1460 HW 1

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(1)

- $\overline{z+w} = \bar{z} + \bar{w}$. Write $z = x+iy$ $w = a+ib$

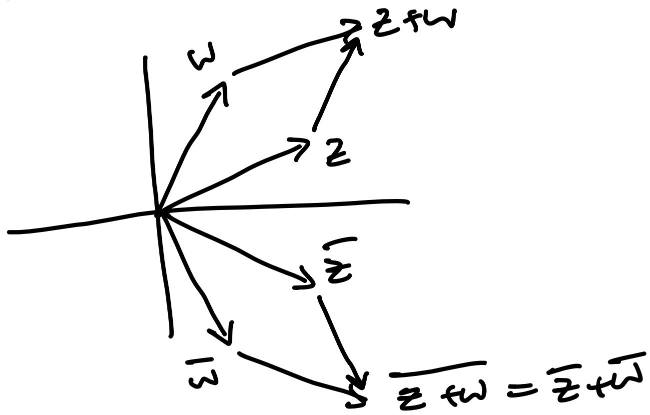
Compute $\overline{z+w} = \overline{x+iy+a+ib} = \overline{x+a+i(y+b)}$

$= x+a-i(y+b)$

$= x-iy + a-ib$

$= \bar{z} + \bar{w}$.

Picture



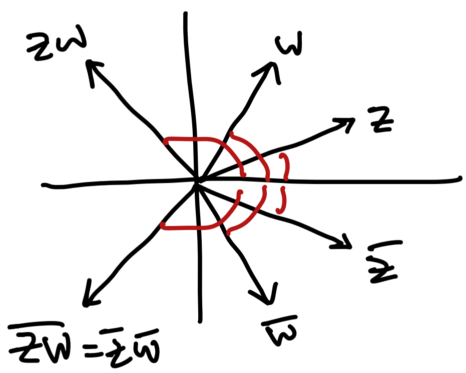
- $\overline{zw} = \bar{z}\bar{w}$. Write $z = re^{i\theta}$ $w = se^{i\phi}$ (polar)

Compute $\overline{zw} = \overline{re^{i\theta}se^{i\phi}} = \overline{rse^{i(\theta+\phi)}}$

$= rse^{-i(\theta+\phi)}$

$= re^{-i\theta}se^{-i\phi} = \bar{z}\bar{w}$.

Picture

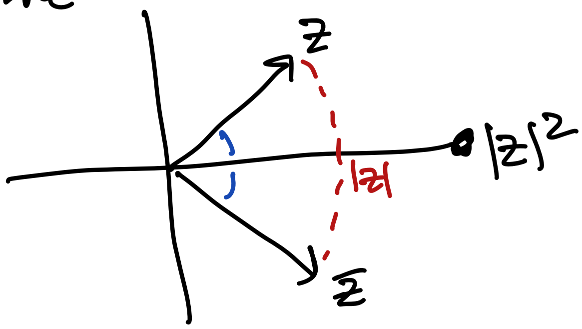


- $|z|^2 = z\bar{z}$. Write $z = x+iy$

Compute $|z|^2 = x^2+y^2$

$z\bar{z} = (x+iy)(x-iy) = x^2+y^2$

Picture



(2) (a) $|Re(z)| \le |z|$ & $|Im(z)| \le |z|$

Write $z = x+iy$

$$|Re(z)| = |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$$

$$|Im(z)| = |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} = |z|.$$

(b) $|z+w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w})$.

Use $|z|^2 = z\bar{z}$ and $Re(z) = \frac{z+\bar{z}}{2}$.

