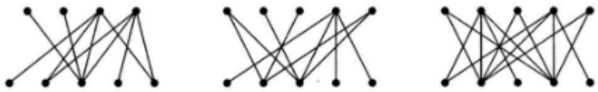
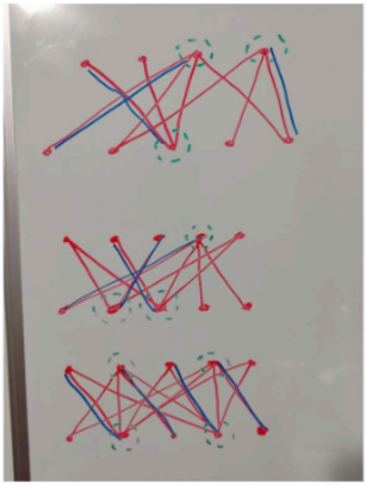
Problem 1. *Find a maximum matching in each graph below. Prove that it is maximum using the dual problem.*



Solution. 

The size of the maximum matchings I selected is 3, 3, and 4 respectively. We know that the size of a matching is upper-bounded by the size of any vertex cover, so if we provide a vertex cover of equivalent size, these matchings must be maximum. The vertex covers are defined by the vertices with dotted green circles.

□

Problem 2. *Let $G = (V, E)$ be a bipartite graph with maximum vertex degree Δ.*

(a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.

(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution.

(a) Let Q be a minimal vertex cover of G. Each $q \in Q$ is contained in no more than Δ unique edges in G, so since every $e \in E$ is incident to some $q \in Q$, this implies that $|Q|\Delta \geq |E|$ and $|Q| \geq \frac{|E|}{\Delta}$. Let M be a maximum matching of G; since Q was minimal, by König's theorem, $|M| \geq \frac{|E|}{\Delta}$ also.

(b) Let $G = (V, E)$ be subgraph of $K_{n,n}$ and Δ be the maximum degree of a vertex in G. Since each vertex has degree n in the complete bipartite graph $K_{n,n}$, we can bound $\Delta \leq n$. Then by (a), G has a matching M such that

$$|M| \geq \frac{|E|}{\Delta} \geq \frac{|E|}{n} > \frac{(k-1)n}{n} = k - 1,$$

and M has at least k edges.

□

Problem 3. *Use König's theorem to prove Hall's theorem: if* $G = (X \sqcup Y, E)$ *is bipartite and* $|S| \leq |N(S)|$ *for each* $S \subset X$*, then* G *has a matching that saturates* X*.* [1]

Solution. Suppose no matching saturates X, so any maximum matching M satisfies $|M| < |X|$. Let C be a minimal vertex cover, so $|C| < |M|$. Let $S = X \setminus (X \cap C)$ be the set of those vertices in X which are not in C. Then

$$|C| = |C \cap X| + |C \cap Y| < |X| \implies |C \cap Y| < |X| - |C \cap X| = |S|$$

A vertex v in S must be connected to a vertex in C since C is a vertex cover and $v \notin U$, so $N(S) \subset U \cap B$ and thus $|N(S)| \leq |C \cap Y|$. This implies $|N(S)| \leq |C \cap Y| < |S|$. By contrapositive, if $|S| \leq |N(S)|$ for each $S \subset X$, there must be a matching which saturates X. □

[1]Hint: Prove the contrapositive. Consider a minimal vertex cover Q. Find a set $S \subset X$ so that $|S| > |N(S)|$. (There are really only two options for S in terms of Q). The proof should be relatively short.

Problem 4. *Give an example of a stable matching problem of a graph* $G = (X \sqcup Y, E)$ *with* $|X| = |Y| = 2$ *in which there is more than one stable matching.*

Solution. Define $x, x' \in X$ and $y, y' \in Y$.

Now we say:

- $x >_{y'} x'$
- $y' >_{x'} y$
- $x' >_y x$
- $y >_x y'$

This results in a graph with two stable matchings, which are shown below. This is because a second and first choices prefer each other. So there is no scenario where two vertices in G both prefer each other to who they are already paired with.

□

Problem 5. *Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.*

Cats $\{u, v, w, x, y, z\}$	Giraffes $\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

Solution

We will begin with cats proposing. I will represent a proposal with a right arrow $\to$, and I will represent a matched pair with a left/right arrow $\leftrightarrow$. Below are the proposal steps:

1. $u \to a, v \to a, w \to c, x \to c, y \to c, z \to d$. This leaves the following sets paired: $a \leftrightarrow u, c \leftrightarrow x, d \leftrightarrow z$.
2. $v \to b, w \to b, y \to d$. This leaves the following sets paired: $a \leftrightarrow u, b \leftrightarrow w, c \leftrightarrow x, d \leftrightarrow y$.
3. $v \to c, z \to e$. This leaves the following sets paired: $a \leftrightarrow u, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z$.
4. $x \to a$. This leaves the following sets paired: $a \leftrightarrow x, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z$.
5. $u \to b$. This leaves the following sets paired: $a \leftrightarrow x, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z$.
6. $u \to d$. This leaves the following sets paired: $a \leftrightarrow x, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z$.
7. $u \to c$. This leaves the following sets paired: $a \leftrightarrow x, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z$.
8. $u \to f$. This leaves the following sets paired: $\boxed{a \leftrightarrow x, b \leftrightarrow w, c \leftrightarrow v, d \leftrightarrow y, e \leftrightarrow z, f \leftrightarrow u}$. Everyone is matched and the algorithm terminates here.

Below is the result of giraffes proposing:

1. $a \to z, b \to y, c \to v, d \to w, e \to u, f \to u$. This leaves the following sets paired: $a \leftrightarrow z, b \leftrightarrow y, c \leftrightarrow v, d \leftrightarrow w, f \leftrightarrow u$.
2. $e \to v$. This leaves the following sets paired: $a \leftrightarrow z, b \leftrightarrow y, c \leftrightarrow v, d \leftrightarrow w, f \leftrightarrow u$.
3. $e \to x$. This leaves the following sets paired: $\boxed{a \leftrightarrow z, b \leftrightarrow y, c \leftrightarrow v, d \leftrightarrow w, e \leftrightarrow x, f \leftrightarrow u}$. Everyone is matched and the algorithm terminates here.

□