Problem 1. *Let X be a finite graph, and let $T \subset X$ be a maximal tree. Prove that $H^1(X;\mathbb{Z})$ is isomorphic to the group of cochains supported on the edges of $X \setminus T$.* [1]

Solution. As in Hatcher, we denote the set of all functions from vertices of X to $\mathbb{Z}$ by $\Delta^0(X;\mathbb{Z})$ and the set of all function from edges of X to $\mathbb{Z}$ by $\Delta^1(X;\mathbb{Z})$. We note that both sets form abelian groups.

Let $G \subset \Delta^1(X;\mathbb{Z})$ be the subgroup of cochains supported on the edges of $X \setminus T$, that is, $G = \{f \in \Delta^1(X;\mathbb{Z}) : f(e) = 0 \ \forall e \in T\}$. Consider the map $\varphi : G \to H^1(X;\mathbb{Z})$ defined by $\varphi(f) = [f]$. We will see that φ is a group isomorphism by seeing that it is a bijective group homomorphism. φ is an homomorphism because it is the restriction of the quotient map $\Delta^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z})$, which is a (surjective) group homomorphism. Let us see that it is bijective:

- *φ is injective:* If $[f] = [g]$ for $f, g \in G$, then there is $h \in \Delta^0(X;\mathbb{Z})$ such that $f - g = \delta h$. Let $e \in X \setminus T$ be an edge from u to v. Then, $f(e) - g(e) = \delta h(e) = h(\partial e) = h(v) - h(u)$. Since T is a maximal tree, there exists a unique path from u to v in the tree. By induction along this path, we see that $h(u) = h(v)$ because f and g are zero on the edges of T. Thus, $f(e) = g(e)$, and we conclude that φ is injective.

- *φ is surjective:* Let $[f] \in H^1(X;\mathbb{Z})$ for $f \in \Delta^1(X;\mathbb{Z})$. Let u_0 be any vertex of T and let $n \in \mathbb{Z}$. Define $h \in \Delta^0(X;\mathbb{Z})$ in the following way: set $h(u_0) = n$ and define $h(v)$ for every edge $e \in T$ by induction on the edges of the unique path in T from u_0 to v, imposing that $f(\tilde{e}) = h(\partial\tilde{e})$ for every edge $\tilde{e}$ in the path. This uniquely determines h on each vertex of X up to a constant n. Now, define $g \in G$ by $g(e) = 0$ if $e \in T$ and $g(e) = h(u) - h(v) + f(e)$ if $e \in X \setminus T$ from u to v. Then, $f - g = \delta h$, so $[f] = [g]$ and we conclude that φ is surjective.

□

[1] Hint: Using cellular chains/cochains/cohomology, there is a map from the latter to the former. Show it is an isomorphism.

Problem 2. *Let $S_*(X)$ denote the singular chain complex of a connected space X. Regard a homomorphism $f : S_1(X) \to M$ as an M-valued function on paths in X. Assume f is a cocycle, and prove the following:*

(a) $f(\alpha * \beta) = f(\alpha) + f(\beta)$ ($*$ = concatenation of paths)

(b) f is zero on constant paths

(c) f takes the same value on homotopic paths

(d) f is a coboundary if and only if $f(\alpha)$ depends only on the endpoints of α, for every α.

Deduce that there is a homomorphism $H^1(X; M) \to Hom(\pi_1(X), M)$. What does the universal coefficient theorem say about it?

Solution. (a) As f is a cocycle, we have that $\delta f = 0$. There is a natural 2-simplex $p : \Delta^2 \to X$ for which the images of the oriented sides are the paths α, β and the concatenation $\alpha * \beta$. The boundary of this 2-simplex is equal to $\beta - (\alpha * \beta) + \alpha$. Since

$$0 = \delta f(p) = f(\partial p) = f(\beta) - f(\alpha * \beta) + f(\alpha),$$

we get the desired formula.

(b) Given any constant path α equal to a point, we consider the 2-simplex $p : \Delta^2 \to X$ that is constantly equal to this point. Naturally $\partial p = \alpha - \alpha + \alpha = \alpha$. Then

$$0 = \delta f(p) = f(\partial p) = f(\alpha),$$

showing the desired property.