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$$

$$= |z|^2 + |w|^2 + \underbrace{z\bar{w} + \overline{z\bar{w}}}_{=2Re(z\bar{w})}$$

✓

(c) $|z+w| \le |z| + |w|$

Use (a) & (b)

$$|z+w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w}) \le |z|^2 + |w|^2 + 2|z|\cdot|w|$$
$$= (|z|+|w|)^2$$

Now take square roots ✓

(d) $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$

Use $|z|^2 = z\bar{z}$

$$|z+w|^2 + |z-w|^2 = (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w})$$

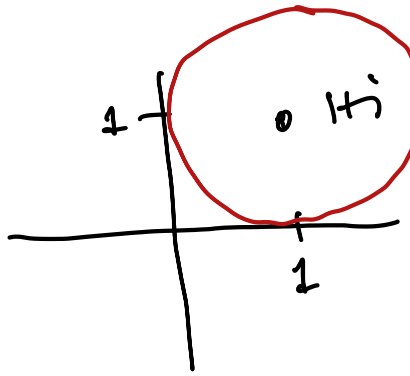
$$= z\bar{z} + w\bar{w} + 2\mathrm{Re}(z\bar{w})$$

$$+ z\bar{z} + w\bar{w} + 2\mathrm{Re}(-z\bar{w})$$

$$= 2\left(|z|^2 + |w|^2\right)$$

(3) (a) $|z-1-i| = 1$ circle radius 1

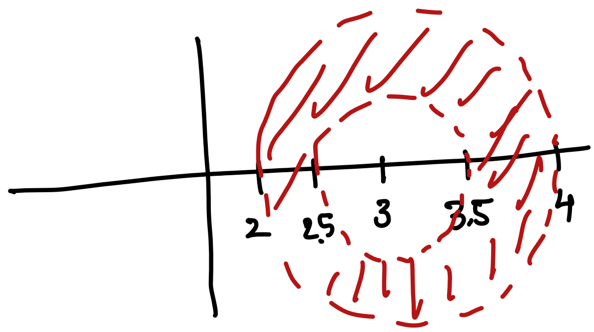
centered at $1+i$



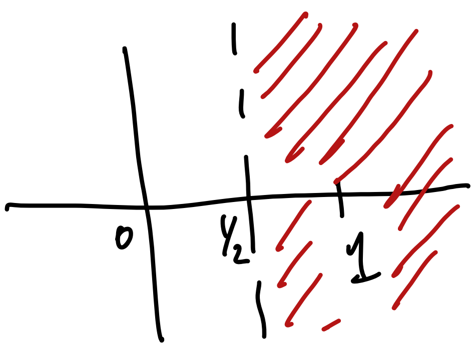
(b) $1 < |2z-6| < 2$ annulus

$1 < 2|z-3| < 2$

$\frac{1}{2} < |z-3| < 1$



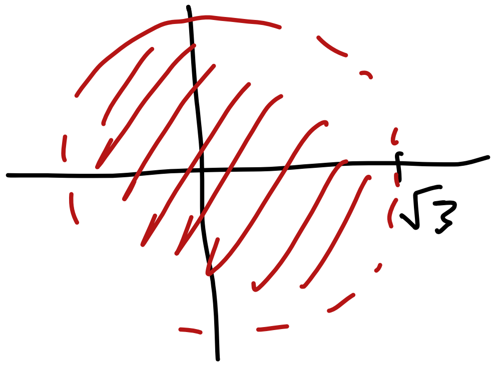
(c) $|z-1| < |z|$ closer to 1 than 0.



(d) $|z-1|^2 + |z+1|^2 < 8$

parallelogram $\Rightarrow$ LHS $= 2\left(|z|^2 + |1|^2\right)$

$\rightsquigarrow \quad |z|^2 < 3 \quad$ so $\quad |z| < \sqrt{3}$



(4) (a)

<u>Claim</u> $z = a + bi + cj + dk$

has inverse $w = \dfrac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$.

<u>Proof</u> Define $\bar{z} = a - bi - cj - dk$

Direct computation $\Rightarrow$ $z\bar{z} = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}$.

Then $w = \dfrac{\bar{z}}{z\bar{z}}$ and $zw = \dfrac{z\bar{z}}{z\bar{z}} = 1$. ✓

(b) <u>claim</u> $zi = iz \iff z = a + bi$

<u>Pf</u> Compute

$(a + bi + cj + dk)i = ai - b - ck + dj$

$i(a + bi + cj + dk) = ai - b + ck - dj$

These equal $\iff$ $c = d = 0$ ✓

(5) <u>WTS</u> n, m sum of 4 squares

$\Rightarrow$ nm is a sum of 4 squares.

