Problem 1. *Prove or disprove: every tree T has at most one perfect matching.*

Solution. To show this, let M and M' be perfect matchings of a tree $T = (V, E)$ and consider the symmetric difference $M \triangle M'$. Since M and M' are perfect, each $v \in V$ must be adjacent to exactly one edge $m \in M$ and $m' \in M'$. Therefore, each v has degree 0 or 2 in $M \triangle M'$, where v is degree 0 if and only if $m = m'$. Suppose that there exists a v of degree 2 and consider its connected component in $M \triangle M'$. This is a connected graph C consisting of entirely degree 2 vertices, and since every tree has some degree 1 vertex, C is not a tree and must have a cycle. But this contradicts the assumption that T is a tree; hence, no degree-2 vertices exist in $M \triangle M'$, so each edge is the same and $M = M'$. $\square$

Problem 1. *Prove or disprove: every tree T has at most one perfect matching.*

Solution

True.

We will prove this by induction. When $n = 1$ there is no perfect matching, which is less than one perfect matching. When $n = 2$ there is exactly one perfect matching, which again is at most one perfect matching.

Now, for the induction step, assume that for all trees with $\leq k$ nodes, at most one perfect matching exists. Let T be a tree with $k+1$ vertices. If $k+1$ is odd, then T has no perfect matching, since a perfect matching must cover all vertices and therefore requires an even number of vertices. Thus assume $k+1$ is even.

Because T is a tree, it has a leaf v. Let u be the unique neighbor of v. In any perfect matching of T, the edge $\{u, v\}$ must be included, since v has degree 1 and must be matched with its only neighbor.

Remove the vertices u and v and their incident edges from T, and let the remaining graph be $T' = T - \{u, v\}$. Then T' is a a collection of trees whose connected components are trees, each with at most $k - 1$ vertices. Any perfect matching of T restricts to a perfect matching of T' after removing the edge $\{u, v\}$, and conversely any perfect matching of T' extends uniquely to one of T by adding $\{u, v\}$. Hence the number of perfect matchings of T equals the number of perfect matchings of T'.

By the induction assumption, each component tree of T' has at most one perfect matching, so the collection of trees T' has at most one perfect matching in total. Therefore T also has at most one perfect matching.

This completes the induction and proves that every tree has at most one perfect matching. $\square$

Problem 2. *For $k \geq 2$, prove that the hypercube graph Q_k has at least $2^{2^{k-2}}$ perfect matchings.*

Solution. For proof by induction observe our base case of $k = 2$ on Q_2. Then notice that we have $2 = 2^{2^{k-2}}$ perfect matchings by taking opposite edges of Q_2 to be in our matching. Hence our base case holds
Suppose that Q_k has at least $2^{2^{k-2}}$. Now notice that we can construct hypercube Q_{k+1} from 2 hypercubes Q_k by drawing edges from vertices of one to vertices of the other. Suppose we have perfect matching M_1 for Q_{k_1} and perfect matching M_2 for Q_{k_2}. Then matching $M_1 \cup M_2$ is also a perfect matching for the constructed hypercube Q_{k+1}. This is because we only add edges, so each vertex remains saturated in Q_{k+1}. Since each hypercube Q_k has at least $2^{2^{k-2}}$ perfect matchings then looking at the combinations of these matchings we see that we have at least $(2^{2^{k-2}})^2$ perfect matchings for a given hypercube Q_{k+1}. Then $(2^{2^{k-2}})^2 = 2^{2^{(k+1)-2}}$. Therefore hypercube Q_{k+1} has at least $2^{2^{(k+1)-2}}$ perfect matchings. $\square$

Problem 3. *Let m be the maximum size of a matching of G. Prove that every maximal matching of G has at least $m/2$ edges.*

Solution. :
Let M be a maximal matching and let M^* be a maximum matching, with $|M^*| = m$.

Take any edge $e \in M^*$. Beacuese M is maximal, e cannot be added to M, so e must share at least one endpoint with some edge of M (otherwise it would be disjoint from all edges of M and we could add it).

Assign each edge of M^* to one edge of M that shares an endpoint with it. An edge $f \in M$ has two endpoints, and because M^* is a matching, at most one edge of M^* can touch each endpoint of f. Hence at most 2 edges of M^* can be assigned to the same $f \in M$.

Therefore

$$m = |M^*| \leq 2|M|,$$

so

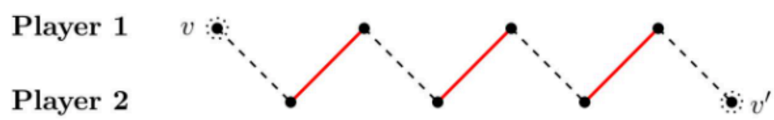
$$|M| \geq \frac{m}{2}.$$

$\square$

Problem 4. *Two people play a game on a graph G, alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins. Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.*

Solution. Suppose first that G has a perfect matching M. Then whenever player 1 picks a vertex, player 2 can respond by picking the other vertex incident to its edge in M. Since M is perfect and vertices cannot be revisited, player 2 can always continue in this manner until player 1 has no more moves (which will take at most until all the vertices of G have been visited).

