluation fibrations as bootstrapping tool.

<u>Warm-up:</u> Reduction to case with boundary.

<u>Prop 1</u>. $S = S_g$ $g \geq 2$. $\mathrm{Diff}(S, *) \hookrightarrow \mathrm{Diff}(S)$ induces π_k-iso $k \geq 1$.

<u>Pf</u>: Fibration $\mathrm{Diff}(S,*) \longrightarrow \mathrm{Diff}(S) \longrightarrow S$.

- $\pi_k(S) = 0$ $k \geq 1$ $\Longrightarrow$ Prop for $k \geq 2$.
- $k = 1$ $\quad 0 \to \pi_1 \mathrm{Diff}(S,*) \to \pi_1 \mathrm{Diff}(S) \to \pi_1(S) \xrightarrow{\delta} \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g \to 1$.

$$\begin{array}{ccc} \pi_1(S) & \xrightarrow{\delta} & \mathrm{Mod}_{g,*} \\ & \searrow & \downarrow \\ & & \mathrm{Aut}(\pi_1(S)) \end{array}$$

δ - inj (c.f. proof of BES). $\square$.

<u>Prop 2</u>. Fix. $(D, 0) \hookrightarrow (S, *)$ embedded disk.
$\mathrm{Diff}(S, D) = \{ \text{diffeos} : f|_D = id \}$. Then $\mathrm{Diff}(S, D) \hookrightarrow \mathrm{Diff}(S, *)$
induces π_k-iso $k \geq 1$.

Proof fibration $\mathrm{Diff}(S,D) \to \mathrm{Diff}(S,*) \to \mathrm{Emb}_+((D,0),(S,*)) =: \mathcal{E}$ 2

Exercise For any mfld M^n

$$\begin{tikzcd} \mathrm{Emb}((D^n,0),(M,*)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Emb}(D^n,M) \arrow[r] \arrow[d, "\simeq"] & M \arrow[d, equal] \\ GL_n\mathbb{R} \arrow[r] & Fr(M) \arrow[r] & M \end{tikzcd}$$

[in gen. can homotope cpt family of emb. to be contained in nbhd of *]

Recall. We showed $\mathrm{Emb}((D^n,0),(\mathbb{R}^n,0)) \sim GL_n\mathbb{R}$.

$\Rightarrow \mathcal{E} \sim SL_2\mathbb{R} \sim SO(2)$. $\Rightarrow$ ~~Thm~~ Prop for $k \geq 2$.

- $k=1$. $0 \to \pi_1 \mathrm{Diff}(S,D) \to \pi_1 \mathrm{Diff}(S,*) \to \underset{\mathbb{Z}.}{\overset{\shortparallel}{\pi_1(\mathcal{E})}} \xrightarrow{\delta} \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g,*} \to 1.$

$\delta(1) =$ Dehn twist about ∂ (acts nontrivial on ~~$\pi_1(S_g)$~~ $\pi_1(S_g^1)$) $\square$.

Note. $\mathrm{Diff}(S,D) \sim \mathrm{Diff}(S')$ $S' = S \setminus \mathring{D}$

$\Rightarrow$ enough to show $\pi_k \mathrm{Diff}(S) = 0$ $k \geq 1$ in case $\partial S \neq \emptyset$.

II. Spaces of arcs.

Goal: reduce problem to simpler surface using fibration

$\left\{ \begin{array}{c} \text{diffeos fixing} \\ \text{an arc } \alpha \end{array} \right\} \to \mathrm{Diff}(S) \to \{\text{arcs}\}.$

$\simeq \mathrm{Diff}(S')$, $S' = S \setminus \alpha$.

reduces problem to understanding space of arcs.

- S compact $\partial S \neq \emptyset$. $p,q \in \partial S$, $\alpha : [0,1] \to S$ nonsep. arc w/ endpoints p,q.

- $A(S,\alpha) := \{ \beta : [0,1] \hookrightarrow S \mid$ endpoints p, q. $\beta \sim \alpha$ isotopic $\}$.

- $\mathrm{Diff}(S,\alpha)$ diffeos s.t. $f \circ \alpha = \alpha$.

- Fibration

$$\mathrm{Diff}(S,\alpha) \cap \mathrm{Diff}_0(S) \longrightarrow \mathrm{Diff}_0(S) \longrightarrow A(S,\alpha)$$
$$f \longmapsto f \circ \alpha.$$

<u>Thm.</u> (Cerf, Gramain) $A(S,\alpha)$ contractible.

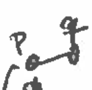
if sketch in

<u>Special case</u>. p, q on diff boundary components.

<u>Recall</u>. General fibration: $V \subset M$ N mflds

$$\left\{ \begin{array}{l} \text{Emb. } M \to N \\ \text{extending a given } V \to N \end{array} \right\} \longrightarrow \mathrm{Emb}(M, N) \longrightarrow \mathrm{Emb}(V, N)$$

Application: $T := S \cup D$.

Emb(

, S) ⟶ Emb( , T) ⟶ Emb( , $T \setminus \partial T$)

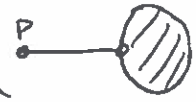
– As above $\mathrm{Emb}(D, T \setminus \partial T) \simeq \mathrm{Fr}(T)$ has $\pi_k = 0$ $k \geq 2$. $k = 0$.

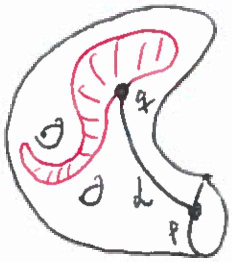
<u>Claim</u>. Emb( , T) contractible.

<u>Corollary</u> $\pi_k \, \mathrm{Emb}(I, S) = 0$ $k \geq 1$ $\pi_0 \, \mathrm{Emb}(I,S) \sim \pi_1(\mathrm{Fr}\, T)$.

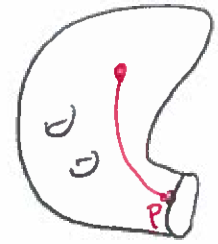
$\Rightarrow$ Components of $\mathrm{Emb}(I,S) \sim *$ $\Rightarrow$ $A(S,\alpha) \sim *$.

Approach to ~~Proving~~ the claim:

$\mathrm{Emb}($  $, T\setminus\alpha) \longrightarrow \mathrm{Emb}($  $, T) \longrightarrow \mathrm{Emb}($  $, T).$

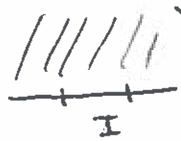


Contractible (exercise).



Contractible - shrink arc to a nbhd of ∂

Then consider $f_t = \frac{1}{t}\cdot f_{[0,}$

related Exercise: $\mathrm{Emb}($  $,$  $)$ contractible.

Rmk. Case α connects p, q on same ∂-comp similar (trick to reduce to case above)

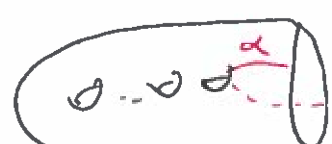
III. Finishing the proof.

Thm $\chi(S) < 0 \implies \pi_k \mathrm{Diff}_0(S) = 0 \quad k \geq 0.$

Proof.

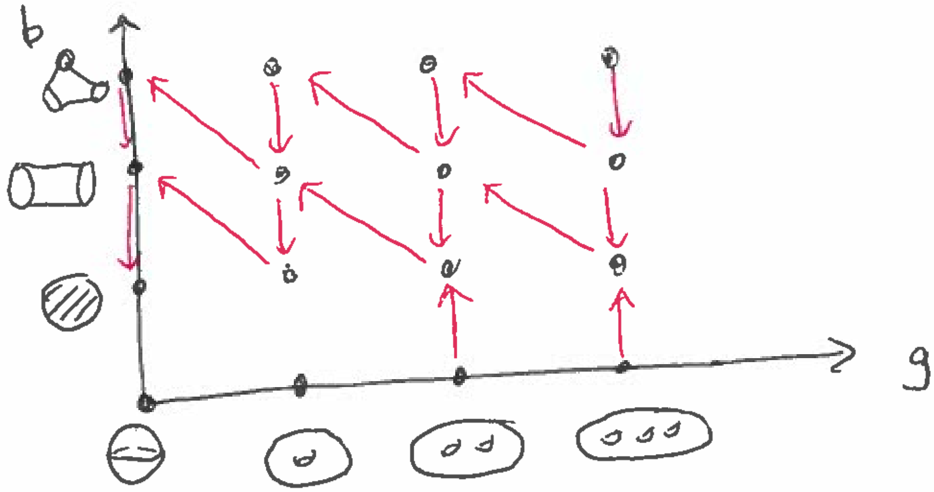
- wlog $\partial S \neq \emptyset$ (by warmup)
- choose nonsep. arc α. $\quad \mathrm{Diff}_0(S,\alpha) \longrightarrow \mathrm{Diff}_0(S) \longrightarrow A(S,\alpha).$

$A(S,\alpha) \sim * \implies \pi_k \mathrm{Diff}_0(S) \cong \pi_k \mathrm{Diff}_0(S') \quad S' = S\setminus\alpha.$ compact.



$S_g^b \rightsquigarrow S_g^{b-1} \qquad\qquad S_g^b \rightsquigarrow S_{g-1}^{b+1}$

In this way we reduce to $S = \mathbb{D}^2$



□.

$\pi_k \mathrm{Diff}_\partial(D^2) \cong 0 \quad \forall k \geq 0$ by last time

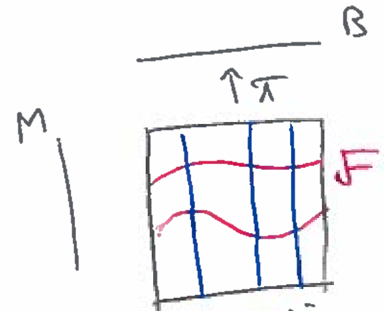
Lecture 29

I. Flat bundles

M, B manifolds $\quad M \to E \to B$ smooth bundle.