Key observation n sum of 4 squares $\Leftrightarrow$

$\exists$ $n = z\bar{z}$ for some quaternion z w/ integer coefficients

So suffices to show if $n = z\bar{z}$ & $m = w\bar{w}$

Then we can write $nm = a\bar{a}$ ...

Well $nm = (z\bar{z})(w\bar{w}) = z(w\bar{w})\bar{z}$

$= zw\bar{w}\bar{z}$

$= (zw)\overline{zw}$

So $nm = a\bar{a}$ is sum of 4 squares ✓

(Warning: quaternions are not commutative!
but $w\bar{w} \in \mathbb{R}$ is central)

<u>check</u> $\overline{zw} = \bar{w}\bar{z}$

write $z = a+bj \quad w = c+dj \qquad a,b,c,d \in \mathbb{C}$

$\bar{z} = \bar{a} - bj \qquad \bar{w} = \bar{c} - dj$

$zw = (a+bj)(c+dj) = (ac - b\bar{d}) + (ad + b\bar{c})j$

$\overline{zw} = (\bar{a}\bar{c} - \bar{b}d) - (ad + b\bar{c})j$

$\bar{w}\bar{z} = (\bar{c} - dj)(\bar{a} - bj)$

$= (\bar{a}\bar{c} - \bar{b}d) - (b\bar{c} + ad)j \quad \checkmark$

Alternative proof:

Observe if $z = a + ib + cj + dk$

then $z\bar{z} = |a+ib|^2 + |c+id|^2$

So $nm = \Big(\underbrace{|a+ib|}_{P}^2 + \underbrace{|c+id|}_{Q}^2\Big)\cdot\Big(\underbrace{|e+if|}_{R}^2 + \underbrace{|g+ih|}_{S}^2\Big)$

$= (P^2 + Q^2)(R^2 + S^2)$

$= (PR)^2 + (PS)^2 + (QR)^2 + (QS)^2$

(b) Counterexample:

$n = 3 = 1^2 + 1^2 + 1^2$, $m = 5 = 0^2 + 1^2 + 2^2$

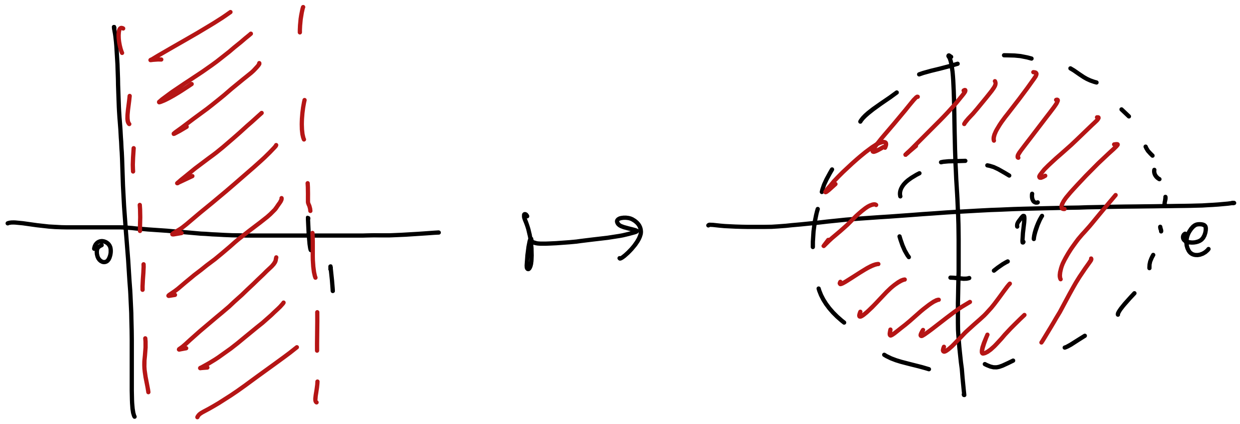
Claim $nm = 15$ not sum of 3 squares

Squares less than 15 are $0, 1, 4, 9$

Observe no 3 sum to 15 ✓.

(6) $e^{x+iy} \mapsto e^x(\cos y + i\sin y)$

(a)



(b)