Problem 1. *The complete bipartite graph $K_{n,m}$ is the graph with $n+m$ vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$ and edges $\{v_i, u_j\}$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. Determine the values n, m so that $K_{n,m}$ is Eulerian.*

Solution. We learned in class that a graph is "Eulerian" if it has an Euler tour, which is a closed path that starts and ends at the same vertex, and visits each edge exactly once. We then proved the theorum that a graph G is Eulerian if and only if each vertex in G has an even degree. In the complete bipartite graph $K_{n,m}$, each vertex $v \in v_1, \ldots, v_n$ shares one edge $\{v, u\}$ with all $u \in u_1, \ldots, u_m$. It is also true that $\forall u \in u_1, \ldots, u_m$, u share an edge $\{u, v\}$ with each $v \in v_1, \ldots, v_n$. Therefore, each vertex $u \in u_1, \ldots, u_m$ has degree n and each vertex $v \in v_1, \ldots, v_n$ has degree m. Thus, we know that if both m and n are even numbers, than all vertices in $K_{n,m}$ will have even degree, and $K_{n,m}$ will be Eulerian as a result of this.

$\square$

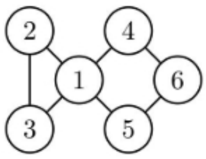
Problem 2. *Prove or disprove:*

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian graph with an even number of vertices has an even number of edges.

Solution. (a.) Consider a Eulerian bipartite graph G. From class, we proved that since G is bipartite, it cannot contain any odd cycles. In addition to this, since G is Eulerian, there must exist Euler tour, we'll call E, that visits every edge exactly once.

If G had an odd number of edges, then E would be a closed walk through an odd number of edges. In class, it was proven that every walk of odd length must have an odd cycle. This would mean that E and as a result, G has at least one odd cycle, causing a contradiction with the fact that G is bipartite. As a result, G must have an even number of edges.



(b.) This statement is false. As a counterexample, consider the graph above which is a Eulerian graph with an even number of vertices, but an odd number of edges. □

Problem 3. [1]

(a) *Classify trees with exactly two vertices of degree* 1.

(b) *What can you say about the shape of trees with either* 3 *or* 4 *vertices of degree* 1*? (Give a qualitative statement – you do not need to provide a formal argument.)*

Solution.

(a) Let $T = (V, E)$ be a tree with exactly two vertices u and w of degree 1. Since T is a tree, we have $|V| = |E| + 1$, and hence

$$\begin{aligned}\sum_{v \in V \setminus \{u,w\}} \deg v &= \sum_{v \in V} \deg v - 2 \\ &= 2|E| - 2 \\ &= 2(|V| - 2),\end{aligned}$$

so the $(|V| - 2)$ remaining vertices must have an average degree of 2. None have degree 0, since T is connected, and none have degree 1 (since we supposed that u and w were the only degree 1 vertices in V); therefore, every other vertex is of degree 2. Since T is a connected graph with two vertices of degree 1 and the rest of degree 2, T is the path $P_{|V|}$.

(b) Any tree with exactly three vertices of degree 1 must have a single 'joining' vertex of degree 3 with the remaining vertices of degree 2. A tree with exactly four vertices of degree 1 has either two vertices of degree 3 or one of degree 4, with all the rest of degree 2.

□

[1] Please wait until after the lecture on Monday (2/2) to solve this.

Problem 4. *Determine the number of graphs with* 7*-vertices, each of degree* 4 *(up to isomorphism).*
[2]

Solution Let G be a 7-vertex graph where every vertex has degree 4. The complement of G is a graph with 7 vertices, where each of the vertices has degree 2 (each vertex can have a maximum degree of 6, if each vertex in G has degree 4, then every vertex in the complement of G is connected to the 2 other vertices that they're not connected to in G, meaning each vertex in the complement of G has degree 2). A graph where every vertex has degree 2 is similar to part a of the previous question, except that there are now no leaves. We see that the only graph type where every vertex has degree 2 is a cycle (take the graph from part a of the previous question and then connect the two leaves, changing them from vertices of degree 1 to degree 2). Now, all we need to do is count the number of graphs with 7 vertices that are only made up of cycles. This leaves us with 2 such graphs, either the graph is a 7-vertex cycle, or the graph is 2 separate cycles, one with 3 vertices and one with 4 vertices. The complements of these two types of graphs have 7 vertices, where each vertex has degree 4. Therefore, there are **2** graphs with 7-vertices, each of degree 4 up to isomorphism. □

[2]Hint: consider the complement. Relate this problem to the previous one. Your solution should be short.

Problem 5.

(a) *Prove that removing opposite corner squares from an 8×8 checkerboard leaves a sub-board that cannot be partitioned into 1×2 and 2×1 rectangles.* [3]

(b) *Translate your solution to the language of bipartite graphs.*

Solution. a) If we color the 8×8 checkerboard into black and white squares, we notice that opposite corner squares must have the same color. However, since each 1×2 or 2×1 rectangle covers two squares of different color, removing two squares of the same color results in an imbalance of squares and there is no possible way for rectangles to cover the board such that two more squares of one color than the other are covered.

b) We can represent the black squares as vertices in one partite and the white squares as vertices of the other partite. Each rectangle must map one of the white vertices to the black vertices such that each vertex is mapped to only one rectangle. If there are two more vertices in one partite, each of the vertices in the smaller partite will form an edge with one vertex in the other partite and leave two vertices left with no valid vertices to form edges with. This means there will always be two squares left of that color unable to be covered by a rectangle. □

[3] Hint: your solution should be short.