Problem 2. *Compute the cohomology ring of the Klein bottle using simplicial homology with the (simplicial) method we used in class for the torus.*

Solution. Recall the homology of the Klein bottle K:

$$H_k(K) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & \text{else} \end{cases}$$

Then we use the universal coefficient theorem to get that the cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients:

$$H^0(K; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

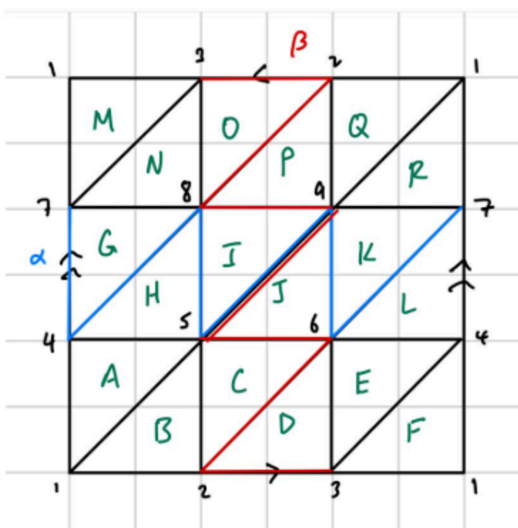
$$H^1(K; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$H^2(K; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

The cohomology ring (as a group) is therefore

$$H^\bullet(K; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\{1\} \oplus \mathbb{Z}/2\mathbb{Z}\{\alpha\} \oplus \mathbb{Z}/2\mathbb{Z}\{\beta\} \oplus \mathbb{Z}/2\mathbb{Z}\{\gamma\}$$

To get the ring structure, we need to consider the cup products between the generators. We know that 1 is the multiplicative identity, and h cupped with anything else is 0 for dimension reasons. Thus the only cup products of interest are $\alpha \smile \alpha$, $\alpha \smile \beta = \beta \smile \alpha$ (the equality follows since $-1 \equiv 1$ modulo 2), and $\beta \smile \beta$. Endowing K with a simplicial structure, we get:



I claim α is the cochain which is 1 on the blue 1-simplices and 0 on the other 1-simplices. Likewise, I claim β is the cochain which is 1 on the red 1-simplices and 0 on the other 1-simplices. Observe that α is a cocycle, as every 2-simplex either touches the blue twice or zero times (and is therefore 0 mod 2); the same reasoning shows that β is a cocycle. If α were a coboundary, then the 0-cochain σ it came from must satisfy $\sigma(7) \neq \sigma(4)$ since $\alpha([4, 7]) = 1$. On the other hand, because $\alpha([1, 7]) = \alpha([1, 4]) = 0$, then we know that $\sigma(7) = \sigma(1) = \sigma(4)$, a contradiction. An analagous argument proves β is not a coboundary using the 0-simplices 3 and 2. When we compute $\alpha \smile \alpha$, we get a cochain which is 0 on all 2-simplices. On the other hand, observe that $(\beta \smile \beta)(P) =$

$\beta([2,8]) \cdot \beta([8,9]) = 1$, which shows that $\alpha \neq \beta$, so the proposed forms of α and β are the two generators for the first cohomology. The full computation of $\beta \smile \beta$ shows that it is 0 on every other 2-simplex (ie $\beta \smile \beta$ is the indicator function on P). We know that the generator of $H_2(K)$ is the sum of the 2-simplices, ie $[K] = A + B + ... + R$. Since $(\beta \smile \beta)([K]) = 1$ this implies that $\beta^2 = \gamma$ (recall γ is the generator of the second cohomology). Finally when we compute $\alpha \smile \beta$, we get the indicator on I. Using the same reasoning, as for β^2, we get that $\alpha \smile \beta = \gamma$. This determines the ring structure of the cohomology ring. Another presentation is given

$$H^\bullet(K; \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \alpha\beta - \beta^2, \beta^3)}$$

