Midterm practice problems

Problem 1. *True or false. Explain your answer. In your explanation, you may want to cite relevant results from class; please do this clearly.*

(a) *Up to isomorphism, there is a unique graph with degree sequence* $(2, 2, 2)$.

(b) *A graph with* 10 *vertices and* 9 *edges is connected.*

(c) *A graph with a perfect matching has an even number of vertices.*

(d) *A* 3*-regular bipartite graph has at least* 3 *different perfect matchings.*

(e) *If* G *is Eulerian with edges* e, f *that share a vertex, then* G *has an Euler tour where* e, f *appear consecutively.*

(f) *Isomorphic trees have the same Prüfer code.*

Problem 2. *Give an example or explain why no example exists.*

(a) *A graph with degree sequence* $(6, 5, 5, 4, 3, 2, 1)$

(b) *A bipartite graph with at least 5 edges, and such that removing any edge decreases the size of a maximum matching.*

(c) *A weighted graph with two different minimal spanning trees.*

(d) *A tree with an even number of vertices and no perfect matching.*

(e) *A graph with* ≥ 8 *edges such that removing any edge gives a tree.*

Problem 3. *Let* T *be the labeled tree with Prüfer code* $(3, 3, 5, 3)$*. What is the degree sequence of* T*?*

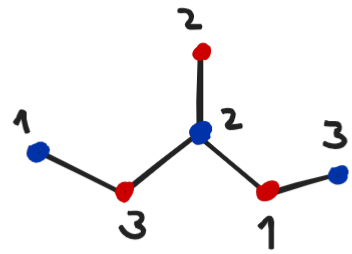
Problem 4. *What is the maximum number of connected components possible for a graph with* 10 *vertices and* 15 *edges?*

Problem 5. *Let* $G = (X \sqcup Y, E)$ *be a bipartite graph. Enumerate* $X = \{x_1, \ldots, x_n\}$ *and* $Y = \{y_1, \ldots, y_m\}$*. Form an* $n \times m$ *matrix* $A = (a_{ij})$*, where* $a_{ij} = 1$ *if* $\{x_i, y_j\} \in E$ *and* $a_{ij} = 0$ *otherwise. Translate König's theorem to a general statement about (certain) matrices.*

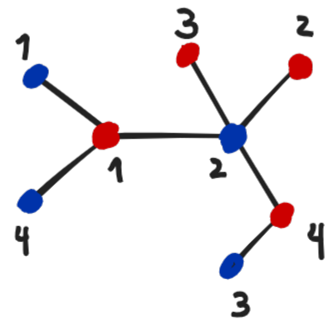
Problem 6. *Use the degree-sum formula to prove that a tree with at least two vertices has at least two leaves.*

Problem 7. *Prove that the symmetric difference of two matchings is a union of paths and even cycles. Use this to prove that a tree has at most one matching.*

Problem 8. *Let* $K_{n,n}$ *be the complete bipartite graph with* n *red vertices and* n *blue vertices. Label the red vertices* $1, \ldots, n$ *and also label the blue vertices* $1, \ldots, n$*. To a spanning tree* T *of* $K_{n,n}$ *we associate a Prüfer-like code, as follows. To start, consider the smallest red leaf and the smallest blue leaf; let* a_1 *be the label of the neighbor of the smallest red leaf, and let* b_1 *be the label of the neighbor of the smallest blue leaf. We delete these two leaves from* T *and record the pair* (a_1, b_1)*. This leaves a tree with* $2n - 2$ *vertices (half red, half blue), and we repeat this process inductively until we obtain a sequence* $(a_1, b_1; \ldots; a_{n-1}, b_{n-1})$*, where* $a_i, b_i \in \{1, \ldots, n\}$*. For example, the tree below has code* $(2, 3; 2, 1)$*.*



(a) Use this procedure to give the code for the following tree.



(b) Working backwards, find the tree with code $(3,3;2,2;1,2)$.

(c) Prove the above procedure defines a bijection between sequences $(a_1, b_1, \ldots, a_{n-1}, b_{n-1})$ *as above and spanning trees of* $K_{n,n}$. *Hint: follow the proof of Cayley's theorem from class as closely as possible.*

(d) Use this to determine the number of spanning trees of $K_{n,n}$.