For the converse, suppose that G has a maximum matching M which is not perfect. Then M has some unsaturated vertex v; let player 1 begin by selecting this vertex. I claim that each subsequent vertex selected by player 2 must then be adjacent to some edge $e \in M$, and player 1 can again pick the neighboring vertex according to e. If this were not the case, and player 2 were able to pick an unsatisfied vertex v', then this would create an M-augmented path in G (illustrated below), contradicting the hypothesis that M is maximum.



Thus, player 1 can continue to pick neighboring vertices according to M, and will eventually exhaust player 2's moves by the same argumentation as before. □

Problem 5. *Recall that a square matrix is called a permutation matrix P if it has exactly one* 1 *in each row and each column and the remaining entries are* 0*. Here's an example:*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) *Use matchings to prove that a square matrix with nonnegative integer entries can be expressed as the sum of k permutation matrices if and only if all the row sums and column sums equal k.*[1]

(b) *Use this to construct your own original 4×4 magic square. Make sure the diagonals also have the same value (you will need to think about how to ensure this). Also make sure your example is not boring.*[2]

Solution.

(a) First, observe that there is a bijection between $n \times n$ permutation matrices and perfect matchings on $K_{n,n}$. This is given by identifying the rows with vertices in X and the columns with vertices in Y, where a 1 in the matrix indicates a shared edge between its row and column vertices. These are matchings since no vertex (row or column) is incident to multiple edges, and perfect since each is incident to one. It is obvious that each matching on $K_{n,n}$ can be converted back into a unique permutation matrix in the same fashion.

In similar vein, an $n \times n$ "magic square" whose rows and columns sum to k corresponds to a k-regular bipartite graph (possibly with multiple edges between pairs of vertices), since each vertex is incident to k edges. We can apply this fact to the statement of the problem above. The forward direction is obvious: if a matrix is the sum of k permutation matrices, then each row and each column has seen exactly k 1's added, and the result is a magic square with row and column sum k. To show that all such magic squares are sums of k permutation matrices, it suffices to show that all k-regular bipartite graphs with $|X| = |Y| = n$ can be "stripped down" into k matchings on $K_{n,n}$.

This is shown by induction on k. In the case $k = 1$, a k-regular graph *is* a matching, so the statement obviously holds. Now assume for inductive hypothesis that all the k-regular graphs can be broken up in this way, and consider the $(k+1)$-regular graph G with vertices in $X \sqcup Y$. We can apply Hall's theorem to find a matching M in G.

Pick some collection of vertices $S \subseteq X$ arbitrarily; since each vertex in S is connected to $(k+1)$ edges, and no edge is shared between two vertices in X, the set of edges E_S incident to vertices in S satisfies $|E_S| = (k+1)|S|$. Similarly, since $N(S)$ is just a subset of vertices in Y, $|E_{N(S)}| = (k+1)|N(S)|$ by the same reasoning. But since all the neighbors of a vertex $v \in S$ are incident to the edges of v, $E_S \subseteq E_{N(S)}$ and $|E_S| \leq |E_{N(S)}|$. Hence

$$(k+1)|S| = |E_S| \leq |E_{N(S)}| = (k+1)|N(S)|,$$

so $|S| \leq |N(S)|$, and there is a perfect matching M by Hall's Theorem. Since each vertex in G is incident to exactly one edge in M, the graph $G \setminus M$ is a k-regular graph which can be

[1] Hint: Here it may help to consider graphs with multiple edges.

[2] There is no formal definition of boring, but permutation matrices are boring, as are matrices with repeated entries..

further divided into k matchings by inductive hypothesis. Therefore, any matrix whose rows and columns sum to k can be represented as a sum of k permutation matrices.

(b) I constructed the following square by summing 4×4 permutation matrices, and used trial and error for the diagonals.

$$\begin{array}{cccc} 15 & 2 & 1 & 12 \\ 8 & 5 & 6 & 11 \\ 4 & 9 & 10 & 7 \\ 3 & 14 & 13 & 0 \end{array}$$

$\square$

Problem 6. *A deck with mn cards with m values and n suits consists of one card for each value in each suit. The cards are dealt into an $n \times m$ array. Use matchings to prove that there is a set of m cards, one in each column, having distinct values.*

Solution. We can represent the m values as one partite and the m columns in the array as the other partite together in a bipartite graph. Let us define each card as an edge between the value and the column it is located, where there can be multiple edges. Since each of the m values appears n times and each column has n entries, all of the vertices therefore have the same degree and a perfect matching must exist between the columns and the values due to Hall's Marriage Theorem. Each subset of k vertices in each partite has a degree sum of kn which means there must be at least $kn/n = k$ vertices the subset is adjacent to in the other partite. This means that each partite is saturated and there exists a perfect matching. Thus, there must exist a set of m cards with one card from each column corresponding to a m distinct values. $\square$