Problem 1. *Let X, Y be cell complexes, and give $X \times Y$ the product cell structure. Prove (with care!) that $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$ are isomorphic chain complexes.*

Solution. **Claim:** $\partial : C_k(X\times Y) \to C_{k-1}(X\times Y)$ is given by $\partial(e^p_\alpha \times e^q_a) = (\partial e^p_\alpha)\times e^q_a + (-1)^p e^p_\alpha \times \partial e^q_a$. Please see proof of this claim at the end of question 1 solution. We now assume this claim.

Then for $p+q=k$, define $C_p(X)\times C_q(Y) \to C_k(X\times Y)$ via $(e^p_\alpha, e^q_a) \mapsto e^p_\alpha \times e^q_a$ on basis element and extend by bilinearly. Then, bilinearity gives a map from the tensor product. Take the direct sum of these maps to obtain a map $\phi : \bigoplus_{p+q=k} C_p(X)\otimes C_q(Y) \to C_k(X\times Y)$. This map is indeed an isomorphism because it gives a bijection on generators

$$\{(e^p_\alpha, e^q_a)\forall \alpha, a, p+q=k\} \xrightarrow[\sim]{\phi} \{e^p_\alpha \times e^q_a \forall \alpha, a, p+q=k\}$$

Since ∂ on tensor product of chains is given by

$$\partial(e^p_\alpha \otimes e^q_a) = (\partial e^p_\alpha)\otimes e^q_a + (-1)^p e^p_\alpha \otimes \partial e^q_a,$$

we have $\phi\partial = \partial\phi$ and thus ϕ is a chain map.

Proof of claim: I will use d to denote the ∂ map above and below, to differentiate between taking the boundary of a cell. Consider attaching maps of non-zero degrees, which will either be of the form

$$S^{n-1} \cong \partial(e^p_\alpha \times e^q_a) \to (X\times Y)^{n-1} \to e^p_\alpha \times e^{q-1}_b \cong S^{n-1} \forall b, \tag{1}$$

or

$$S^{n-1} \cong \partial(e^p_\alpha \times e^q_a) \to (X\times Y)^{n-1} \to e^{p-1}_\beta \times e^q_a \cong S^{n-1} \forall \beta. \tag{2}$$

The boundary of the n-cell can be written as the fibered product

$$\partial(e^p_\alpha \times e^q_a) = \partial(e^p_\alpha)\times e^q_a \cup_{\partial e^p_\alpha \times \partial e^q_a} e^p_\alpha \times \partial(e^q_a).$$

Observe that

$$\partial(e^p_\alpha)\times e^q_a \xrightarrow{\psi_X \times id} e^{p-1}_\beta \times e^q_a$$
$$e^p_\alpha \times \partial(e^q_a) \xrightarrow{id\times\psi_Y} e^p_\alpha \times e^{q-1}_b$$

where ψ_X, ψ_Y are the attaching maps from X, Y with degree $d_{\alpha\beta}, d_{ab}$ respectively. In contrast, the following two maps are both the map collapsing down to a point.

$$\partial(e^p_\alpha)\times e^q_a \to e^p_\alpha \times e^{q-1}_b$$
$$e^p_\alpha \times \partial(e^q_a) \to e^{p-1}_\beta \times e^q_a$$

Hence, the magnitude of the degrees of maps (1) and (2) are $d_{ab}, d_{\alpha\beta}$ respectively.
Since $\partial e^q_a \subset e^q_a$, it also inherits its orientation, which is fixed by some ordered basis of its tangent space. Let $(y_1, \ldots, y_{q-1})$ be a basis of the tangent space of ∂e^q_a, then completing the basis to obtain $(v, y_1, \ldots, y_{q-1})$ for the tangent space of e^q_a. This is the ordered basis used in the attaching map ψ_Y.

Let $(x_1, \ldots, x_p)$ be a basis of the tangent space of e^p_α. Hence, the tangent space of $e^p_\alpha \times e^q_a$ has basis $(v, x_1, \ldots, x_p, y_1, \ldots, y_{q-1})$. This is the ordering of the basis used in $id \times \psi_Y$. In order to write the $D(id \times \psi_Y)$ as a block diagonal matrix, we need to first permute the basis via p number of transpositions

$$(v, x_1, \ldots, x_p, y_1, \ldots, y_{q-1}) \mapsto (x_1, \ldots, x_p, v, y_1, \ldots, y_{q-1}).$$

We know that the determinant of block diagonal matrix is the product of determinant of each block. Combining with the determinant of the permutation matrix, we obtain degree of $id \times \psi_Y = (-1)^p d_{ab}$.