__Defn__ $E \to B$ is __flat__ if $\exists\ \rho : \pi_1(B) \to \mathrm{Diff}(M)$

and iso
$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr] & & \frac{\tilde{B}\times M}{\pi_1(B)} =: E_\rho . \arrow[dl] \\ & B & \end{tikzcd}$$



__Prop.__ TFAE.

(1) $M \to E \xrightarrow{\pi} B$ flat

(2) E has ~~horizontal~~ foliation $\mathcal{F}$ whose leaves are transverse to fibers and each leafe projects to B as covering space.

(3) the structure group reduces to $B\mathrm{Diff}(M)^\delta \equiv K(\mathrm{Diff}M, 1)$

where $\mathrm{Diff}(M)^\delta = \mathrm{Diff}\,M$ w/ discrete topology.

ie
$$\begin{tikzcd} & B\mathrm{Diff}(M)^\delta \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & B\mathrm{Diff}(M) \end{tikzcd}$$

~~Remark/Defn~~ __Proof sketch__

- (1) $\Rightarrow$ (2). E_ρ has foliation by $L_x = \mathrm{im}\left(\tilde{B}\times\{x\} \to \frac{\tilde{B}\times M}{\pi_1 B}\right)_{x\in M}$.

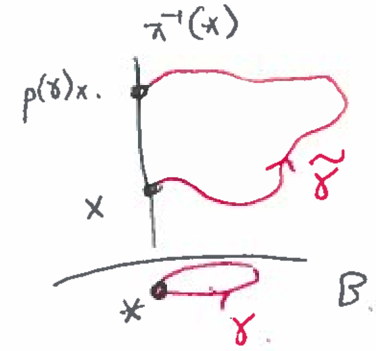
Note. $\tilde{B}\times\{x\} \to \frac{\tilde{B}\times M}{\pi_1 B} \to B$ is universal cover.

- $(2) \Rightarrow (1)$. Monodromy. Define $\rho : \pi_1(B,*) \longrightarrow \mathrm{Diff}(\overset{\pi^{-1}(*)}{M})$

$$[\gamma] \longmapsto \left[x \mapsto \tilde{\gamma}(1) \right]$$

where $\tilde{\gamma}$ lift of γ w/ $\tilde{\gamma}(0) = x$.

well-defined ~~b/c [illegible]~~. leaf containing x is a cover of B.



- $(1) \Leftrightarrow (3)$

Given $E_\rho \to B$, $\rho \rightsquigarrow$

$$\begin{tikzcd} & K(\mathrm{Diff}M, 1) \equiv B\mathrm{Diff}(M)^\delta \arrow[d] \\ B \arrow[ur] \arrow[r] & B\mathrm{Diff}(M) \end{tikzcd}$$

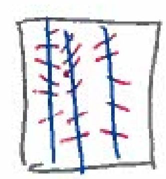
is lift.

OTOH given

$$\begin{tikzcd} & B\mathrm{Diff}(M)^\delta \arrow[d] \\ B \arrow[ur] \arrow[r] & B\mathrm{Diff}M \end{tikzcd}$$

Consider $\rho : \pi_1(B) \to \mathrm{Diff}(M)$ induced on π_1. $\square$.

<u>Rmk</u> / <u>Defn</u> A <u>connection</u> on $M^n \to E \to B^d$ is a d-plane distribution H on E, everywhere transverse to the fibers



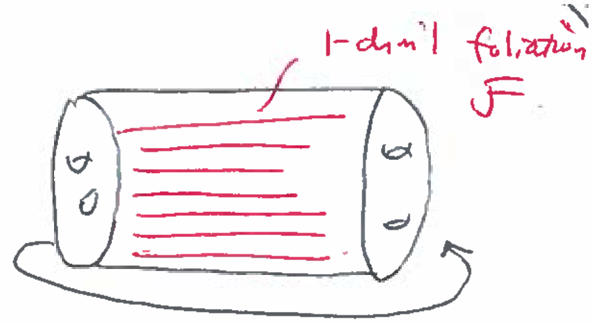
H defines parallel transport $\pi_1(B) \longrightarrow \pi_0 \mathrm{Diff}(M)$.

$$[\gamma] \mapsto \left[x \longmapsto \tilde{\gamma}(1) \right]$$

diffeo depends in general on ~~loop~~ representative of $[\gamma]$. Only isotopy class well-defined

If H <u>integrable</u> (tangent to a foliation) then parallel ... $[\gamma]$ call cnx ...

Examples.



(1) Any bundle over S^1 is flat

$M \to E = E_\rho$
$\downarrow$
S^1

$\rho : \pi_1(S^1) = \mathbb{Z} \longrightarrow \mathrm{Diff}(M)$
$1 \longmapsto \phi$

$\phi \in \mathrm{Diff}(M)$.

(2) Euclidean / Flat manifolds.

$M = \mathbb{E}^n / \Gamma$ where $\Gamma \overset{f}{\hookrightarrow} \mathrm{Isom}\, \mathbb{E}^n \simeq \mathbb{R}^n \rtimes O(n)$.

Then $\mathbb{R}^n \to TM \to M$ is flat: $TM \simeq \dfrac{\mathbb{E}^n \times \mathbb{R}^n}{\Gamma} \left(= \dfrac{T\mathbb{E}^n}{\Gamma} \right)$.

where $\Gamma \curvearrowright \mathbb{E}^n$ by ρ & $\Gamma \curvearrowright \mathbb{R}^n$ by $\bar{\rho} : \Gamma \to \mathbb{R}^n \rtimes O(n) \to O(n)$

(3) ~~Note $T\mathbb{E}^n$~~ Hyperbolic manifolds.

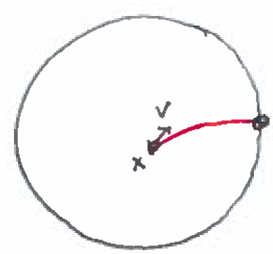
$M = \mathbb{H}^n / \Gamma$ $\quad \Gamma < \mathrm{Isom}\, \mathbb{H}^n$ discrete.

Claim. ~~Then~~ $S^{n-1} \to T^1 M \to M$ flat.

Consider $\rho : \pi_1(M) \to \mathrm{Diff}(S^{n-1})$ induced by $\Gamma \curvearrowright \partial \mathbb{H}^n$.

Pf Identify $T^1 \mathbb{H}^n \xrightarrow{\simeq} \mathbb{H}^n \times \partial \mathbb{H}^n$ via exponential.

$(x, v) \longmapsto (x, \lim_{t \to \infty} \exp_x(tv))$



Then $T^1 M \simeq T^1 \mathbb{H}^n / \Gamma \cong \dfrac{\mathbb{H}^n \times \partial \mathbb{H}^n}{\Gamma} = E_\rho$. $\square$

(4) $S^1 \to T^1S^2 \longrightarrow S^2$ not flat.
$S^1 \longrightarrow S^3 \longrightarrow S^2$ not flat

$\pi_1(B) = 1 \Longrightarrow$ the only flat bundles over are trivial.

<u>Problem / Question</u> When does $E \to B$ admit a flat connection?

II. Circle bundles.

$S = S_g$ closed surface (oriented).

$$\begin{array}{c} S^1 \longrightarrow E \\ \downarrow \\ S \end{array}$$
oriented. circle bundle.

<u>Classification</u>

$$\left\{ \begin{array}{c} S^1 \to E \\ \downarrow \\ S \end{array} \right\} / \text{iso} \xleftrightarrow{\simeq} [S, B\mathrm{Diff}(S^1)]$$

$$\simeq [S, BSO(2)] \simeq [S, \mathbb{CP}^\infty]$$

$$\simeq [S, K(\mathbb{Z}, 2)].$$

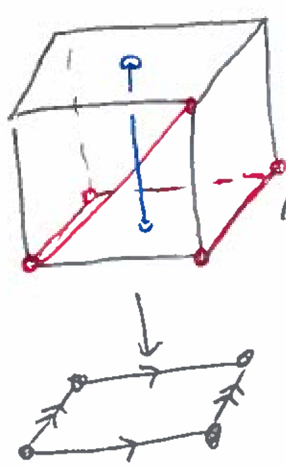
$SO(2) \hookrightarrow \mathrm{Diff}(S^1)$ h.e.

$$\simeq H^2(S; \mathbb{Z}).$$

$\Rightarrow$ $\begin{array}{c} S^1 \to E \\ \downarrow \\ S \end{array}$ determined up to isomorphism by Euler class $e(E) \in H^2(S; \mathbb{Z}) \simeq \mathbb{Z}$.

<u>Defn</u> (Euler class).

<u>Ex</u>. $\begin{array}{c} S^1 \longrightarrow E \\ \downarrow \\ T^2 \end{array}$

$E =$  $/\sim$.

glue
left / right, top / bottom by translation
front / back by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

<u>Defn</u> (primary obstruction to section $\sigma : S \to E$).

- pick σ on 0-skeleton
- interpolate on 1-skeleton (S^1 connected).
- over 2-cell define $\sigma|_{\partial c} : S^1 \simeq \partial c \longrightarrow S^1$. by.

$$\begin{tikzcd} S^1 & \partial c \times S^1 \arrow[l] \arrow[r] \arrow[d] & \pi^{-1}(c) \simeq c \times S^1 \arrow[d] \\ & \partial c \arrow[u, bend right, "\sigma"'] \arrow[r] & c \end{tikzcd}$$

$\deg(\sigma|_{\partial c})$ obstruction to extending σ over c.

$\rightsquigarrow$ 2-cochain cocycle!
$$C_2(S) \xrightarrow{\phi} \mathbb{Z}.$$
$$c \longmapsto \deg(\sigma|_{\partial c}).$$

$\rightsquigarrow$ $e(E) = [\phi] \in H^2(S;\mathbb{Z})$.

Giving S cell decomp w/ one 2-cell have
$$e(E) = \deg(\sigma|_{\partial c}) \in \mathbb{Z}.$$

For example above $e(E) = 1$.