The map $H^\bullet(K; \mathbb{Z}/2\mathbb{Z}) \to \frac{\mathbb{Z}/2\mathbb{Z}[\alpha,\beta]}{(\alpha^2, \alpha\beta-\beta^2, \beta^3)}$ sending $1 \mapsto 1$, $\alpha \mapsto \alpha$, $\beta \mapsto \beta$, and $\gamma \mapsto \beta^2$ is a bijection of rings (both rings have 16 elements) so it is an isomorphism. □

Problem 3. *Use homology or cohomology to prove that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are not homotopy equivalent.*

Solution. We show that $X = \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \vee S^3$ are not homotopy equivalent by showing that their cohomology rings with $\mathbb{Z}_2$ coefficients are not isomorphic. We start by showing that

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$$

for $n = 2, 3$.

Fix $0 \leq k \leq n$ an arbitrary manifold M. Note that

$$\mathrm{Ext}_{\mathbb{Z}_2}(H_{n-k-1}(M; \mathbb{Z}_2); \mathbb{Z}_2) = 0$$

since $\mathbb{Z}_2$ is a field. Therefore by the universal coefficients theorem, the map

$$h : H^{n-k}(M; \mathbb{Z}_2) \to \mathrm{Hom}_{\mathbb{Z}_2}(H_{n-k}(M; \mathbb{Z}_2), \mathbb{Z}_2)$$

is an isomorphism. Also, by Poincaré duality, since every manifold is $\mathbb{Z}_2$-orientable,

$$[M] \frown : H_{n-k}(M; \mathbb{Z}_2) \to H^k(M; \mathbb{Z}_2)$$

is an isomorphism as well. Therefore, as we discussed in class, for any $f \in H^{n-k}(M; \mathbb{Z}_2)$ primitive, there is some $g \in H^k(M; \mathbb{Z}_2)$ so that $f \smile g \neq 0$.

Now for $\mathbb{RP}^2$, recall from Problem 1 that the cohomology is given by

$$H^*(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2\{1\} \oplus \mathbb{Z}_2\{f\} \oplus \mathbb{Z}_2\{g\}.$$

We claim that $f \smile f = g$. Write $f \smile f = tg$, $t \in \mathbb{Z}_2$. By the discussion above, there is some $st \in H^1(\mathbb{RP}^2; \mathbb{Z}_2)$ so that $f \smile (sf) \neq 0$. Therefore

$$0 \neq s(f \smile f) = stg.$$

Therefore $st = 1$ so $s = t = 1$ and $f \smile f = g$. This shows that

$$H^*(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^3).$$

For $H^*(\mathbb{RP}^3; \mathbb{Z}_2)$,

$$H^*(\mathbb{RP}^3; \mathbb{Z}_2) = \mathbb{Z}_2\{1\} \oplus \mathbb{Z}_2\{f\} \oplus \mathbb{Z}_2\{g\} \oplus \mathbb{Z}_2\{h\}.$$

We can use the same argument above to show that $g \smile f = h$ and use the embedding $\mathbb{RP}^2 \to \mathbb{RP}^3$ to show that $f^2 = g$ (as we did in class). This shows that

$$H^*(\mathbb{RP}^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^4).$$

Finally, we are ready to show that X and Y are not homotopy equivalent. In Y, let

$$f \in H^1(\mathbb{RP}^2; \mathbb{Z}_2) = H^1(Y; \mathbb{Z}_2), \quad g \in H^2(\mathbb{RP}^2; \mathbb{Z}_2) = H^2(Y; \mathbb{Z}_2).$$

Here we are using that the cohomology of a wedge sum is the direct sum of the cohomologies (in degrees > 0). Technically, we should represent f and g in $H^*(Y; \mathbb{Z}_2)$ by images of the induced map of the retraction $r : Y \to \mathbb{RP}^2$. Then

$$r^*(f) \smile r^*(g) = r^*(f \smile g) = 0$$

since $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$ vanishes in degree 3. Therefore

$$H^1(Y;\mathbb{Z}_2) \smile H^2(Y;\mathbb{Z}_2) = 0.$$

This is not true of $H^*(X;\mathbb{Z}_2)$ because of the polynomial ring structure mentioned above. Therefore X and Y are not homotopy equivalent.