For the other map $\psi_X \times id$, no permutation matrix is needed for

$$(v, x_1, \ldots, x_{p-1}, y_1, \ldots, y_q) \mapsto (v, x_1, \ldots, x_{p-1}, y_1, \ldots, y_q).$$

Hence we have

$$\begin{aligned} d(e^p_\alpha \times e^q_a) &= \sum_b (-1)^p d_{ab} e^p_\alpha \times e^{q-1}_b + \sum_\beta d_{\alpha\beta} e^{p-1}_\beta \times e^q_a \\ &= (-1)^p e^p_\alpha \times (\sum_b d_{ab} e^{q-1}_b) + (\sum_\beta d_{\alpha\beta} e^{p-1}_\beta) \times e^q_a \\ &= (-1)^p e^p_\alpha \times (\partial e^q_a) + (\partial e^p_\alpha) \times e^q_a \end{aligned}$$

□

Problem 2. *Prove that Euler characteristic of compact spaces is multiplicative*

$$\chi(X \times Y) = \chi(X)\chi(Y),$$

where $\chi(X) = \sum(-1)^i \beta_i$ *and* β_i *is the rank of* $H_i(X)$ *(known as the i-th Betti number).*

Solution. For a topological space Z, let's denote $\beta_i(Z) := \text{rank } H_i(Z)$. Since X and Y are compact cell complexes, $X \times Y$ is also a compact cell complex and the homology groups $H_k(X), H_k(Y)$'s and $H_k(X \times Y)$'s are finitely generated, hence their ranks are finite and only finitely many homology groups have non-zero ranks. Therefore all these spaces has well defined Betti numbers; hence Euler characteristics. Also $\text{Tor}(A, B) = \text{Tor}(T_A, B)$ where $T_A \oplus \mathbb{Z}^\alpha = A$ is decomposition into torsion and free parts because for free groups $\text{Tor}(\mathbb{Z}^\alpha, B) = 0$. Since the torsion groups T_A and T_B of $A = H_p(X)$ and $B = H_q(Y)$ are finite direct sum of finite cyclic groups, $\text{Tor}(A, B)$ is a torsion group because for cyclic groups we have $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \ker[\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}]$ is a finite cylic group; hence $\text{rank}(\text{Tor}(A, B)) = 0$.

By the topological Künneth theorem, we have following split short exact sequence;

$$0 \to \bigoplus_{p+q=k} H_p(X) \otimes H_q(Y) \to H_k(X \times Y) \to \bigoplus_{p+q=k} \text{Tor}(H_p(X), H_q(Y)) \to 0.$$

Since this splits and rank is additive over direct sum, if $G_k = \bigoplus_{p+q=k} H_p(X) \otimes H_q(Y)$ we have,

$$\begin{aligned}
\text{rank}(H_k(X \times Y)) &= \text{rank}(G_k) + \sum_{p+q=k} \text{rank}(\text{Tor}(H_p(X), H_q(Y)) \\
&= \text{rank}(G_k).
\end{aligned}$$

Now let's compute the rank of G_k using the additivity of rank over direct sums and multiplicativity over the tensor product;

$$\text{rank } G_k = \sum_{p+q=k} \text{rank}(H_p(X) \otimes H_q(Y)) = \sum_{p+q=k} \beta_p(X)\beta_q(Y)$$

Therefore,

$$\begin{aligned}
\chi(X \times Y) &= \sum_{k \geq 0} (-1)^k \text{rank } G_k \\
&= \sum_{k \geq 0} (-1)^k \sum_{p+q=k} \beta_p(X)\beta_q(Y) \\
&= \sum_{p,q \geq 0} (-1)^p \beta_p(X)(-1)^q \beta_q(Y) \\
&= \chi(X)\chi(Y).
\end{aligned}$$

□

Problem 3. *Fix n, and let X_d be the space obtained from attaching an n-cell to S^{n-1} by a map of degree d. Use Künneth to compute the homology of $X_d \times X_{d'}$ for any d, d'.* [2]

Solution. First we'll compute the homology of X_d:

$$\tilde{H}_p(X_d) = \begin{cases} \mathbb{Z}/d\mathbb{Z} & p = n-1 \\ \mathbb{Z} & p = 0 \\ 0 & \text{else} \end{cases}$$