<u>Thm</u> (Milnor–Wood 70s)

$$\begin{array}{c} S^1 \to E \\ \downarrow \\ S_g. \end{array} \text{ flat} \iff |e(E)| \le 2g-2$$

$$\chi(S_g) \le e(E) \le -\chi(S_g)$$

Next time: flat [illegible] bundles

I. Milnor - Wood inequalities

<u>Defn.</u> $G < \mathrm{Diff}(M)$ a smooth bundle $M \to E \to B$ is <u>G-flat</u> if any of the following hold.

(i) $E \simeq E_\rho = \frac{\tilde{B} \times M}{\pi_1 B}$ for some $\rho: \pi_1(B) \to G < \mathrm{Diff}(M)$.
$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[dr] & E_\rho \arrow[d] \\ & B \end{tikzcd}$$

(ii) E has a transverse foliation ~~or~~ ... w/ holonomy in G.

(iii) $\exists$ lift
$$\begin{tikzcd} & BG^\delta \arrow[r] \arrow[d] & B\mathrm{Diff}(M)^\delta \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & BG \arrow[r] & B\mathrm{Diff}(M) \end{tikzcd}$$

<u>Thm</u> $S^1 \to E \to S_g$ (closed, or. $g \geq 1$) flat $\iff$ $|e(E)| \leq -\chi(S_g)$

G-flat $\qquad$ $|e(E)| \leq f_G(S_g)$.

where

	G	f_G
(Chern-Weil)	$SO(2)$	0
(Milnor)	$SL_2\mathbb{R}$	$-\frac{1}{2}\chi(S_g)$
(Wood)	$PSL_2\mathbb{R}$, $\mathrm{Homeo}(S^1)$	$-\chi(S_g)$

<u>Example</u> • $g \geq 2$ $\quad E = T^1 S_g$ unit tangent bundle.

$e(T^1 S_g) = \chi(S_g) \Rightarrow$ $T^1 S_g$ has no flat $SO(2)$-connection and no flat $SL_2\mathbb{R}$-connection but does have flat $PSL_2\mathbb{R}$-connection $\subseteq^{\text{flat}}$ $\mathrm{Homeo}(S^1)$-connection

• $g = 1$. $\forall \rho: \mathbb{Z}^2 \to \mathrm{Homeo}(S^1)$ $\quad e(E_\rho) = 0$.

<u>Rmk.</u> $SO(2) \hookrightarrow \mathrm{Homeo}(S^1)$ $\Rightarrow$ every $\begin{array}{c} S^1 \to E \\ \downarrow \\ S_g \end{array}$ iso to linear S^1 bundle.

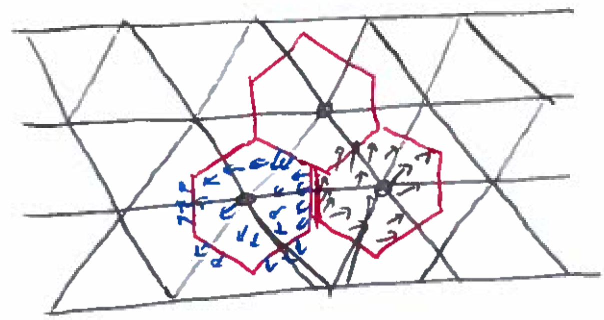
but if E flat, not nec. flat wrt linear bundle structure.

<u>Sullivan approach to MW</u> $S^1 \to E \to S_g$ flat.

Show $e(E) \in \mathbb{Z}$ bounded by (m.n) # triangles in Δ'lation of S

<u>Euler class for flat $S^1 \to E \to S_g$</u>

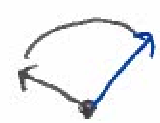
- triangulate S_g, consider dual.



<u>want</u> to define cocycle $C_2(S_g) \to \mathbb{Z}$ rep'ing Euler class using flat cnxn.

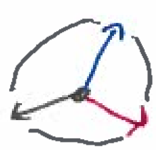
- choose section on vertex. spread to dual face using cnxn.
- on dual edge e, have disagreement.

pick pt on ~~edge~~ e, in fiber choose arc btwn two sections
spread to ~~dual~~ e using connection

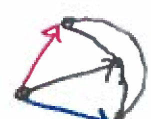


- at dual vertex have

Winding number

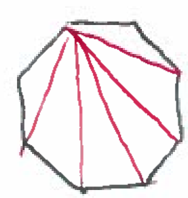


± 1



0.

$\Rightarrow |e(E)| \le$ # Δ's in Δ'lation



4g-gon

$\rightsquigarrow$ Euler cocycle $C_2(S) \to \mathbb{Z}$

2 cell $\longmapsto$ winding # at dual vertex

II. Cohomological perspective on flat bundles

Rmk.

$$\begin{tikzcd} & BG^\delta \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & BG \end{tikzcd} \rightsquigarrow \begin{tikzcd} & H^\bullet(BG^\delta) \arrow[dl, dashed] \\ H^\bullet(B) & H^\bullet(BG) \arrow[l] \arrow[u, "\phi"'] \end{tikzcd}$$

One approach to Q of which bundles admit flat G-cnxns is to study ϕ.

Example (Chern-Weil theory) G Lie group w/ Lie alg $\mathfrak{g}$.

- $\exists$ hom $\underset{\substack{\| \\ \text{invariant polys on } \mathfrak{g}}}{I^\bullet(G)} \longrightarrow H^{2\bullet}(BG;\mathbb{R})$

  eg $a \in \mathfrak{g}$ $\det(\lambda I - a) = \sum Q_k(a)\lambda^{\cdots}$ $Q_k \in I^k(\cdots$ ~~eg~~ ~~Chern~~ polys.

  $$\left\{ \begin{array}{c} P : \mathfrak{g} \times \cdots \mathfrak{g} \to \mathbb{R} \\ \text{symmetric, multi-linear} \\ \text{invt under adjoint action of } G. \end{array} \right\} \simeq \mathbb{R}[x_1, \ldots, x_N]^G$$

  $x_i \in \mathfrak{g}^*$ ~~dual~~ ~~to~~ basis

- For $Q \in I^k(G)$ and $\begin{array}{c} G \to P \\ \downarrow \\ B \xrightarrow{f} BG \end{array}$ principal, connection ∇, curvature $\omega \in \Omega^2(P;\mathfrak{g})$

  $$f^* \circ \alpha(Q) = [Q(\omega^k)] \quad \omega^k \in \Omega^{2k}(P; \mathfrak{g}^{\otimes k}).$$
  $$\in H^{2k}(B).$$

- $\mathrm{Im}(\alpha) < \ker(\phi)$.

  if $\begin{array}{c} G \to P \\ \downarrow \\ B \end{array}$ flat then $\omega = 0 \Rightarrow f^* \circ \alpha(Q) = [Q(\omega^k)] = \cdots$

- If G compact α is isomorphism $\Rightarrow \underset{\text{zero}}{H^\bullet(BG;\mathbb{R}) \to H^\bullet(BG^\delta;\mathbb{R})}$

  $\Rightarrow$ any $a \in H^\bullet(BG)$ is obstruction to flat G-cnxn.

- For G noncompact α typically not surj

e.g. $G = SL_2\mathbb{R}$. $\quad I^\cdot(G) \xrightarrow{\alpha} H^{2\cdot}(BSL_2\mathbb{R}) \simeq H^{2\cdot}(BSO(2)) \simeq \mathbb{R}[e]$.

$\mathrm{Im}(\alpha)$ generated by e^2. (so e not in $\mathrm{im}(\alpha)$)

<u>Thm</u> (Gromov) $\mathrm{im}\left(\phi : H^\cdot(BG;\mathbb{R}) \longrightarrow H^\cdot(BG^\delta;\mathbb{R})\right)$

consists of <u>bounded classes</u>.

<u>Defn</u>. X space $\quad C^k(X) = \mathrm{Hom}(C_k(X), \mathbb{R})$ singular cochains.

L^∞ norm $\quad |f|_\infty = \sup\limits_{\sigma : \Delta^k \to X} f(\sigma)$.

$C^k_b(X) = \{ f \in C^k(X) : |f|_\infty < \infty \}$ bounded cochain complex

$\rightsquigarrow H^k_b(X)$ bounded coho, seminorm $\quad \|c\|_\infty = \inf\limits_{[f]=c} |f|_\infty \in [0,\infty)$

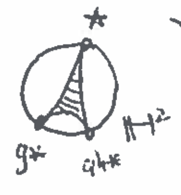
forgetful map $\quad H^k_b(X) \longrightarrow H^k(X)$ in general neither inj/surj

<u>Thm</u> $\quad$ $$\begin{array}{ccc} H^\cdot(BG) & \xrightarrow{\phi} & H^\cdot(BG^\delta) \\ & & \uparrow{\scriptstyle \text{forget}} \\ & & H^\cdot_b(BG^\delta) \end{array}$$ $\quad \mathrm{im}(\phi) < \mathrm{im}(\text{forget})$.

<u>Ex</u>. $G = PSL_2\mathbb{R}$ $\quad H^\cdot(BG;\mathbb{R}) \simeq \mathbb{R}[e] \xrightarrow{\phi} H^\cdot(BG^\delta;\mathbb{R})$

- $\phi(e) \neq 0$ $\quad$ ($\exists$ flat S^1 bundles w/ nonzero euler class, eg $T^1S_g \to S_g$)

bounded rep for $\phi(e)$: $\quad c : G \times G \longrightarrow \mathbb{R}$.

$(g,h) \longmapsto \frac{1}{2\pi}$ Area(

) $\in \{0, \pm\frac{1}{2}\}$

$\longrightarrow \|\phi(e)\|_\infty \leq \frac{1}{2}$.