$\square$

Problem 5. *(a) Prove that every oriented surface S is the boundary $S = \partial M$ of some compact 3-manifold M.*

(b) Prove that $\mathbb{R}P^2$ is not the boundary of any 3-manifold. [1] [2]

Solution.

(a) The boundary of a compact manifold is compact, so a non-compact surface cannot be the boundary of a compact manifold. We can also reduce to the connected surface case by considering each connected component separately. Thus, we let S be an oriented connected compact surface. By the classification theorem of closed surfaces, there is a homeomorphism φ from S to either the sphere or a genus g surface. We will show that spheres and genus g surfaces are the boundary of a compact 3-manifold M and give a CW structure on M. Since φ is a homeomorphism from S to ∂M, this procedure will induce a cell structure on a manifold N such that $S = \partial N$.

If S is homeomorphic to S^2, we take M to be the ball D^3, that has a cell structure given by 1 0-cell, 1 2-cell and 1 3-cell. If S is homeomorphic to the torus T, we take M to be the solid torus, that has a cell structure given by 1 0-cell, 2 1-cells, 2 2-cells and 1 3-cell. In general, for a genus g surface, we take M to be the solid genus g surface, that has a cell structure given by 1 0-cell, $2g$ 1-cells, $(g+1)$ 2-cells and 1 3-cell. Indeed, we start from a cell structure of a genus g surface given by 1 0-cell, $2g$ 1-cells and 1 2-cell, then we attach a 2 cell to half of the 1-cells (see the drawing) and then we attach a 3 cell the $(g+1)$ 2-cells by the natural maps. This induces a cell structure on a manifold N such that $S = \partial N$ since it gives a procedure to fill the surface S with a 3-cell. Note that the cell structure on M induces a cell structure on N since φ is a homeomorphism between their boundaries and the attaching maps of M induce the attaching maps of N.

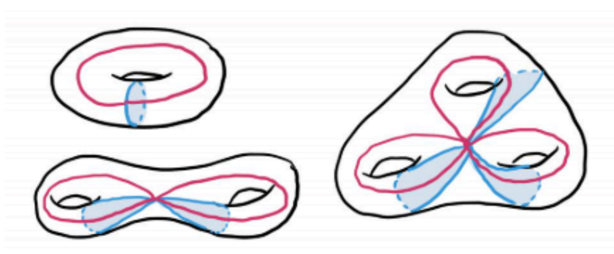


Figure 1: The g 2-cells are shaded in blue.

(b) Since $\partial(\mathbb{R}P^2 \times [0,\infty)) = \mathbb{R}P^2 \times \partial[0,\infty) = \mathbb{R}P^2$, $\mathbb{R}P^2$ is the boundary of a non-compact 3-manifold. We will now show by contradiction that $\mathbb{R}P^2$ is not the boundary of any compact 3-manifold.

Assume $\mathbb{R}P^2$ is the boundary of a compact 3-manifold M, so that $\partial M = \mathbb{R}P^2$. Consider the double $D = M \cup_{\partial M} M$, which is also a 3-manifold, and therefore $\chi(D) = 0$ because it is a compact manifold of odd dimension. On the other hand, we can give D a cell structure by taking the cell structure induced by M on each copy of M in D and identifying the cells

[1] Hint: Proceed by contradiction. Note that a 3-manifold with boundary can be doubled to get a closed 3-manifold.

[2] Hint: compute the Euler characteristic of the double in two ways to reach a contradiction. It may help to use the previous exercise.

contained in the boundary ∂M. Thus, the number of k-cells in D is equal to twice the amount of k-cells in M minus the amount of k-cells contained in ∂M, since we have counted them twice but they are identified in D. Therefore, by Problem 4a, $\chi(D) = 2\chi(M) - \chi(\partial M) = 2\chi(M) - \chi(\mathbb{R}P^2) = 2\chi(M) - 1 \neq 0$, so we have reached a contradiction. We conclude that $\mathbb{R}P^2$ is not the boundary of any compact 3-manifold.

$\square$