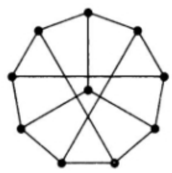
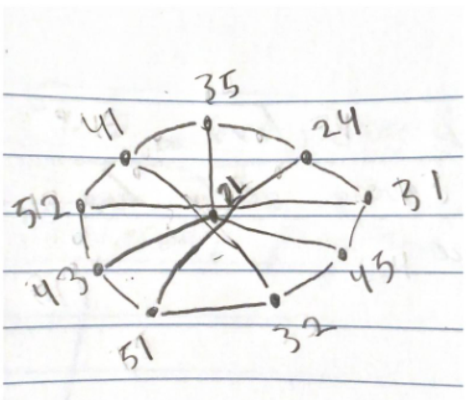
Problem 1. *Prove that the following graph is isomorphic to the Petersen graph.*



Solution. As seen below we can label each vertex with a 2 element subset of $\{1, 2, 3, 4, 5\}$ such that each vertex has a unique subset, we use all the subsets, and edges are pairs of disjoint subsets. Thus there exists an isomorphism from this graph to the Peterson graph which implies that the two graphs are isomorphic. $\square$



Problem 2. *How many cycles of length n are there in the complete graph K_n? (Explain your answer.)*

Solution. Notice first that in K_1 and K_2 there are no cycles. Now we consider K_n for $n > 2$.

I claim that there are $(n-1)!/2$ cycles of length n in the complete graph K_n. Arbitrarily label the vertices of K_n as $v_1, v_2, \ldots, v_n$. We want to create a cycle containing all of these vertices, that is for each vertex we add to the subgraph we must also add the edge connecting it to the previously added vertex. WLOG begin with the vertex v_1. Now we may pick any other vertex $v_2, \ldots, v_n$ as the next vertex, hence there are $n-1$ possible options for our new subgraph. Suppose we pick v_k as our vertex. For the next vertex we may now choose any vertex $\{v_j | j \neq 1, k\}$, as picking a previously chosen vertex would create a cycle of size less than n. Hence we have $n-2$ choices for the third element. Repeating this process of adding new vertices we see that for the j-th vertex, we have $n-j+1$ choices of new vertices to add. When n vertices are added we simply connect the last vertex to the first to create a cycle. Hence we may count the possible methods of cycle creation as

$$(n-1)(n-2)\cdots(n-j+1)\cdots(2)(1) = (n-1)!$$

Notice thought that each cycle created this way is created twice, once forwards (picking v_1 then v_2, ..., then v_n) and once backwards (picking v_1 then v_n, ... , then v_2). Logically notice that each cycle can only be "walked" in two ways, hence therefore we double count each cycle using this method. Hence we see that there are $(n-1)!/2$ cycles of length n in the complete graph K_n. □

Problem 3. *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \ldots, a_k)$ with coordinates $a_i \in \{0, 1\}$ and with an edge between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if they differ in exactly one coordinate.*

(a) *Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.*

Solution. Notice that 4-cycles in Q_k use exactly two dimensions out of the k that are available. Consider the left half of the image below, which contains an example 4-cycle between vertices A, B, C, D (where $A = (a_1, \ldots, a_k)$ etc.). By the definition of a hypercube graph, A differs from B in exactly one tuple entry at index i. Similarly, C differs from B in exactly one tuple entry at index j. Crucially, $i \neq j$ because $A \neq C$ (i.e. these are two edits along distinct dimensions). Notice that once A, B, and C are defined, our choice of D is fixed: we must make *both* edits at indices i and j to reach D.

Suppose for contradiction that two distinct 4-cycles in Q_k share more than one vertex.

- If these two cycles share four vertices, then they are the same cycle, contradicting the fact that they are distinct.
- If these two cycles share three vertices, then the above example illustrates that the choice of the fourth vertex is already fixed (they actually have to share four vertices) leading to the same contradiction.
- If these two cycles share two vertices, then this pair of shared vertices is either 1 edit apart or 2 edits apart. If they are 2 edits apart, then both cycles constitute edits along the same two dimensions i and j, meaning the cycles are identical. Once again, this contradicts them being distinct. So, the two vertices must be 1 edit apart, which is equivalent to them sharing a single edge.

Therefore, we have proven that two non-disjoint 4-cycles must either share exactly one vertex or exactly one edge. $\square$

(b) *Let $K_{2,3}$ be the complete bipartite graph with 2 red and 3 blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.*

Solution. Recall our example from part (a), which demonstrated that in a hypercube graph Q_k, distinct 4-cycles cannot share exactly three vertices. Now, consider the labeling of $K_{2,3}$ given in the right half of the image below. In $K_{2,3}$, there are two distinct four cycles that share three vertices (namely A, B, C). Therefore, $K_{2,3}$ cannot be a subgraph of Q_k. $\square$

Problem 4. *Let $G = (V, E)$ be a graph. The complement of G is the graph $\bar{G} = (V, \bar{E})$, where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$.*

(a) Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices.)

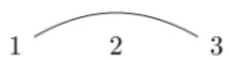
(b) We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n or $n-1$ is divisible by 4.