Lecture 31

Last time. G Lie group (eg. $G = PSL_2\mathbb{R}$)

- $H^\cdot(BG) \xrightarrow{\phi} H^\cdot(BG^\delta)$ useful/important for understanding which G bundles are G-flat
- (Chern-Weil, Milnor) $I^\cdot(G) \longrightarrow H^\cdot(BG) \xrightarrow{\phi} H^\cdot(BG^\delta)$ exact for $\cdot > 0$. (See Milnor "Homology of Lie groups made discrete")
- (Gromov) $\mathrm{Im}(\phi)$ consists of coho classes w/ bounded representatives

I. Bounded Cohomology

- X space, $C^k(X) = \mathrm{Hom}(C_k(X), \mathbb{R})$ singular cochains.
- L^∞ norm $|f|_\infty := \sup_{\sigma : \Delta^k \to X} f(\sigma)$.
- $C_b^k(X) = \{ f \in C^k(X) : |f|_\infty < \infty \}$. $\partial : C_b^k(X) \longrightarrow C_b^{k+1}(X)$

  bounded cochain complex.

$\rightsquigarrow$ $H_b^k(X)$ bounded cohomology, seminorm $\|c\| = \inf_{[f]=c} |f|_\infty$
$\in [0, \infty)$.

Properties.

(1) functorial $h : X \longrightarrow Y$ $\rightsquigarrow$ $h^* : H_b^\cdot(Y) \longrightarrow H_b^\cdot(X)$.

(2) h^* norm non-increasing ~~decrease~~ $\|h^* c\| \le \|c\|$. [cocycle rep for c gives rep for h^*c but maybe h^*c has more efficient rep...]

(3) homotopy invt (in fact depends only on $\pi_1(X)$)

(4) not a homology Theory (excision fails)

(5) <u>Comparison map</u> $\psi : H^k_b(X) \longrightarrow H^k(X)$

in general not inj/surj.

<u>Ex</u>. $H^1_b(X) \longrightarrow H^1(X)$ zero $\forall X$ (in fact $H^1_b(X)=0$ $\forall X$)

Fix $c \in H^1(X)$ nonzero.

<u>Claim</u>. $\forall f \in C^1(X)$ $[f]=c$, $\exists$ $\sigma_k : \Delta^1 \longrightarrow X$ $k \geq 1$

st. $f(\sigma_k) \longrightarrow \infty$ as $k \to \infty$.

<u>Pf</u>: $c \neq 0 \Rightarrow \exists$ ~~closed~~ cycle $\sigma : \Delta^1 \longrightarrow X$ st. $\langle c, \sigma \rangle \neq 0$

$$\begin{tikzcd} \Delta^1 \arrow[rr] \arrow[dr] & & X \\ & S^1 \arrow[ur] & \end{tikzcd}$$

define $\sigma_k : \Delta^1 \longrightarrow S^1 \xrightarrow{\deg k} S^1 \longrightarrow X$. Note $[\sigma_k] = k[\sigma]$ in $H_1(X)$.

$[f]=c \Longrightarrow f(\sigma_k) = \langle c, \sigma_k \rangle = k \langle c, \sigma \rangle \longrightarrow \infty$.

(6) For group G, can define $H^\bullet_b(G)$ in terms of bounded cochains $G \times \cdots \times G \longrightarrow \mathbb{Z}$.

For $X = K(G,1)$ $H^\bullet_b(X) \simeq H^\bullet_b(G)$.

<u>Rmk</u>. Non injectivity of $H^2_b(G) \longrightarrow H^2(G)$ related to existence of <u>quasimorphism</u> on G, ie (unbounded) function 函数

$\alpha : G \longrightarrow \mathbb{Z}$ st. $\exists D > 0$ st. $\underbrace{|\alpha(g) + \alpha(h) - \alpha(gh)|}_{= \delta\alpha} < D$. $\forall g,h \in G$

Thm (Gromov) G Lie group. $H^\cdot(BG) \xrightarrow{\phi} H^\cdot(BG^\delta) \xleftarrow{\Psi} H^\cdot_b(BG^\delta)$

$\mathrm{Im}(\phi) < \mathrm{Im}(\Psi)$.

Example. $G = PSL_2\mathbb{R}$. $H^\cdot(BG) \simeq \mathbb{R}[e] \xrightarrow{\phi} H^\cdot(BG^\delta)$

$e^\delta := \phi(e) \neq 0$ (e.g. $\exists$ $PSL_2\mathbb{R}$-flat S^1-bundles w/ nonzero Euler class
eg $T^1S_g \to S_g$ $g \geq 2$)

Show e^δ bounded.
$\in H^2(PSL_2\mathbb{R})$

- e^δ corresp. to central ext $\{f:\mathbb{R}\to\mathbb{R}$ or. pres homeo $\mid f(x+1) = f(x)+1\}$ $= \widetilde{\mathrm{Homeo}}(S^1)$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \widetilde{\mathrm{Homeo}}(S^1) \arrow[r, shift left] & \mathrm{Homeo}(S^1) \arrow[r] \arrow[l, shift left, "\sim"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \widetilde{PSL_2\mathbb{R}} \arrow[r] \arrow[u] & PSL_2\mathbb{R} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

- cocycle rep: Choose set section $\widetilde{\ }$: $\mathrm{Homeo}(S^1) \longrightarrow \widetilde{\mathrm{Homeo}}(S^1)$
$f \longmapsto \tilde{f}$ lift of f s.t. $\tilde{f}(0) \in [0,1)$.

e^δ rep'd by $c(f,g) = \tilde{f}\circ\tilde{g}\circ(\widetilde{fg})^{-1}(0) \in \mathbb{Z}$.

Exercise c takes values $\{0,1\}$, i.e. $|c|_\infty = 1$.

$\Rightarrow$ e^δ bounded.

- $c'(f,g) = c(f,g) - \frac{1}{2}$ $c' \sim c$ cohomologous. $|c'|_\infty = \frac{1}{2}$.

$\longrightarrow$ $\|e^\delta\| \leq \frac{1}{2}$.

II. Dual norm & Milnor-Wood

<u>Defn</u>. (Gromov norm) $|\cdot|_1 : C_k(X;\mathbb{R}) \longrightarrow \mathbb{R}$.

$$a = \sum c_i \sigma_i \longmapsto \sum |c_i|.$$

$\rightsquigarrow$ seminorm on $H_k(X)$. $\quad \|z\| = \inf_{[a]=z} |a|_1$.

<u>Rmk</u>. • For $X = M^3$ $k=2$, closely related to Thurston norm.

~~Defn~~. • For $X = M^n$ closed, or. $\|M\| := \|[M]\|$ called simplicial volume.

<u>Thm</u> (Gromov-Thurston) M^n closed hyperbolic. Then.

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n} \qquad v_n = \text{volume of regular ideal } n\text{-simplex in } \mathbb{H}^n.$$

$\mathbb{H}^2$

<u>Rmk</u>. (Mostow rigidity)

M, N closed hyperbolic homotopy equiv $\iff$ isometric.

$\Rightarrow$ geometric invts are htpy invts.

<u>Cor</u> S_g closed, $g \geq 2$ $\quad \|S_g\| = \dfrac{\mathrm{Vol}(S_g)}{v_2} = \dfrac{-2\pi\chi(S_g)}{\pi} = -2\chi(S_g) = 4g-4$.

<u>Easier pf of upper bound</u>: $S_g =$

$4g$-gon.

$(4g-2)$ triangles.

$\rightsquigarrow$ rep for $[S_g]$ w/ norm $4g-2$. $\Rightarrow \|S_g\| \leq 4g-2$.

- $S_h \xrightarrow[d]{\deg} S_g$ $\quad [S_h]$ has rep w/ norm $4h-2$.

  $\rightsquigarrow$ rep of $d[S_g]$ w/ norm $4h-2$.

$$\Rightarrow \|S_g\| \le \frac{4h-2}{d}.$$

- $\chi(S_h) = d\,\chi(S_g) \Rightarrow h = d(g-1)+1.$

$$\Rightarrow \|S_g\| \le \frac{4[d(g-1)+1]-2}{d} = 4(g-1) + \frac{2}{d}.$$

As $d \to \infty$ get. $\|S_g\| \le -2\chi(S_g)$.

<u>Basic duality</u>. $c \in H^k(X)$ $\quad z \in H_k(X)$

$$|\langle c, z\rangle| \le \|c\|_\infty \|z\|_1$$

(if $\|c\| = \infty$ no info... no content.)

<u>Thm</u> (Milnor-Wood) $S^1 \to E \to S_{g \ge 1}$ G-flat, $G = PSL_2\mathbb{R}$ or $\mathrm{Homeo}(S^1)$ $\Rightarrow |e(E)| \le -\chi(S_g)$

<u>Pf</u>. $|e(E)| = |\langle e^b(E), S_g\rangle| \le \|e^b\| \cdot \|S_g\| \le \frac{1}{2} \cdot -2\chi(S_g) = -\chi(S_g)$

$\square$

<u>Next time</u>. Flat surface bundles.

Lecture 32

I. Flat surface bundles

Recall $\begin{matrix} S_g \to E \\ \downarrow \\ B \end{matrix}$ flat $\iff$

$$\begin{tikzcd} & BDiff(S_g)^\delta \sim K(Diff(S_g),1) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\exists \text{ lift}"] & BDiff(S_g) \sim K(Mod_g,1) \end{tikzcd}$$

$\iff$ $\exists$ lift

$$\begin{tikzcd} & Diff(S_g) \arrow[d] \\ \pi_1(B) \arrow[r] \arrow[ur, dashed] & Mod_g \end{tikzcd}$$

<u>Examples.</u>

(1) $\begin{matrix} S_g \to E \\ \downarrow \\ S_{h,p} \end{matrix}$ $p>0$ always flat since any hom

$$\begin{tikzcd} & Diff(S_g) \arrow[d] \\ F_k \arrow[r] \arrow[ur] & Mod_g \end{tikzcd}$$

lifts

(2) <u>Thm</u> (Kerckhoff, Nielsen realization) Every finite $G < Mod_g$ lifts to $Diff(S_g)$

- eg $G = \mathbb{Z}/n$: Thm says if $\phi \in Diff(S_g)$ and $\phi^n \sim id$ isotopic, then $\phi \sim \psi$ st. $\psi^n = id$.

- in fact Kerckhoff shows $\exists$ hyperbolic metric st. $G < Isom(S_g)$

- Thm $\Rightarrow$ every S_g bundle over B w/ $\pi_1(B)$ finite is flat.

