Instructions (read these carefully!):

- This assignment must be submitted on Gradescope by the due date. Gradescope entry code: PRXD46
- Please **include a title page** with your name and the names of your collaborators.
- Homework is graded anonymously, so please avoid putting your name on each page of the assignment.
- If you are stuck, please ask for help (from me, Sarah, a classmate, ...). Use Ed discussions! AI use is generally **not** recommended for completing the homework.
- You may freely use any fact proved in class (mention it in the appropriate spot). You *may not* freely use facts from the book or other sources.
- As a general rule, try to restrict your solution to each problem to a single page. Often solutions can be short, while still being complete. If your solution is very long, you should think more about how you might express it more concisely.

HW1: complex arithmetic, complex plane, exponential map

Due: Friday, Sept 18 by 5pm

1. For complex number $x+iy$, define the complex conjugate by $\overline{x+iy} = x - iy$. Verify from the definition: $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $|z|^2 = z\bar{z}$. Draw pictures to illustrate these.

2. (a) Show $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
   (b) Show $|z+w|^2 = |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w})$.
   (c) Prove the triangle inequality[1] $|z+w| \leq |z| + |w|$.
   (d) Prove the parallelogram law[2] $|z+w|^2 + |z-w|^2 = |z|^2 + |w|^2$.

3. Sketch the following regions in the complex plane. Each of these should have a clear geometric meaning, possibly after a little algebraic manipulation.

   (a) $|z - 1 - i| = 1$
   (b) $1 < |2z - 6| < 2$
   (c) $|z - 1| < |z|$
   (d) $|z-1|^2 + |z+1|^2 < 8$ (hint[3])

4. A quaternion is a 4-tuple of real numbers (a, b, c, d), typically written (in analogy with $\mathbb{C}$) as $a + bi + cj + dk$. Quaternions are added in the usual way; multiplication is defined using the relations $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$.

   (a) Prove that every nonzero quaternion has a multiplicative inverse.[4]
   (b) Quaternion multiplication is not commutative. Determine all the quaternions that commute with i.

5. Show that if n, m are integers that can be expressed as a sum of four square integers, then the product nm is also a sum of four squares. Show that the same statement is false if "four" is replaced by "three".

6. Sketch each region and its image under the exponential map.

   (a) $0 < \mathrm{Re}(z) < 1$
   (b) $5\pi/3 < \mathrm{Im}(z) < 8\pi/3$

[1] Be sure to understand the reason for this name.
[2] Be sure to understand the reason for this name.
[3] parallelogram law
[4] hint: follow the proof from class in the complex case.