Exam practice problems

Problem. True or false. Be sure to explain your answer. (E.g. if the answer is true, you might give a direct argument or an explanation that refers to some theorem from class or homework. If the answer is false, you most likely should give a counterexample.)

(a) Gauss–Bonnet can be used to compute the area of the unit sphere.

(b) A geodesic $c : [a, b] \to S$ on a surface S has constant speed, i.e. $|c'|$ is constant.

(c) The covariant derivative $\nabla_c w$ of a tangent vector field w along a curve c on a surface is always tangent to the surface.

(d) Let $c = c(t)$ is a geodesic on a surface S and assume $w = w(t)$ is a parallel vector field along c. Then the angle between $w(t)$ and $c(t)$ is constant.

(e) Let S be the surface $x^2 + y^2 + 2z^2 = 1$. For any plane P through the origin, the intersection of P with S is a geodesic (when given a constant speed parameterization).

(f) Let $c : [0, 1] \to S$ and $\hat{c} : [0, 1] \to S$ be two curves with the same endpoints, i.e. $c(0) = p = \hat{c}(0)$ and $c(1) = q = \hat{c}(1)$. Fix a tangent vector $w_0 \in T_pS$ and write $w, \hat{w}$ for the parallel transport of w along $c, \hat{c}$, respectively. Then $w(1) = \hat{w}(1)$.

(g) The surfaces $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$ are isometric.

(h) By Gauss–Bonnet, there is no embedding of the 2-sphere in $\mathbb{R}^3$ with a point with negative Gaussian curvature.

Problem. Given an example or explain why there is no example.

(a) A surface S with a point $p \in S$ such that there are infinitely many geodesics that pass through p.

(b) A surface S with a point p such that there is no geodesic on S that passes through p.

(c) A unit speed curve on a cylinder that is not a geodesic.

(d) A tangent vector field w on the torus that is nowhere zero (i.e. $w(p) \neq 0$ for all $p \in S$).

Problem. Show that stereographic projection chart for the unit sphere is isothermal.

Problem. Fix $\theta \in (0, \pi)$, and let R be a triangle on the sphere whose sides are geodesics and with interior angles $\frac{\pi}{2}, \frac{\pi}{2}, \theta$. Compute the angle of parallel transport of vectors along the boundary of R.

Problem. Compute Christoffel symbols for a surface that's the graph of a function.