(3) Every ~~surface bundle. S_g bun~~ $\begin{matrix} S_g \to E \\ \downarrow \\ T^k \end{matrix}$ flat.

(exercise in Nielsen Thurston classification)

Q Is every S_g bundle flat?

$$S \longrightarrow \frac{EDiff(S) \times S}{Diff(S)}$$
$$\downarrow$$
$$BDiff(S)$$

ie is the universal bundle flat?

ie does $Diff(S_g) \xrightarrow{\pi} Mod_g$ split?

$(\exists ?^{hom}\ \sigma : Mod_g \longrightarrow Diff(S_g)$ s.t. $\pi \circ \sigma = id)$

<u>Note</u>. yes for $g=1$: $\quad Diff(T^2) \longrightarrow Mod_1 \simeq SL_2\mathbb{Z}$

<u>Thm</u> (Morita nonlifting; Morita, Franks-Handel, Bestvina-Church-Souto)

$Diff(S_g) \longrightarrow Mod_g$ not split for $g \geq 2$.

<u>Rmk</u> (Morita's proof) As in S^1 bundle case can try to understand

$$H^\cdot(Mod(S)) \simeq H^\cdot(BDiff(S)) \xrightarrow{\phi} H^\cdot(BDiff(S)^\delta) \simeq H^\cdot(Diff(S))$$

Any $0 \neq c \in \ker(\phi)$ is obstruction to flat ~~cxsda~~ cnxn on S bundles.

Morita: $\quad e_3 \in \ker\left[H^6(BDiff(S_g)) \longrightarrow H^6(BDiff(S_g)^\delta)\right]$.

$$e_3\begin{pmatrix} S_g \to E \\ \downarrow \\ B \end{pmatrix} = \int_{S_g} e(T_\pi E)^4$$

$\Rightarrow \exists \quad \begin{matrix} S_g \to E \\ \downarrow \\ B^6 \end{matrix} \quad$ not flat

II. Flat connections & foliations

Recall. $\begin{matrix} S_g \to E \\ \downarrow \\ B \end{matrix}$ flat $\iff$ E has ~~foliation~~ k-dim'l foliation whose leaves are covering spaces of B.

(smooth)

(haven't explored foliation POV)

Defn. Fix $n = k+q$.

- A k-dim'l foliation $\mathcal{F}$ on M is integrable k-plane field on M, ie ~~TM~~ $E \subset TM$ rank-k subbundle s.t. $\forall X, Y \in \Gamma(E)$ $[X,Y] \in \Gamma(E)$.

- Equivalently (Frobenius) $\mathcal{F}$ is decomp.
$$M = \bigsqcup \left(\begin{matrix} \text{leaves: k-dim'l} \\ \text{immersed submflds} \end{matrix} \right)$$
w/ local model
$$\mathbb{R}^n = \mathbb{R}^{k+q} = \bigsqcup_{y \in \mathbb{R}^q} \mathbb{R}^k \times \{y\}$$

- Foliation cocycle $M = \bigcup U_\alpha$ atlas.

$U_\alpha \xrightarrow[y_\alpha]{} \mathbb{R}^q$ submersion

$$\begin{matrix} U_\alpha \cap U_\beta & \xrightarrow{y_\beta} & \mathbb{R}^q \\ \downarrow y_\alpha & \nearrow & \\ \mathbb{R}^q & \gamma_{\beta\alpha} & \end{matrix}$$
$\gamma_{\beta\alpha}$ local diffeo.

cocycle: $\gamma_{\delta\beta} \circ \gamma_{\beta\alpha} = \gamma_{\delta\alpha}$.

Example. Reeb foliation of S^3

periodic fol of $\mathbb{R} \times [0,1]$

$\rightsquigarrow$ per. fol of $\mathbb{R} \times D^2$

$\rightsquigarrow$ fol of $S^1 \times D^2$

$\rightsquigarrow$ fol. of $S^3 = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$

w/ one compact leaf: $S^1 \times S^1$.

Homotopy viewpoint on foliations

$n = k+q$.

- Warmup: when does M^n admit a k-plane field?

i.e. $TM \simeq \overset{\text{rank } k}{E_1} \oplus \overset{\text{rank } q}{E_2}$.

Recall. $\mathrm{Vect}_m(X) := \left\{ \begin{array}{c} \mathbb{R}^m \to E \\ \downarrow \\ X \end{array} \right\}/\text{iso} \simeq [X, BO(m)]$

$BO(m) \sim Gr_m \mathbb{R}^\infty$.

$\Rightarrow \exists$ of k-plane field is lifting prob:

$$\begin{tikzcd} & BO(k) \times BO(q) \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "\tau"', "\text{tangent bundle}"'] & BO(n) \end{tikzcd}$$

Want (phrase integrability as lifting prob).

(i) functor $\mathrm{Fol}_q : \mathrm{Top} \longrightarrow \mathrm{Set}$

$\mathrm{Fol}_q(X) = \left\{ \begin{array}{c} \text{codim-}q \\ \text{fol. on } X \end{array} \right\} / \text{htpy}$.

(ii) space $B\Gamma_q$ & natural isos $\mathrm{Fol}_q(X) \simeq [X, B\Gamma_q]$.

(iii) map $B\Gamma_q \longrightarrow BO(q)$ s.t.

$\exists$ lift

$$\begin{tikzcd} & BO(k) \times B\Gamma_q \arrow[d] \\ M \arrow[ur, dashed] \arrow[r] & BO(k) \times BO(q) \end{tikzcd}$$

precisely when the k-plane field is integrable

Problems

(1) "foliation" doesn't make sense if X not mfld.

(2) foliations don't pull back under cts maps.

Haefliger solution

- Haefliger cocycle: on space X ~~[illegible]~~. is cover $X = \bigcup U_\alpha$ w/

$$y_\alpha : U_\alpha \longrightarrow \mathbb{R}^q$$

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{y_\beta} & \mathbb{R}^q \\ {\scriptstyle y_\alpha}\downarrow & \nearrow & \\ \mathbb{R}^q & & \end{array} \quad \{\gamma^w_{\beta\alpha}\}_{w \in U_\alpha \cap U_\beta} \text{ germs of diffeos}$$

cocycle $\gamma^w_{\delta\beta} \circ \gamma^w_{\beta\alpha} = \gamma^w_{\delta\alpha}$ ~~[illegible]~~ $\forall w \in U_\alpha \cap U_\beta \cap U_\delta$.

- ~~Thm~~ $\mathcal{H}_q(X) = \left\{ \begin{matrix} \text{eq. classes} \\ \text{of H-cocycles} \end{matrix} \right\}$ "codim q fol. w/ singularities"

- <u>Thm</u> For Γ_q groupoid of germs of diffeos $\mathbb{R}^q \longrightarrow \mathbb{R}^q$.

  For any space X, $\mathcal{H}_q(X) \simeq [X, B\Gamma_q]$.

- <u>Thm</u>. (h-principle) For M open mfld.

$$\left\{ \text{liftings} \quad \begin{array}{ccc} & & BO(k) \times B\Gamma_q \\ & \nearrow & \downarrow \\ & & BO(k) \times BO(q) \\ & & \downarrow \\ M & \longrightarrow & BO(n) \end{array} \right\} \simeq \left\{ \begin{matrix} \text{integrable } k\text{-plane} \\ \text{field on } M \end{matrix} \right\}.$$

$\Rightarrow$. $H^\cdot(BO(q)) \longrightarrow H^\cdot(B\Gamma_q)$ gives obstructions for plane field being integrable.

<u>Thm</u> (Haefliger) no obstructions in low deg: $B\Gamma_q \longrightarrow BO(q)$ $\pi_i \to$ iso $i \leq q$.

<u>Thm</u> (Bott vanishing) obstr. in high deg: $n = k + q$. $TM = E^k \oplus \nu^q$

E^k [illegible] $\Rightarrow \langle p_1(\nu) \dots p_{[q/2]}(\nu) \rangle \subset H^\cdot(M)$ vanishes in deg $> 2q$

Lecture 33

I. Bott vanishing

M^n mfld, $E^k \subset TM$ k-plane distribution, $n = k+q$.

Q Can E be homotoped to an integrable distribution?

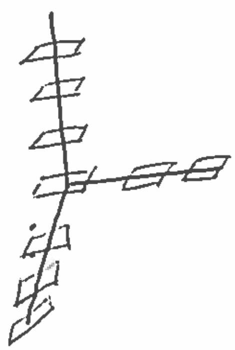
Thm (Haefliger) ~~[illegible]~~. M^{k+q} open mfld w/ homotopy type of $(q+1)$ complex, then every k-plane distribution on M homotopic to integrable one

Example $M = \mathbb{R}^3$ $\quad H(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$.

plane field on $H(\mathbb{R}) \cong \mathbb{R}^3$:

$$E_{(0,0,0)} = \mathbb{R}\{e_1, e_2\}.$$

$$E_{(x,y,z)} = \begin{pmatrix} \text{Left-mult} \\ \text{by } \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \end{pmatrix}_* E_{(0,0,0)}.$$



$$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 1 & t \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

E not integrable but is homotopic to ~~integrable~~ Constant plane field

Thm (Bott vanishing) ~~$E^k \subset TM$~~ $TM \simeq E^k \oplus Q^q$.

$$E \text{ integrable} \Rightarrow \langle p_1(Q), \dots, p_{\lfloor q/2 \rfloor}(Q) \rangle_{\deg i} = 0 \quad \text{for } i > 2q.$$

Rmk. For Γ_q = groupoid of germs of diffeos $\mathbb{R}^q \to \mathbb{R}^q$

$\Gamma_q \to O(q)$ derivative $\rightsquigarrow$ $B\Gamma_q \longrightarrow BO(q)$ Haefliger str $\longmapsto$ normal bundle.