(c) If two paths α and β are homotopic relative to their boundary, their homotopy can also be seen through the map of a 2-simplex $p : \Delta^2 \to X$ whose oriented sides are α, the constant path at one of the endpoints, and β. As again $\delta f(p) = 0$ and $\partial p = \gamma - \beta + \alpha$, where γ is the constant path at the endpoint, we see from the previous properties that $f(\alpha) = f(\beta)$.

(d) Suppose that f is a coboundary, $f = \delta g$. Then for any path α, with endpoints oriented from u to v, we get that

$$f(\alpha) = \delta g(\alpha) = g(v) - g(u),$$

so that the value of α under f depends only it is endpoints. Conversely, assume that for every path α, $f(\alpha)$ depends only on its endpoints. We fix a point $u_0 \in X$, and define a function $g : X \to M$ given by $g(v) = f(\alpha)$, where α is any oriented path joining u_0 to v (this assumes path-connectedness of X). This is well-defined by the hypothesis that f only depends on the endpoints of the path. Let β be an oriented path with endpoints u and v. If γ is a path from u_0 to u, $\gamma * \beta$ is a path from u_0 to v, so that

$$f(\beta) = f(\gamma * \beta) - f(\gamma) = g(v) - g(u) = g(\partial \beta) = \delta g(\beta),$$

so that $f = \delta g$.

With all of these facts proven, We see that any cocycle f induces a group homomorphism from $\pi_1(X)$ to M, since the cocycle respects concatenation of loops viewed as paths, is zero on constant paths and is well-defined with respect to the quotient by homotopy classes of loops. Moreover, if the cocycle $f = \delta g$ is a coboundary, since the evaluation of depends only on the endpoints of the path, it will be evaluated to 0 on any loop; so this map descends to the quotient as a group homomorphism $H^1(X;M) \to \mathrm{Hom}(\pi_1(X), M)$.

We recall that, assuming path-connectedness, there is a natural surjective map $\pi_1(X) \to H_1(X)$ that induces an isomorphism $\pi_1(X)^{ab} \cong H_1(X)$, and also that there is a natural identification $\mathrm{Hom}(\pi_1(X), M) \cong \mathrm{Hom}(\pi_1(X)^{ab}, M) \cong \mathrm{Hom}(H_1(X), M)$, since M is abelian. Moreover, since $H_0(X)$ is free abelian, we get that $\mathrm{Ext}(H_0(X), M) = 0$, so that by the universal coefficient theorem we have $H^1(X;M) \cong \mathrm{Hom}(H_1(X), M)$. The map $H^1(X;M) \to \mathrm{Hom}(\pi_1(X), M)$ described previously in fact coincides with the map from the universal coefficient theorem up to these isomorphisms, showing that it itself must be an isomorphism.

□

Problem 3. *Let M be an abelian group. Show that $Hom(-, M)$ preserves split short exact sequences of abelian groups, i.e. if $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is a split short exact sequence, then*

$$0 \to Hom(C, M) \xrightarrow{p^*} Hom(B, M) \xrightarrow{i^*} Hom(A, M) \to 0$$

is also a split short exact sequence.[2] *Deduce*[3] *from this (and the zig-zag lemma) that there is a long exact sequence in cohomology, i.e. given a pair (X, A) (that is X is a space and $A \subset X$ is a subspace, possibly with some mild assumptions), then there is a long exact sequence*

$$\cdots \to H^k(X, A; M) \to H^k(X; M) \to H^k(A; M) \to H^{k+1}(X, A; M) \to \cdots$$

Here $H^k(X, A; M)$ is defined by dualizing the chain complex $S_k(X, A) := S_k(X)/S_k(A)$. [4]

Solution. We first show that the dual sequence is exact. From the Moral exercise we know that p^* is injective, and $\text{Ker}(i^*) = \text{Im}(p^*)$, so it suffices to show that i^* is surjective. First, since $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ splits, there exists $s : B \to A$ such that $si = \mathbb{1}_A$. Now, suppose we have $f : A \to M$, define $g : B \to M$ as $g = s^*f$. Notice that $i^*g = i^*(s^*f) = (si)^*f = \mathbb{1}_A^*f = f$. This shows i^* is surjective, so the sequence

$$0 \to \text{Hom}(C, M) \xrightarrow{p^*} \text{Hom}(B, M) \xrightarrow{i^*} \text{Hom}(A, M) \to 0$$

is exact. To see that it splits, observe that $s^* : \text{Hom}(A, M) \to \text{Hom}(B, M)$ satisfies $i^*s^* = (si)^* = (\mathbb{1}_A)^*$ is the identity on $\text{Hom}(A, M)$, so by the splitting lemma the above short exact sequence splits.

