Midterm 2 practice problems

1. True/false: If two holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ agree at every point of the unit circle, then they agree everywhere.
2. Express $f(z) = 1/(3-z)$ as a power series centered at $z = 1$. What is the largest disk around $z = 1$ on which this power series converges?
3. True/false: Every nonconstant polynomial $p : \mathbb{C} \to \mathbb{C}$ is surjective.
4. Use Cauchy integral formula to prove that if f is holomorphic on a neighborhood of $\{|z| \leq 2\}$ and $|f(z)| \leq 5$ when $|z| = 2$. Prove $|f^{(3)}(0)| \leq 15/4$.
5. Give an example or prove that none exists: A nonconstant bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$.
6. Give an example or prove that none exists: A nonconstant bounded holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.
7. True/false: if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| = |z|$ for all z with $|z| = 1$, then f is constant.
8. True/false: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ and $f'(0) = 1/2$, then $f(z) = z/2$.
9. Give an example or prove that none exists: $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$ and $f(1/2) = 3/4$.
10. True/false: if two holomorphic functions $\mathbb{C} \to \mathbb{C}$ agree on $\mathbb{Z}$, then they are equal.
11. True/false: if $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f(1/n) = 0$ for each n, then $f = 0$.
12. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\int_{|z|=r} f(z)\, dz = 0$ for each $R > 0$, then $f = 0$.
13. True/false: if f has an isolated singularity at 0 and $\mathrm{Res}(f, 0) = 0$, then 0 is a removable singularity of f.
14. True/false: If $f : D \to D$ is holomorphic, $f(0) = 0$, and $f'(0) = e^{i\pi/3}$, then $f(z) = e^{i\pi/3}z$ for every $z \in D$.
15. True/false: if f has a pole at 0, then $|f(z)| \to \infty$ as $z \to 0$.
16. Compute $\mathrm{Res}(f, 0)$ for $f(z) = e^{1/z^2}$.
17. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Assume that $|f(z)| \leq |z| + 2$ for all z. Show that f is a polynomial of degree ≤ 1. [20]
18. Let $f(z) = \sum_{n=0}^{\infty}((z-1)/2)^n$ for $|z-1| < 2$. Find a holomorphic analytic continuation of f to a $\mathbb{C} \setminus \{3\}$, and explain why f cannot be analytically continued to all of $\mathbb{C}$.
19. Give an example or prove that none exists: a holomorphic function with an essential singularity at zero and such that $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

[20] Hint:Show $f'' = 0$.

20. For the Gamma function, use the relations $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$ to prove that Γ has a pole at each nonpositive integer $k \leq 0$ and compute the residue $\mathrm{Res}(\Gamma, k)$.

21. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic then we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$ (not just near 0).

22. If f and g are holomorphic functions $\mathbb{D} \to \mathbb{D}$ and $f(0) = g(0) = 0$ and $f(1/2) = g(1/2) = 1/4$, then $f = g$.

23. True/false: the power series $f(z) = 1 + z + z^2 + \cdots$ can be analytically continued to $\mathbb{C} \setminus \{1\}$.

24. Assume $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and suppose $f(0) = f'(0) = f''(0) = 0$. Show that $|f(z)| \leq |z|^3$ for all $z \in \mathbb{D}$.

25. True/false. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ and $f(a) = a$ for some $a \neq 0$, then f is the identity function $f(z) = z$.

26. True/False: $f(z) = \frac{1}{e^z - 1} - \frac{1}{z}$ has a removable singularity at $z = 0$.

27. Find the maximum modulus of $f(z) = z^2 + 1$ on the unit disk.

28. Assume f and g are holomorphic functions defined on a neighborhood of the closed unit disk $\{|z| \leq 1\}$. Assume that $|f(z)| \leq |g(z)|$ for $|z| = 1$. Show that if g has no zeros inside the disk, then $|f(z)| \leq |g(z)|$ for all z in the disk.

29. Let f be holomorphic and nonvanishing on a neighborhood of the closed unit disk $\{|z| \leq 1\}$. Suppose that $|f(z)| \geq 3$ for $|z| = 1$. Show that $|f(z)| \geq 3$ for all z in the disk.

30. Show that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\mathrm{Re}\, f(z) \geq 0$ for all z, then f is constant.

31. Suppose $f : D \to D$ is holomorphic and $f(0) = a$. Prove that

$$|f'(0)| \leq 1 - |a|^2.$$

32. Show that every holomorphic invertible map $f : \mathbb{D} \to \mathbb{D}$ satisfying $f(0) = 0$ is a rotation.

33. Suppose $f, g : \mathbb{C} \to \mathbb{C}$ are entire and satisfy $f(z)^2 = g(z)^2$ for all z. What can you say about f and g?

34. Let f, g be holomorphic near 0, and suppose $g(0) = 0$ and $g'(0) \neq 0$. Prove that

$$\mathrm{Res}\left(\frac{f}{g}, 0\right) = \frac{f(0)}{g'(0)}.$$

35. Suppose f has an isolated singularity at a. Prove that

$$\mathrm{Res}(f', a) = 0.$$

36. True/false: if f has an essential singularity at zero, then there exists a sequence $z_n \to 0$ such that $f(z_n) \to 17$.

37. Compute $\int_{|z|=1} \frac{e^z}{(z - i/2)^3}\, dz$.

38. Compute $\int_{|z|=2} \frac{e^z}{z(z-1)}\,dz$.
39. Compute $\int_{|z|=1} \frac{1}{z^2(z-\frac{1}{2})}\,dz$.
40. Compute $\int_{|z|=1} \frac{e^{1/z}}{z^2}\,dz$.
41. Classify the singularity at zero for $f(z) = \frac{\sin z - z}{z^4}$.
42. Classify the singularity at zero for $f(z) = \frac{1-\cos z}{z^2}$.
43. Let $f(z) = \frac{1}{z(z-1)}$. Compute $\int_{|z|=1/2} f(z)\,dz$ and $\int_{|z|=2} f(z)\,dz$.
44. Compute $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ using residue calculus.
45. Compute $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\,dx$ using residue calculus.
46. Compute $\int_{0}^{\infty} \frac{x}{1+x^5}\,dx$ using residue calculus.
47. Compute $\int_{0}^{\infty} \frac{x^{1/3}}{1+x^2}\,dx$ using residue calculus.