Then $H^i(BO(q); \mathbb{Q}) \longrightarrow H^i(B\Gamma_q; \mathbb{Q})$ zero for $i > 2q$

Examples

(1) $q=1$. Gives No obstruction. $H^i(BO(1);\mathbb{Q}) = H^i(\mathbb{RP}^\infty;\mathbb{Q}) = 0 \quad i>0$

Thm (Thurston) Every $(n-1)$-plane field on M^n homotopic to integrable one. (don't need M open!)

Cor. M^n closed. M^n has codim-1 foliation $\Leftrightarrow \chi(M)=0$.

Pf: codim 1-fol $\Leftrightarrow$ $(n-1)$ plane field. $\Leftrightarrow$ line field $\Leftrightarrow$ $\chi(M)=0$.

($\hat{M}\to M$ has or. line field $\rightsquigarrow$ nonvanishing vector field.

(2) $q=2$. Recall $H^\cdot(BO(2)) \simeq \mathbb{Q}[p_1]$.
$H^\cdot(BSO(2)) \simeq \mathbb{Q}[p_1,e]/(e^2=p_1)$

If $TM^n = E^{n-2}\oplus Q^2$ then $p_1(Q)^i = 0$ in $H^{4i}(M)$ for $4i>2q=4$
i.e. $i\geq 2$.

If Q orientable, the $e(Q)^{2i}=0$ for $i\geq 2$.

(3) Non integrable plane field on $M=\mathbb{CP}^4\times\mathbb{T}^2$.

Claim $\exists$ $TM = E^8\oplus Q^2$ with $p_1(Q)^2\neq 0$.

Note: • $TM \simeq T\mathbb{CP}^4 \oplus T\mathbb{T}^2$ $\qquad T\mathbb{T}^2 \simeq \mathbb{R}^2\times\mathbb{T}^2$ trivial.

• M is complex mfd. $TM \simeq T\mathbb{CP}^4\oplus\underline{\mathbb{C}}$.

Recall $T\mathbb{CP}^4 \simeq \mathrm{Hom}(\gamma,\gamma^\perp)$ where $\begin{array}{c}\mathbb{C}\to\gamma\\ \downarrow\\ \mathbb{CP}^4\end{array}$ tautological line bdle

$\mathbb{C}^4 \to \gamma^\perp \to \mathbb{CP}^4$ hyperplane bundle.

$$\Rightarrow T\mathbb{CP}^4 \oplus \mathbb{C} \simeq \mathrm{Hom}(\gamma,\gamma^\perp) \oplus \mathrm{Hom}(\gamma,\gamma)$$
$$\simeq \mathrm{Hom}(\gamma,\mathbb{C}^5) \simeq \mathrm{Hom}(\gamma,\mathbb{C})^{\oplus 5} \simeq (\gamma^*)^{\oplus 5}$$

- $E := (\gamma^*)^{\oplus 4}$ $\quad Q := (\gamma^*)$ $\quad c_1(Q) = a$ $\quad H^*(\mathbb{CP}^4) = \frac{\mathbb{Q}[a]}{(a^5)}$

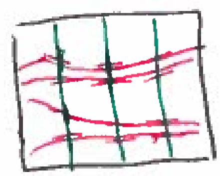
$$\Rightarrow p_1(Q) = \pm a^2 \quad \Rightarrow p_1(Q)^2 = a^4 \neq 0.$$
in $H^8(\mathbb{CP}^4) \subset H^8(M)$.

$\Rightarrow$ E not integrable.

(4) Surface bundles.

$$\begin{array}{c} S_g \to M \\ \downarrow \\ B \end{array} \text{ flat} \Rightarrow TM \simeq E \oplus T_\pi M \quad \text{w/ } E \text{ integrable.}$$

(ie normal bundle is vertical tangentbundle)



$\Rightarrow$ ~~$p_1(T_\pi M)$~~
$$e(T_\pi M)^{2k} = p_1(T_\pi M)^k = 0 \quad k \geq 2.$$
$$\Rightarrow e_{2k-1} = \int_{S_g} e(T_\pi M)^{2k} = 0 \text{ in } H^{4k-2}(B) \quad k \geq 2.$$

(Harer, Madsen-Weiss) $e_3 \in H^6(\mathrm{Mod}_g;\mathbb{Q})$ nonzero for $g \geq 10$.

~~$\Rightarrow H^*(B\mathrm{Diff}(S_g);\mathbb{Q}) \to H^*(B\mathrm{Diff}($~~

$$\Rightarrow \exists \begin{array}{c} S_g \to E \\ \downarrow \\ B^6 \end{array} \text{ not flat for } g \geq 10.$$

<u>Thm</u> (Morita) $\mathrm{Diff}(S_g) \to \mathrm{Mod}_g$ not split for $g \gg 0$.

Pf. If split, then every surface

If split then $H^*(B\mathrm{Diff}(S_g)) \to H^*(B\mathrm{Diff}(S_g)^\delta)$ injective

II. Proof of Bott vanishing.

Connections, curvature, Pontryagin classes

- Fix $\mathbb{R}^q \longrightarrow Q \longrightarrow M$ vector bundle.
- Horizontal distribution $\rightsquigarrow$ parallel transport $\leftrightsquigarrow$ connection.

  connection $\nabla : \Gamma(Q) \longrightarrow \Gamma(\mathrm{Hom}(TM, Q))$.

  equivalently for $X \in \Gamma(TM)$ $\quad \nabla_X : \Gamma(Q) \longrightarrow \Gamma(Q)$.

  defined using parallel transport: For $\gamma : [0,1] \longrightarrow M$ trajectory of X

  P_γ parallel transport along γ back to $\gamma(0)$

$$(\nabla_X s)_{\gamma(0)} = \lim_{t \to 0} \frac{P_\gamma s(\gamma(t)) - s(\gamma(0))}{t}$$

- curvature: $X, Y \in \Gamma(TM)$ $\quad s \in \Gamma(Q)$.

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

  failure of parallel transport along X, Y to commute.

  locally R given by $\Omega = (\omega_{ij})$ matrix of 2-forms on M.

- Pontryagin classes $p_j(Q)$ expressed as polys in $\mathrm{tr}(\Omega^i)$.

Idea of Bott's Thm

Suppose $TM \cong E \oplus Q$ E integrable

(1) parallel transport using foliation $\mathcal{F}$

$U_\alpha \xrightarrow{y_\alpha} \mathbb{R}^q$.

– Define parallel transport of vectors in Q along paths tangent to $\mathcal{F}$.

Pick any connection $\tilde{\nabla}$ on TM.

For $X \in \Gamma(TM)$ $s \in \Gamma(Q)$ define $X = X_E + X_Q$.

$$\nabla_X s = \tilde{\nabla}_{X_Q} s + [X_E, s]_Q .$$

Note. $X \in \Gamma(E) \Rightarrow \nabla_X s = [X_E, s]_Q$.

Check. If $X, X' \in \Gamma(E)$ $s \in \Gamma(Q)$ then $R(X,X')s = 0$.

(EX) ie. ∇ flat in directions along the foliation.

(2) locally $\Omega = (\omega_{ij})$ contained in ideal

$I = \langle\langle dy_1, \ldots, dy_q \rangle\rangle$ where $E = \ker dy_1 \cap \cdots \cap \ker dy_q$.

$\Rightarrow \omega_{ij} = \sum \alpha_k \wedge dy_k$. for some 1-forms α_k.

$\Rightarrow \Omega^{q+1} = 0$ since any wedge prod. of w_{ij}'s of length $q+1$ is zero.

$\Rightarrow$ polys in $\mathrm{tr}(\Omega^i)$ vanish in deg $> q$ $\square$.

Lecture 34

I. Construction of nonflat surface bundles.

Last time • If $S_g \to E \to B$ has $e_3(E) \neq 0 \in H^6(B;\mathbb{Q})$, then $E \to B$ not flat. (Bott vanishing)

• (Harer stability, Madsen-Weiss) $e_3 \neq 0 \in H^6(\mathrm{Mod}_g;\mathbb{Q})$ for $g \geq 10$.

$\Rightarrow$ nonflat bundles exist, but doesn't give construction.

Morita m-construction (sketch)

Combines 3 operations

(i) Given $S_g \to E \xrightarrow{\pi} M$

$$\begin{tikzcd} E^* \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd}$$

$$E^* = \{(u,v) \in E \times E \mid \pi(u) = \pi(v)\}$$

$s: E \longrightarrow E^*$ "diagonal" section.
$u \longmapsto (u,u)$

(ii) fiberwise m-fold cover. Given $S_g \to E \to M$, $S_h \xrightarrow[\text{cover}]{\mathbb{Z}/m} S_g$

Construct.

$$\begin{tikzcd} S_h \arrow[d] & S_g \arrow[d] & S_g \arrow[d] \\ E' \arrow[r, "\mathbb{Z}/m \text{ cover}"', "\text{fiberwise}"] \arrow[d] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ M' \arrow[r, equal] & M' \arrow[r, "f", "\text{cover}"'] & M \end{tikzcd}$$

~~Note~~ • If $E \to M$ has section, can ~~assum~~ arrange for $E' \to M'$ to have m disjoint sections.

(iii) ~~Given~~ fiberwise branched cover.

Given $S_g \to E \to M$, $s_1,\dots,s_m$ disjoint sections, $S_h \to S_g$ $\mathbb{Z}/m$ branched cover branched at m point.

$$\begin{tikzcd}
S_h \arrow[d] & S_g \arrow[d] & S_g \arrow[d] \\
E' \arrow[r, "\text{fiberwise } \mathbb{Z}/m"', "\text{branched cover.}"] \arrow[d] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
M' \arrow[r, equal] & M' \arrow[r, "f", "\text{cover}"'] & M
\end{tikzcd}$$

branched over $f^*s_1,\dots,f^*s_m : M' \to f^*E$.