To show that there is a long exact sequence of cohomology, we start by showing that $0 \to S_k(A) \to S_k(X) \to S_k(X, A) \to 0$ splits. Recall that $S_k(X)$ is free, so we may define a projection homomorphism $p : S_k(X) \to S_k(A)$ by sending each singular k-simplex $\sigma \in S_k(X)$ to σ if $\text{Im}(\sigma) \subset A$, or 0 if $\text{Im}(\sigma) \not\subset A$. It clearly follows that $pi = \mathbb{1}_{S_k(A)}$, where $i : S_k(A) \to S_k(X)$ is the inclusion homomorphism. By the splitting lemma, the exact sequence $0 \to S_k(A) \to S_k(X) \to S_k(X, A) \to 0$ splits. This implies we have a dual split short exact sequence

$$0 \to \text{Hom}(S_k(X, A), M) \to \text{Hom}(S_k(X), M) \to \text{Hom}(S_k(A), M) \to 0.$$

Then, by the zigzag lemma, the homology groups of these three chain complexes form a long exact sequence, which is the one desired in the problem statement. □

[2] You may assume that $Hom(-, M)$ is left exact, c.f. the related Moral Exercise.
[3] You will need to explain why the exact sequence $0 \to S_k(A) \to S_k(X) \to S_k(X, A) \to 0$ splits.
[4] Hint: it may help to first remember how you derived the long exact sequence in homology in 2410.

Problem 6. *Let G be a group and let M be a $\mathbb{Z}[G]$ module, where $\mathbb{Z}[G]$ denotes the group ring. The cohomology of G with coefficients in M is defined as*

$$H^k(G; M) := Ext^k_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

(a) *Prove that $H^0(G; M)$ is isomorphic to the fixed submodule*

$$M^G := \{x \in M : gx = x \text{ for all } g \in G\}.$$

(b) *For $G = \mathbb{Z}/2\mathbb{Z}$, identifying $\mathbb{Z}[G] \cong \mathbb{Z}[t]/(t^2-1)$, there is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$ modules given by*

$$\cdots \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{t+1} \mathbb{Z}[G] \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is the augmentation map (in terms of polynomials, it is evaluation at 1), and the other maps are multiplication by the element given in the label.

Use this resolution to compute $H^*(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z})$.

Solution. (a)

$$H^0(G; M) = Ext^0_{\mathbb{Z}[G]}(\mathbb{Z}, M) = Hom_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

Let $f \in Hom_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. Then $f(1) = x$ for some $x \in M$. For any $g \in G$, $gx = gf(1) = f(g \cdot 1) = f(1) = x$. Hence, $x \in M^G$. Hence, there is a map $\phi : H^0(G; M) \to M^G$ mapping $f \mapsto f(1)$.

(This is indeed a $Z[G]-$module homomorphisms since for any $f, h \in H^0(G; M)$ and any $g \in G$, $\phi(f+h) = (f+h)(1) = f(1)+h(1) = \phi(f)+\phi(h)$ and $\phi(g \cdot f) = (g \cdot f)(1) = gf(1) = g\phi(f)$.)

For injectivity, if $f(1) = h(1)$ for some $f, h \in H^0(G; M)$, then $f(n) = nf(1) = nh(1) = h(n)$ for all $n \in \mathbb{Z}$. So $f = h$.

For surjectivity, take any $x \in M^G$ and define $f : \mathbb{Z} \to M$ via $f(n) = nx$ all $n \in \mathbb{Z}$. Then, for any $m, n \in \mathbb{Z}$ and $g \in G$, $f(n+m) = (n+m)x = nx+mx = f(n)+f(m)$ and $f(g \cdot n) = f(n) = nx = gnx = gf(n)$ since $x \in M^G$.