By Kunneth, we know that

$$H_k(X_d \times D_{d'}) \cong \left(\bigoplus_{p+q=k} (H_p(X_d) \otimes H_q(X_{d'})) \right) \oplus \left(\bigoplus_{p+q=k-1} \operatorname{Tor}(H_p(X_d), H_q(X_{d'})) \right)$$

Let's start by computing the tensor products. Observe that if either $H_p(X_d) = 0$ or $H_q(X_{d'}) = 0$ then the tensor product will be 0. Thus we only get a few nonzero terms:

- If $p = q = 0$, then we have a term $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$.
- If $p = 0$ and $q = n-1$ we get $\mathbb{Z} \otimes \mathbb{Z}/d'\mathbb{Z} \cong \mathbb{Z}/d'\mathbb{Z}$.
- If $p = n-1$ and $q = 0$ we get $\mathbb{Z}/d\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.
- if $p = q = n-1$, then we have $\mathbb{Z}/d\mathbb{Z} \otimes \mathbb{Z}/d'\mathbb{Z} \cong \mathbb{Z}/\gcd(d,d')\mathbb{Z}$. For convenience I denote $\ell = \gcd(d, d')$.

When we consider the torsion component, we know that $\operatorname{Tor}(A, B)$ is nontrivial if and only if A and B both have a nontrivial torsion component. In our setup, this only happens when $p = q = n-1$. In this case, we have that

$$\operatorname{Tor}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/d'\mathbb{Z}) \cong \ker(\mathbb{Z}/d'\mathbb{Z} \xrightarrow{d} \mathbb{Z}/d'\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}$$

For the second isomorphism, write $d' = \ell \cdot r$. Then we know that $n \in \mathbb{Z}/d'\mathbb{Z}$ is in the kernel if and only if $d' | dn$ so that both $\ell | dn$ and $r | dn$. We know $\ell | d$ and $r \nmid d$ by definition of gcd, $d' | dn$ necessarily implies that $r | n$, and $r | n$ is clearly a sufficient condition. Thus the kernel consists of elements of the form $r \cdot \mathbb{Z}/d'\mathbb{Z}$. An isomorphism $r \cdot \mathbb{Z}/d'\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}$ is given by division by r, with the reverse map being multiplication by r. In any case, this shows that the only contribution from Tor is when $p = q = n-1$, corresponding to $k = 2n-1$. Overall, this tells us that

$$H_k(X_d \times X_{d'}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d'\mathbb{Z} & k = n-1 \\ \mathbb{Z}/\ell\mathbb{Z} & k = 2n-2 \\ \mathbb{Z}/\ell\mathbb{Z} & k = 2n-1 \\ 0 & \text{else} \end{cases}$$

□

[2] The answer should be an (explicit) abelian group that depends on d, d'.

Problem 4. *Use the acyclic models method to prove:*

(a) There is a natural chain map $S_*(X\times Y) \to S_*(X)\otimes S_*(Y)$ with the property that $\theta(x,y) = x\otimes y$ for $(x,y) \in X\times Y \subset S_0(X\times Y)$.

(b) Any two natural chain maps $\phi, \psi : S_*(X)\otimes S_*(Y) \to S_*(X)\otimes S_*(Y)$ that agree in degree 0 are chain homotopy equivalent.

Solution.

(a) Let $(x,y) \in X\times Y \subset S_0(X\times Y)$ and define $\theta(x,y) = x\otimes y$. Then, $\partial\theta(x,y) = 0\otimes 0 = \theta\partial(x,y)$. Now, suppose that θ is defined for degrees less than k for a certain $k > 0$ and that $\partial\theta = \theta\partial$ for those degrees.

Let Δ^k be the standard k-simplex and let $d_k : \Delta^k \to \Delta^k \times \Delta^k$ be the diagonal map, so that $d_k \in S_k(\Delta^k\times\Delta^k)$. Since $\partial d_k \in S_{k-1}(\Delta^k\times\Delta^k)$, by the induction hypothesis $\theta(\partial d_k)$ is defined and $\partial\theta(\partial d_k) = \theta(\partial^2 d_k) = 0$, so $\theta(\partial d_k)$ is a cycle. Since Δ^k is contractible, $S_*(\Delta^k)\otimes S_*(\Delta^k)$ is acyclic, i.e. $\tilde{H}_*(S_*(\Delta^k)\otimes S_*(\Delta^k)) = 0$, so $\theta(\partial d_k)$ is also a boundary, and thus there exists σ such that $\theta(\partial d_k) = \partial\sigma$. Define $\theta(d_k) = \sigma$.