<u>Example</u>. Start w/ $E = S_g \to M = *$

- (i) gives

$$\begin{tikzcd}
S_g \times S_g \arrow[r] \arrow[d] & S_g \arrow[d] \\
S_g \arrow[u, bend left, "\Delta"] \arrow[r] & *
\end{tikzcd}$$

- (ii) ~~Show~~ Let $g=2$. Take $S_3 \to S_2$ $\mathbb{Z}/2$ cover. Can take.

$$\begin{tikzcd}
S_3 \times S_3 \arrow[r] \arrow[d] & S_2 \times S_3 \arrow[r] \arrow[d] & S_2 \times S_2 \arrow[d] \\
S_3 \arrow[u, bend left, "\Gamma_{id}, \Gamma_\tau"] \arrow[r, "\text{multi-section.}"] & S_2 \arrow[u, bend left] \arrow[r] & S_2
\end{tikzcd}$$

$\tau : S_3 \to S_3$ deck trans.

- (iii) Atiyah-Kodaira bundle

$$\begin{tikzcd}
S_6 \arrow[d] & S_3 \arrow[d] & S_3 \arrow[d] \\
E \arrow[r] \arrow[d] & S_3 \times S_{129} \arrow[r] \arrow[d] & S_3 \times S_3 \arrow[d] \\
S_{129} \arrow[r, equal] & S_{129} \arrow[r, "\mathbb{Z}/2 \text{ homology cover}"'] & S_3
\end{tikzcd}$$

$e_1(E) \doteq \mathrm{sig}(E) \neq 0$.

Operations (i)-(iii) allow one to ~~Gen~~ iterate this procedure.

<u>Thm</u> (Morita) This construction produces surface bundles w/ $e_i \neq 0$ for $i \geq 1$. In particular produces nonflat bundles.

II (non)Flat surface bundles over surfaces

<u>Q</u>: $\exists$? $\begin{array}{c} S_g \to E \\ \downarrow \\ S_h \end{array}$ not flat?

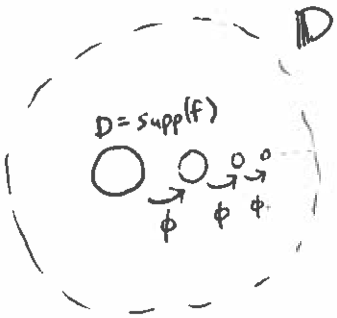
Source of difficulty:

<u>Thm</u> (Mather, Thurston) $\mathrm{Diff}_{0,c}(M)$ = compactly supported diffeos isotopic to the identity is a simple group. (no nontrivial normal subgroup)

$$\Rightarrow \quad 0 = (\mathrm{Diff}_{0,c} M)^{ab} = H_1(\mathrm{Diff}_{0,c} M)$$

<u>Toy case</u> $\mathbb{D} \subset \mathbb{R}^2$ open unit disk. $\mathrm{Homeo}_c(\mathbb{D})$ is perfect.

<u>Proof</u>: Fix $f \in \mathrm{Homeo}_c(\mathbb{D})$. Must show f is product of commutators.



$$\mathrm{supp}(\phi^i f \phi^{-i}) = \phi^i(D).$$

$$\prod_{i=0}^{\infty} \phi^i f \phi^{-i} \in \mathrm{Homeo}_c(\mathbb{D})$$

$$\underbrace{\phi\Big(\prod_{i=0}^{\infty} \phi^i f \phi^{-i}\Big)\phi^{-1}}_{\phi g f \phi^{-1}} = \underbrace{\prod_{i=1}^{\infty} \phi^i f \phi^{-i}}_{g}$$

$$\phi g f \phi^{-1} = g$$

$$\cancel{\Rightarrow f = g^{-1}\phi g \phi^{-1} = [g^{-1}, \phi]} \qquad \Rightarrow \quad f = [g^{-1}, \phi^{-1}]. \qquad \square$$

Q: Is e_1 zero for flat $S_g \to E \to S_n$?

SES $\quad 1 \to \mathrm{Diff}_0(S_g) \to \mathrm{Diff}(S_g) \to \mathrm{Mod}_g \to 1$

$\rightsquigarrow$ 5 term SES

$$0 \to H^1(\mathrm{Mod}_g) \to H^1(\mathrm{Diff}(S_g)) \to \underbrace{H^1(\mathrm{Diff}_0(S_g))^{\mathrm{Mod}_g}}_{=0} \to \underset{\langle e_1 \rangle}{\overset{}{H^2(\mathrm{Mod}_g)}} \hookrightarrow H^2(\mathrm{Diff}(S_g))$$

$\Rightarrow$ ~~that~~ $e_1 \neq 0 \in H^2(\mathrm{Diff}(S_g)) \quad (g \gg 0)$

$\Rightarrow \exists \; S_g \to E \to S_n$ flat w/ $e_1(E) \neq 0$ for $g \gg 0$.

Explicit construction

Prop For any $S_g \to E \to S_n \quad \exists \; S_{n+N} \xrightarrow{f} S_n$ so that (i) $\mathrm{sig}(E) = \mathrm{sig}(f^*E)$ (ii) $f^*E \to S_{n+N}$ flat

Proof. ~~and~~ Consider monodromy $\pi_1(S_n) \longrightarrow \mathrm{Mod}_g$ (lifting to $\mathrm{Diff}(S_g) \to \mathrm{Mod}_g$)

$a_1, b_1, \dots, a_n, b_n$, $\prod [a_i, b_i] = 1$.

Choose any lifts $\tilde a_1, \tilde b_1, \dots, \tilde a_n, \tilde b_n \in \mathrm{Diff}(S_g)$

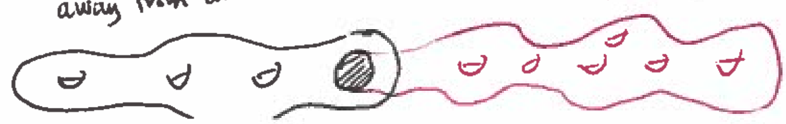
Then $f = \prod [\tilde a_i, \tilde b_i] \in \mathrm{Diff}_0(S_g) \underset{\mathrm{Diff}_0 \text{ perfect}}{\Rightarrow} f = \prod_{j=1}^{N} [\phi_j, \varphi_j]$

$\phi_j, \varphi_j \in \mathrm{Diff}_0(S_g)$

$$\begin{tikzcd} & \mathrm{Diff}(S_g) \arrow[d] \\ \pi_1(S_{n+N}) \arrow[r] \arrow[ur, dashed] & \mathrm{Mod}_g \end{tikzcd}$$

∎

On bundle level. $E \to S$. put flat cnxn away from disk... nontrivial foliation... $S_g \times S_N \to S_N$



signature adds, over S_N have trivial bundle

Rmk being not flat is not robust for $\begin{matrix} S_g \to E \\ \downarrow \\ S_h \end{matrix}$

III. Nonflat surface bundles ~~over~~ w/ section.

~~Given~~ $\begin{tikzcd} S_g \arrow[r] & E \arrow[d, shift right] \\ & B \arrow[u, shift right, "\sigma"'] \end{tikzcd}$ $\exists$ section $\iff$ $\begin{tikzcd} & \mathrm{Mod}_{g,*} \arrow[d] \\ \pi_1(B) \arrow[r] \arrow[ur, dashed] & \mathrm{Mod}_g \end{tikzcd}$

For bundles w/ section can ask $\begin{tikzcd} & \mathrm{Diff}(S_g,*) \arrow[d] \\ \pi_1(B) \arrow[r] \arrow[ur, dashed, "\exists ?"] & \mathrm{Mod}_{g,*} \end{tikzcd}$

If lift exists $E \to B$ has flat cnxn where section is <u>parallel</u>

ie E has horizontal foliation where $\sigma(B) \subset E$ is one of the leaves

Example. Birman exact seq. $1 \to \pi_1(S_g) \xrightarrow{P} \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g \to 1$.

$P: \pi_1(S_g) \to \mathrm{Mod}_{g,*}$ is monodromy of $\begin{matrix} S_g \times S_g \\ \downarrow \\ S_g \end{matrix}$ wrt diagonal section $\sigma(x) = (x,x)$

Thm (Bestvina–Church–Souto) For $g \geq 2$ $\nexists$ lift $\begin{tikzcd} & \mathrm{Diff}(S_g,*) \arrow[d] \\ \pi_1(S_g) \arrow[r] \arrow[ur, dashed] & \mathrm{Mod}_{g,*} \end{tikzcd}$

Cor Although $S_g \times S_g \to S_g$ has ~~a~~ flat connection,

it has no flat cnxn where $\Delta \subset S_g \times S_g$ is parallel.

Lecture 35

I. Lifting problem for point-pushing subgroup.

<u>Thm</u> (Bestvina - Church - Souto) S_g closed

$$g \geq 2 \implies \nexists \text{ lift } \quad \begin{tikzcd} & \mathrm{Diff}(S_g,*) \arrow[d] \\ \pi_1(S_g) \arrow[r] \arrow[ur, dashed] & \mathrm{Mod}_{g,*} \end{tikzcd} \qquad \pi_1(S_g) = \ker\begin{pmatrix} \mathrm{Mod}_{g,*} \\ \downarrow \\ \mathrm{Mod}_g \end{pmatrix}$$

$\curvearrowleft$ point-pushing.

<u>Corollaries</u>

(1) $S_g \times S_g \to S_g$ has no flat cnxn where $\Delta : S_g \to S_g \times S_g$ parallel

(2) $\nexists$ lift

$$\begin{tikzcd} & \mathrm{Diff}(S_g, n\,\mathrm{pts}) \arrow[d] \\ B_n(S_g) \arrow[r] \arrow[ur, dashed] & \mathrm{Mod}_{g,n} \end{tikzcd} \qquad g \geq 2,\ n \geq 1.$$

where $B_n(S) = \pi_1(\mathrm{Conf}_n(S))$ is $\ker[\mathrm{Mod}_{g,n} \to \mathrm{Mod}_g]$.

"multi-point-pushing"

(3) $\mathrm{Diff}(S_g) \to \mathrm{Mod}_g$ not split for $g \geq 8$.

(4) Atiyah - Kodaira bundle $\quad \begin{matrix} S_g \to & E \circlearrowleft \mathbb{Z}/m \\ & \downarrow \\ & S_h \end{matrix} \quad$ has no flat cnxn invariant under $\mathbb{Z}/m$ action $m \geq 3$.

