Instructions (read these carefully!):

- This assignment must be submitted on Gradescope by the due date. Gradescope entry code: PRXD46
- Please **include a title page** with your name and the names of your collaborators.
- Homework is graded anonymously, so please avoid putting your name on each page of the assignment.
- If you are stuck, please ask for help (from me, Sarah, a classmate, ...). Use Ed discussions! AI use is generally **not** recommended for completing the homework.
- You may freely use any fact proved in class (mention it in the appropriate spot). You *may not* freely use facts from the book or other sources.
- As a general rule, try to restrict your solution to each problem to a single page. Often solutions can be short, while still being complete. If your solution is very long, you should think more about how you might express it more concisely.

HW3: Cauchy–Riemann equations

Due: Friday, Oct 2 by 5pm

1. Derive the Cauchy–Riemann equations in polar coordinates.[5]

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

2. Prove $f(z) = \sqrt{z}$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

3. Visualize $f(z) = \bar{z}^2$ by writing $f = u + iv$ and plotting the level sets of u, v (probably best to use a computer for this step). You should find that the level sets of u and v meet orthogonally, despite the fact that f is not holomorphic. Explain how (in this case) you can use the level sets, with a little additional (visual) information, to conclude that f is not holomorphic.

4. Show that $\cosh(y)\cos(x)$ is harmonic and find a harmonic conjugate. Recall that $\cosh(y) = \frac{e^y + e^{-y}}{2}$ and $\sinh(y) = \frac{e^y - e^{-y}}{2}$.

5. For complex numbers z, w, we define[6] $z^w = e^{w \log z}$. Determine all values of i^i.

6. Derive Laplace's equation in polar coordinates and show that $u(re^{i\theta}) = \theta \log(r)$ is harmonic.

[5] Hint: This is an application of the chain rule, which you can use to express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ in terms of $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$.

[6] Why is this a reasonable definition?