(b) There is an isomorphism $\psi : Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) \to \mathbb{Z}$ given by $f \to f(1)$. If $f, h \in Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z})$ satisfy $f(1) = h(1)$, then $f(t) = tf(1) = f(1) = h(1) = th(1) = h(t)$ and

$$f(n+mt) = (n+mt)f(1) = (n+m)f(1) = (n+m)h(1) = (n+mt)h(1)) = h(n+mt)$$

for any $n+mt \in \mathbb{Z}[G]$. Given any n, we can also define $f : \mathbb{Z}[G] \to Z$ with $f(n+mt) = n+m$. Hence, ψ is an isomorphism. In other words, $Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) = \mathbb{Z} \cdot \epsilon$.

Dualize the free resolution in part (b) gives

$$\cdots \xrightarrow{(t-1)^*=0} \mathbb{Z} \xrightarrow{(t+1)^*=2} \mathbb{Z} \xrightarrow{(t-1)^*=0} \mathbb{Z} \to 0.$$

with $(t-1)^*(\epsilon) = \epsilon \circ (t-1) = 0$ and $(t+1)^*(\epsilon) = \epsilon \circ (t+1) = 2\epsilon$. Take the homology to obtain:

$$H^k(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \neq 0 \text{ even} \\ 0, & \text{otherwise} \end{cases},$$

with cohomology ring $H^*(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}[\alpha]/(2\alpha)$ where $|\alpha| = 2$, agreeing with$H^*(\mathbb{R}P^\infty; \mathbb{Z})$ from page 222 in Hatcher.

□

Problem 6. *Let G be a group and let M be a $\mathbb{Z}[G]$ module, where $\mathbb{Z}[G]$ denotes the group ring. The cohomology of G with coefficients in M is defined as* [7]

$$H^k(G;M) := Ext^k_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

(a) *Prove that $H^0(G;M)$ is isomorphic to the fixed submodule* [8]

$$M^G := \{x \in M : gx = x \text{ for all } g \in G\}.$$

(b) *For $G = \mathbb{Z}/2\mathbb{Z}$, identifying $\mathbb{Z}[G] \cong \mathbb{Z}[t]/(t^2-1)$, there is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$ modules given by*

$$\cdots \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{t+1} \mathbb{Z}[G] \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is the augmentation map (in terms of polynomials, it is evaluation at 1), and the other maps are multiplication by the element given in the label.

Use this resolution to compute $H^*(\mathbb{Z}/2\mathbb{Z};\mathbb{Z})$. [9]

Solution. $H^0(G;M) := \mathrm{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, M) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. A map $\mathbb{Z} \to M$ is determined by where it sends 1. This gives an injection $i : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) \to M$. A value $m \in M$ is a valid target for 1 if and only if the action of G of m is compatible with that of G on 1. In other words, one requires G to fix m. Thus, $\mathrm{im}\, i = M^G$. Injections are isomorphisms onto images, so part (a) is proven.

To compute $H^*(\mathbb{Z}/2\mathbb{Z};\mathbb{Z})$, we must dualize the given resolution and compute the homology. Dualizing gives

$$... \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(0, \mathbb{Z})$$

The image of 1 may be freely specified, and doing so determines a $\mathbb{Z}[G]$-module map $\mathbb{Z}[G] \to \mathbb{Z}$. Thus, replacing groups with isomorphic ones, we have

$$... \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$$

The maps are pullbacks of $t-1$ and $t+1$. Composing $\epsilon \circ (t-1)$ is the 0 map, so the rightmost $\mathbb{Z} \leftarrow \mathbb{Z}$ is a 0 map. Composing $\epsilon \circ (t+1)$ sends 1 to 2, so the second rightmost $\mathbb{Z} \leftarrow \mathbb{Z}$ is a multiplication-by-2 map. This pattern repeats, so the resolution goes

$$... \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0.$$

Taking homology of this chain complex is straightforward, and gives

$$H^k(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is non-zero even.} \end{cases}$$

□

[7] Small clarification from class for the general definition of $Ext^k_R(N, M)$. The formula $Ext^k_R(N, M) := H^k(C_*, M)$ from class needs to be interpreted properly: here C_* is a chain complex of R-modules, and $H^k(C_*; M)$ should be defined as the cohomology of the chain complex $Hom_R(C_k, M)$ of ***R*-module** maps (the groups $Hom_R(C_k, M)$ form a chain complex of abelian groups). This more general interpretation of $H^k(C_*; M)$ might not be standard.