<u>Pf sketch of Thm</u>

• Recall <u>Thm</u> (Milnor, Wood) $\quad \begin{matrix} S^1 \to & V \\ & \downarrow \\ & S_g \end{matrix} \quad$ circle bundle. $g \geq 1$.

(i) if $\mathrm{Homeo}\,S^1$ flat, then $|e(V)| \leq 2g-2$.

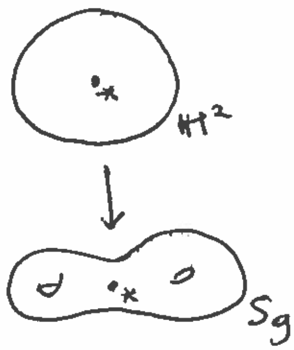
(ii) if $GL_2^+\mathbb{R}$-flat, then $|e(V)| \leq g-1$.

$$\begin{tikzcd} & \mathrm{Diff}(S_g,*) \arrow[r] \arrow[d] & \mathrm{Diff}(\mathbb{H}^2,*) \\ \pi_1(S_g) \arrow[ur, dashed, "\tilde{P}"] \arrow[r, "P"'] & \mathrm{Mod}_{g,*} \arrow[r] & \mathrm{Homeo}(\partial\mathbb{H}^2) \end{tikzcd}$$

- Suppose lift ~~existed~~ exists

$\pi_1(S_g) \curvearrowright$ $\mathbb{H}^2$

$\pi_1(S_g) \curvearrowright$ S_g



Two circle actions

$$\begin{tikzcd} & PSL_2\mathbb{R} \arrow[dr] & \\ \rho_0 : \pi_1(S_g) \arrow[ur] \arrow[dr] & & \mathrm{Homeo}(S^1) \\ \rho_1 & GL_2^+\mathbb{R} \arrow[ur] & \end{tikzcd}$$

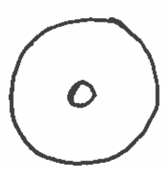
$\rightsquigarrow$ two flat bundles $S^1 \to E_0 \to S_g$

$S^1 \to E_1 \to S_g$

- $E_0 \simeq T^1 S_g \Rightarrow e(E_0) = \chi(S_g) = 2-2g.$
- E_1 is $GL_2^+\mathbb{R}$ flat $\Rightarrow |e(E_1)| \leq g-1.$

OTOH $e(E_0) = e(E_1)$ b/c E_0, E_1 are fiberwise bordant:

$\pi_1(S_g) \curvearrowright$



$Bl_* \mathbb{H}^2 \simeq S^1 \times [0,1]$ $\rightsquigarrow$

$$\begin{tikzcd} S^1\times\{1\} \arrow[r] & E_1 \arrow[r] \arrow[d] & S_g \arrow[d, equal] \\ S^1\times[0,1] \arrow[r] & E \arrow[r] & S_g \\ S^1\times\{0\} \arrow[r] & E_0 \arrow[r] \arrow[u] & S_g \arrow[u, equal] \end{tikzcd}$$

$\cancel{\times}$

$\square$

II. Lifting problem for braid groups

BCS: $\nexists$ lift

$$\begin{tikzcd} & \mathrm{Diff}(S_g, n\text{ pts}) \arrow[d] \\ B_n(S_g) \arrow[ur, dashed] \arrow[r] & \mathrm{Mod}_{g,n} \end{tikzcd}$$

S_g closed

$g \geq 2$, $n \geq 1$.

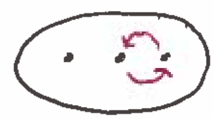
<u>Q</u>: What about $g = 0, 1$ or $\partial S \neq \phi$?

Ex

$$\begin{array}{c} \mathrm{Diff}(\mathbb{D}, n\,\text{pts}) \\ \downarrow \ \ \overset{?}{\uparrow} \\ B_n = B_n(\mathbb{D}) \simeq \mathrm{Mod}(\mathbb{D}, n\,\text{pts}) \end{array}$$

Does this split?

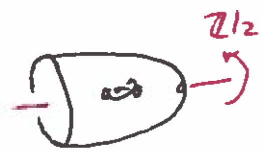
e.g. for $n=3$ $\quad B_3 = \langle \sigma, \tau \mid \sigma\tau\sigma = \tau\sigma\tau \rangle$

as mapping classes σ:  τ: 

Q $\exists$? $f, g \in \mathrm{Diff}(\mathbb{D}, 3\,\text{pts})$ s.t. (i) $[f] = \sigma$, $[g] = \tau$ $\in \mathrm{Mod}(\mathbb{D}, 3\,\text{pts})$

(ii) $fgf = gfg$?

A (Thurston) yes.

Pf: • $SL_2\mathbb{Z} \curvearrowright$  $\rightsquigarrow$ $SL_2\mathbb{Z} \curvearrowright$  $\rightsquigarrow$ $PSL_2\mathbb{Z} \curvearrowright$ 

wrong group, doesn't fix ∂.

$$1 \to \mathbb{Z} \to \underset{\langle u,v \mid u^2 = v^3\rangle}{\overset{\shortparallel}{B_3}} \to \underset{\langle x,y \mid x^2 = 1 = y^3\rangle}{\overset{\shortparallel}{PSL_2\mathbb{Z}}} \to 1.$$

• homotope action of x, y on ∂ (···) to id, preserving relation $x^2 = y^3$.

so $x|_\partial$ & $y|_\partial$ commute.

- homotope $y|_\partial$ through order-3 rots

- homotope $x|_\partial$ & $y|_\partial$ to id in $SO(2)$ preserving $x^2 = y^3$.

$\square$.

$\rightsquigarrow$ action $B_3 \curvearrowright$  giving a lift

Thm (Nariman, 15) $\mathrm{Diff}(\mathbb{D}^2 \setminus n\,\text{pts}) \longrightarrow \underset{\mathrm{Mod}(\mathbb{D}^2, n\,\text{pts})}{\overset{\shortparallel}{B_n}}$ splits cohomologically.

(unlike higher genus case)

<u>Thm</u> (Salter-T) $\mathrm{Diff}(D^2, n\text{pts}) \to \mathrm{Mod}(D^2, n\text{pts})$ not split for $n \geq 5$.

The obstruction:

<u>Thm</u> (Thurston stability) M mfld, $p \in M$. The group

$$\mathrm{Diff}(M, T_pM) = \left\{ f \in \mathrm{Diff}(M) \;\middle|\; \begin{array}{l} f(p) = p \\ df_p = id \end{array} \right\}$$

is <u>locally indicable</u> ie. $\forall$ f.g. $\Gamma < \mathrm{Diff}(M, T_pM)$ $\exists$ surj. $\Gamma \twoheadrightarrow \mathbb{Z}$.

E.g. A locally indizable group doesn't contain a f.g. perfect subgroup $\Gamma = [\Gamma, \Gamma]$

Idea: If $B_n \curvearrowright (\mathbb{D}, n\text{pts})$ then Show: $\mathrm{Diff}(\mathbb{D}, T_p\mathbb{D})$ contains perfect subgroup.

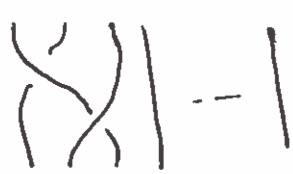
<u>Fact</u> B_n not perfect $[B_n, B_n] \longrightarrow \underset{\langle \sigma_1, \ldots, \sigma_{n-1} \rangle}{\overset{\|}{B_n}} \longrightarrow \mathbb{Z}$.

$\sigma_i \longmapsto 1.$

hom since all relations have zero total exponent sum.

$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$

- eg $\sigma_1 =$  not a ~~commutator~~ commutator.

- but $\sigma_1 \sigma_2^{-1} =$  is commutator

$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \Rightarrow \sigma_1 \sigma_2^{-1} = \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 = [\sigma_2^{-1}, \sigma_1^{-1}].$

Moreover if $n \geq 4$ have $\sigma_4 \in B_n$ commutes w/ σ_1, σ_2 so

$$\sigma_1 \sigma_2^{-1} = [\sigma_4 \sigma_2^{-1}, \sigma_4 \sigma_1^{-1}]$$ commutator of commutators.

<u>Thm</u> (Gorin-Lin) For $n \geq 5$ $[B_n, B_n]$ is f.g. perfect group.

$[B_n, B_n] \quad [[B_n, B_n], [B_n, B_n]]$

<u>Rmk</u>. False for $n=3,4$. $\quad B_4 \twoheadrightarrow B_3 \twoheadrightarrow PSL_2\mathbb{Z} \twoheadrightarrow \mathbb{Z}$.

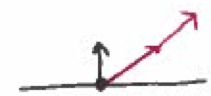
- Finding $[B_n,B_n] < \mathrm{Diff}(D, T_pD)$

$$GL_2\mathbb{R} = \mathrm{Aut}(T_pD)$$

if $B_n \curvearrowright (D, n\text{ pts})$ choose $p \in \partial D$.

$$B_n \longrightarrow \mathrm{Aut}(T_pD)$$
$$\tau \longmapsto \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

$[B_n,B_n]$ perfect $\Rightarrow$ $[B_n,B_n] \to GL_2\mathbb{R} \xrightarrow{\det} \mathbb{R}$ trivial.

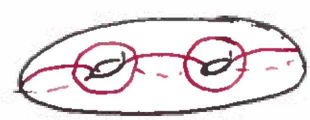


$\rightsquigarrow$ $[B_n,B_n] \longrightarrow GL_2\mathbb{R}$

$\tau \longmapsto \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \simeq \mathbb{R}$ again trivial. $\Rightarrow$ $[B_n,B_n] < \mathrm{Diff}(D, T_pD)$

<u>Cor of Proof</u> $\mathrm{Diff}(S_g) \longrightarrow \mathrm{Mod}_g$ not split for $g \geq 2$.

idea: τ:  hyperelliptic.



$B_6 \longrightarrow C(\tau) < \mathrm{Mod}_g$ Centralizer.

$\uparrow$ obstruction to lifting.