Let $\pi_X : X\times Y \to X$ and $\pi_Y : X\times Y \to Y$ be the projection maps. Consider a singular k-simplex $\tau : \Delta^k \to X\times Y$ and the map $\pi_X\tau\times\pi_Y\tau : \Delta^k\times\Delta^k \to X\times Y$ which induces a chain map $(\pi_X\tau, \pi_Y\tau)_\# : S_*(\Delta^k\times\Delta^k) \to S_*(X\times Y)$. We see that $\tau = (\pi_X\tau, \pi_Y\tau)_\#(d_k)$ because $(\pi_X\tau\times\pi_Y\tau)\circ d_k = \tau$. Then, since $\theta(\tau) = \theta((\pi_X\tau,\pi_Y\tau)_\#(d_k))$, θ is a natural chain map if we define $\theta(\tau) = ((\pi_X\tau)_\#\otimes(\pi_Y\tau)_\#)(\theta(d_k))$. Moreover,

$$\begin{aligned}\partial\theta(\tau) &= \partial(((\pi_X\tau)_\#\otimes(\pi_Y\tau)_\#)(\theta(d_k))) \overset{\text{chain map}}{=} ((\pi_X\tau)_\#\otimes(\pi_Y\tau)_\#)(\partial\theta(d_k)) \overset{\text{definition of }\theta}{=} \\ &= ((\pi_X\tau)_\#\otimes(\pi_Y\tau)_\#)(\theta(\partial d_k)) \overset{\text{naturality}}{=} \theta((\pi_X\tau,\pi_Y\tau)_\#(\partial d_k)) \overset{\text{chain map}}{=} \\ &= \theta(\partial(\pi_X\tau,\pi_Y\tau)_\#(d_k)) = \theta\partial(\tau).\end{aligned}$$

Therefore, we have inductively defined θ to be a natural chain map.

(b) Let $\phi,\psi : S_*(X)\otimes S_*(Y) \to S_*(X)\otimes S_*(Y)$ be natural chain maps that agree in degree 0. We will inductively construct a chain homotopy D with $D\partial + \partial D = \phi - \psi$. We start by defining D to be zero on 0-chains. Since ϕ and ψ agree in degree 0, $D\partial + \partial D = 0 = \phi - \psi$ in degree 0. Now, suppose that D is defined for degrees less than k for a certain $k > 0$. Let $i_n \in S_n(\Delta^k)$ be the identity map and let $p + q = k$. Then,

$$\begin{aligned}\partial(\phi-\psi-D\partial)(i_p\times i_q) &= \partial\phi(i_p\times i_q) - \partial\psi(i_p\times i_q) - \partial D\partial(i_p\times i_q) \overset{\text{chain map}}{=} \\ &= \phi\partial(i_p\times i_q) - \psi\partial(i_p\times i_q) - \partial D(\partial(i_p\times i_q)) \overset{\text{induction}}{=} \\ &= \phi\partial(i_p\times i_q) - \psi\partial(i_p\times i_q) - (\phi-\psi-D\partial)(\partial(i_p\times i_q)) \overset{\partial^2=0}{=} 0,\end{aligned}$$

so $(\phi-\psi-D\partial)(i_p\times i_q)$ is a cycle. Since $S_*(\Delta^k)\otimes S_*(\Delta^k)$ is acyclic, there exists σ such that $(\phi-\psi-D\partial)(i_p\times i_q) = \partial\sigma$. We define $D(i_p\times i_q) = \sigma$. Now, consider singular simplices $\tau : \Delta^p \to X$ and $\kappa : \Delta^q \to Y$, so that $(\tau,\kappa)_\#(i_p\times i_q) = \tau_\#(i_p)\times\kappa_\#(i_q) = \tau\times\kappa$, and define $D(\tau\times\kappa) = (\tau_\#\otimes\kappa_\#)(D(i_p\times i_q))$. We extend the definition of D to $S_*(X)\otimes S_*(Y)$ by linearity, which implies that $D\partial + \partial D = \phi - \psi$ is satisfied in degree k.

Thus, we conclude that ϕ and ψ are chain homotopy equivalent.

□