[8] Recall that there is a general fact about $Ext^0_R(N, M)$, c.f. the relevant Moral Exercise.

[9] Aside: your computation should agree with the cohomology of $\mathbb{R}P^\infty$. This is because an equivalent way to define $H^k(G; M)$ (when the module M is trivial) is as the cohomology of the Eilenberg–Maclane space $K(G, 1)$ with coefficients in M, and here $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2\mathbb{Z}, 1)$.

Problem 6. *Let G be a group and let M be a $\mathbb{Z}[G]$ module, where $\mathbb{Z}[G]$ denotes the group ring. The cohomology of G with coefficients in M is defined as* [7]

$$H^k(G;M) := Ext^k_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

(a) Prove that $H^0(G;M)$ is isomorphic to the fixed submodule [8]

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(b) For $G = \mathbb{Z}/2\mathbb{Z}$, identifying $\mathbb{Z}[G] \cong \mathbb{Z}[t]/(t^2-1)$, there is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$ modules given by

$$\cdots \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{t+1} \mathbb{Z}[G] \xrightarrow{t-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the augmentation map (in terms of polynomials, it is evaluation at 1), and the other maps are multiplication by the element given in the label.

Use this resolution to compute $H^*(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z})$. [9]

Proof. (a) To calculate $H^0(G;M)$, we use the following set of equivalences outlined by Hatcher for H^0 and Ext^0 groups:

$$\text{Ext}^0(\mathbb{Z}, M) = H^0(G;M) = \text{Hom}_{Z[G]}(\mathbb{Z}, M). \tag{4}$$

$\mathbb{Z}[G]$-module homomorphisms from $\mathbb{Z} \to M$ are entirely determined by where 1 is sent, and satisfy $f(g \cdot n) = g \cdot f(n) = f(n)$. Then we send each map $f \in \text{Hom}_{Z[G]}(\mathbb{Z}, M) \mapsto f(1)$ such that $gf(1) = f(1)$, meaning that $f(1) \in M^G$. Since f is entirely determined by where 1 is sent, this map is injective. This map is also surjective since for any $x \in M^G$, we may send 1 to $x = f(1)$ such that $gf(1) = f(1)$, giving us a map in $\text{Hom}_{Z[G]}(\mathbb{Z}, M)$ that corresponds to x. Thus we have an isomorphism.

(b) Let $G = \langle t \mid t^2 = 1\rangle$ To compute the cohomology using the free resolution, we dualize and compute $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z})$. Given that we evaluate polynomials in t at 1, for $f \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z})$, we find that $f(1)$ entirely determines the map, and $t \mapsto \varepsilon(t) = 1$. Then $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], \mathbb{Z}) = \mathbb{Z} \cdot \varepsilon \cong \mathbb{Z}$. This gives us a dual chain

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(t-1)^*} \mathbb{Z} \xrightarrow{(t+1)^*} \mathbb{Z} \xrightarrow{(t-1)^*} \cdots$$

However, since t is evaluated by ε to be 1, the maps $(t-1)^*$ and $(t+1)^*$ correspond to the 0 map, and multiplication by 2 respectively.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \cdots$$

[7] Small clarification from class for the general definition of $Ext^k_R(N, M)$. The formula $Ext^k_R(N,M) := H^k(C_*, M)$ from class needs to be interpreted properly: here C_* is a chain complex of R-modules, and $H^k(C_*; M)$ should be defined as the cohomology of the chain complex $Hom_R(C_k, M)$ of ***R*-module** maps (the groups $Hom_R(C_k, M)$ form a chain complex of abelian groups). This more general interpretation of $H^k(C_*; M)$ might not be standard.

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