

Instructions (read these carefully!):

- This assignment must be submitted on Gradescope by the due date. Gradescope entry code: PRXD46
- Please **include a title page** with your name and the names of your collaborators.
- Homework is graded anonymously, so please avoid putting your name on each page of the assignment.
- If you are stuck, please ask for help (from me, Sarah, a classmate, ...). Use Ed discussions! AI use is generally **not** recommended for completing the homework.
- You may freely use any fact proved in class (mention it in the appropriate spot). You *may not* freely use facts from the book or other sources.
- As a general rule, try to restrict your solution to each problem to a single page. Often solutions can be short, while still being complete. If your solution is very long, you should think more about how you might express it more concisely.

HW7: analytic continuation, residue calculus

Due: Friday, Nov 6 by 5pm

1. In class we discussed the function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, which has the following properties: (i) holomorphic on $\{\operatorname{Re}(z) > 0\}$, (ii) $\Gamma(1) = 1$, and (iii) $\Gamma(z+1) = z\Gamma(z)$. Give another example of a function satisfying (i)–(iii) or prove that no such function exists.
2. Prove or disprove: $f(z) = \frac{\sin z}{z}$ has a unique analytic continuation to all of $\mathbb{C}$.
3. The function defined by $f(z) = \sum z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$ is analytic on $\{|z| < 1\}$. Show that it cannot be analytically continued to any larger open set. Hint: observe that $f(z) = z + f(z^2)$ and that $f(x) \rightarrow \infty$ as $x \rightarrow 1$.¹⁴
4. Compute $\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2} dx$ using residue calculus.
5. Compute $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2+1} dx$ using residue calculus.¹⁵

¹⁴Further hint: Show that f similarly blows up at a certain dense set of roots of unity.

¹⁵Hint: First use trig identities to rewrite $\sin^2 x$ as a polynomial in terms of $\cos(2x)$. This should reduce the computation to familiar ones.