

Instructions (read these carefully!):

- This assignment must be submitted on Gradescope by the due date. Gradescope entry code: PRXD46
- Please **include a title page** with your name and the names of your collaborators.
- Homework is graded anonymously, so please avoid putting your name on each page of the assignment.
- If you are stuck, please ask for help (from me, Sarah, a classmate, ...). Use Ed discussions! AI use is generally **not** recommended for completing the homework.
- You may freely use any fact proved in class (mention it in the appropriate spot). You *may not* freely use facts from the book or other sources.
- As a general rule, try to restrict your solution to each problem to a single page. Often solutions can be short, while still being complete. If your solution is very long, you should think more about how you might express it more concisely.

HW6: FTA, Liouville's theorem, residue calculus

Due: Friday, Oct 30 by 5pm

1. Let $p(z)$ be a polynomial, i.e. $p(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_i \in \mathbb{C}$. Assume $n \geq 1$.
 - (a) Prove that if $p(a) = 0$ (we say a is a root of p), then there exists a polynomial $q(z)$ so that $p(z) = (z - a)q(z)$.¹¹ Deduce that given any a (not necessarily a root), there exists a polynomial q so that $p(z) = p(a) + (z - a)q(z)$.
 - (b) Find (by hand) $q(z)$ when $p(z) = 1 + z + z^2 + \cdots + z^n$, and $a = -1$.
2. (a) Prove that for any polynomials $p(z), q(z)$ there exists polynomials $s(z)$ and $r(z)$ such that $p(z) = s(z)q(z) + r(z)$, where the degree of $r(z)$ is less than the degree of $q(z)$.¹²
 - (b) Do the division algorithm (by hand) for $p(z) = z^5$ and $q(z) = z^2 - 1$.
3. Let $p(z) = z^3 + 2z + 5$. Find an explicit $R > 0$ so that $|p(z)| > 1$ for $|z| > R$.¹³
4. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and doubly periodic, i.e. $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$. Show that f is constant.
5. Compute $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ by applying residue calculus to a half-disk, similar to the argument from class (to be discussed Monday).

¹¹Hint: one way to do this is to use the factoring $z^n - a^n = (z - a)(z^{n-1} + az^{n-2} + \cdots + a^{n-2}z + a^{n-1})$.

¹²Hint: use induction on the degree of p . It may help to think about how the division algorithm works to figure out how to reduce to the induction hypothesis.

¹³Recall this this kind of bound was the main technical step in our proof of FTA.