(c) Construct a self-complementary graph for each n such that n or $n-1$ is divisible by 4.[1]

Solution. (a) Recall that P_3 is the graph

$$1 \text{ —— } 2 \text{ —— } 3$$

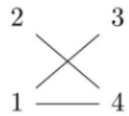
so the complement is



In symbols,

$$\bar{P}_3 = (\{1, 2, 3\}, \quad \{\{1, 3\}\})$$

Similarly, the complement $\bar{P}_4$ is



In symbols,

$$\bar{P}_4 = (\{1, 2, 3, 4\}, \quad \{\{1, 4\}, \{1, 3\}, \{2, 4\}\})$$

(b) If G is self-complementary, E account for exactly half of all possible edges. Thus,

$$|E| = \frac{\binom{n}{2}}{2} = \frac{n(n-1)}{4}$$

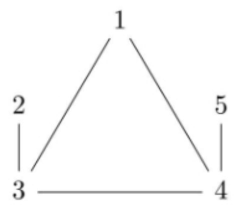
Since exactly one of n and $n-1$ is even, n or $n-1$ must be divisible by 4.

(c) Note that it suffices to prove this is the case for $n = 4$ and $n = 5$, since given a graph self-complementary graph $G = (V, E)$ with n vertices, we can construct a self-complementary graph with $n + 4$ vertices by adding P_4 as well as edges from each $v \in V$ to the vertices of degree two in P_4. This works since both G and P_4 are self-complementary and vertices of degree one in P_4 have degree two in $\bar{P}_4$. Specifically, the only edges between G and P_4 are incident to vertices of degree two when restricted to P_4, and this remains the case when taking the complement by the above.

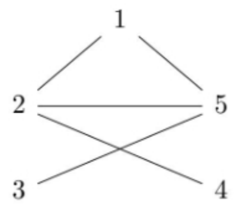
The case $n = 4$ is solved by P_4. As seen above, $\bar{P}_4$ is the path $(3, 1, 4, 2)$, so there is an evident isomorphism $P_4 \cong \bar{P}_4$.

[1]Hint: The first interesting case is $n = 5$. Build an example using P_4 and one more vertex. Study this case carefully and generalize. For $n = 8$ you can start with two copies of P_4.

For $n = 5$, we have G as



whose complement is



Then the following, which we will denote f, is graph isomorphism:

$$1 \mapsto 1, \quad 2 \mapsto 4, \quad 3 \mapsto 2, \quad 4 \mapsto 5, \quad 5 \mapsto 3$$

It is evidently an isomorphism of vertex sets, and here is the painstaking verification that it preserves edge relations:

$$\begin{aligned} \bar{E} &= \{\{1,2\},\{1,5\},\{2,5\},\{3,5\},\{2,4\}\} \\ &= \{\{f(1),f(3)\},\{f(1),f(4)\},\{f(3),f(4)\},\{f(2),f(3)\},\{f(4),f(5)\}\} \\ &= f\left(\{\{1,3\},\{1,4\},\{3,4\},\{2,3\},\{4,5\}\}\right) \\ &= f(E) \end{aligned}$$

□

Problem 5. *True or false: if G is isomorphic to H, then the complements $\bar{G}$ and $\bar{H}$ are also isomorphic.*[2]

Solution. Suppose G is isomorphic to H. Then there is a bijection accross the relation

$$V(G) \to V(H)$$

that preserves edge relationships, meaning

$$uv \in E(G) \iff \text{some } uv \in E(H).$$

In the complement graph of either H or G, two vertices are connected when they are not connected in the original graph. So,

$$uv \in E(\overline{G}) \iff uv \notin E(G).$$

Using the isomorphism, we get

$$uv \notin E(G) \iff \text{some } uv \notin E(H).$$

Therefore,

$$uv \in E(\overline{G}) \iff \text{some } uv \in E(\overline{H}).$$

So the bijection (isomorphism) also preserves edges in the complement graphs. Thus,

$$\overline{G} \text{ is isomorphic to } \overline{H}.$$

□

[2]True/false questions require either a proof (if the statement is true) or a counterexample (if the statement is false).

Problem 6. *For this problem, let G denote the Petersen graph. Use the definition of the Petersen graph given in class (where vertices are 2-element subsets of $\{1, 2, 3, 4, 5\}$) to prove the following.*

(a) G has no cycles of length 3 or 4.

(b) G has no cycle of length 7.[3]

Solution.

(a) Pick any vertex v in the Petersen graph and consider distinct vertices v' and v'' which each share an edge with v. By construction, v' and v'' must be subsets of the three-element set $\{i, j, k\} = \{1, \ldots, 5\} \setminus v$ of size two. Since a three-element set has no disjoint size-two subsets, $\{v', v''\} \notin E$, so v is not contained in any 3-cycles. Now let u be a vertex sharing an edge with both v' and v''. Since $v' \neq v''$, we know that $v' \cup v'' = \{i, j, k\}$—so u must be a two-element subset of $\{1, \ldots, 5\} \setminus \{i, j, k\} = v$, and thus $u = v$. Hence, v's neighbors have no common neighbors besides v itself, and v is not contained in any 4-cycles. Since v was picked arbitrarily, G cannot have any 3- or 4-cycles.

(b) Suppose for contradiction that G has a 7-cycle C. Since each vertex in G has degree 3, and each vertex in C neighbors two edges in C, there must be one edge connected to each vertex of C which is not itself contained in C. If any of these edges connected two vertices of C, it would form a 3- or 4-cycle, which is impossible by (a). Hence, there must be seven distinct edges connecting vertices in C to the three vertices not in C. By the pigeonhole principle, one of these outside vertices must connect to three of the vertices in C; but this is impossible, for any choice of three vertices in C contains two which are one or two edges apart in the chain, and we once again have a 3- or 4-cycle. Hence, G has no 7-cycle.

□

[3]Hint: argue by contradiction. Observe that each vertex of G has degree 3. If C is a 7-cycle in G is it possible that the only edges in G connecting vertices in C are the edges in C?