Some midterm solutions

Problem What is the image of the Gauss map for

- a cylinder

  Reasoning geometrically, for the cylinder $x^2 + y^2 = 1$ the image is the equator.

- a Mobius band

  Here the image is a closed curve, but not the equator. It takes some careful thought to figure out what it is.

- a cone

  For a cone $x^2 + y^2 = c^2 z^2$ the image is a latitude on the sphere in the southern hemisphere.

- the surface $x + y + z = 0$

  This is a plane. The image is a point. The normal is spanned by $(1, 1, 1)$ so it is this point normalized.

Problem True or false. Explain your answer.

- A surface with zero Gauss curvature is contained in a plane.

  False: cylinder.

- The Gauss map for a torus of revolution is surjective.

  True. Reason geometrically. Consider the image of each meridian circle.

- If you scale a surface by a constant, then the Gauss curvature scales by the same constant.

  False. It scales by $1/c^2$. The quick way to arrive here is to remember that a sphere of radius r has constant Gauss curvature $1/r^2$, which follows quickly if your remember the definition of $K = k_1 k_2$ and that (normal) curvature of a circle of radius r is $1/r$.

Problem Give an example or explain why none exists:

- A curve on the sphere with nonzero torsion.

  Recall the result from class that a curve has vanishing torsion if and only if it is planar. Thus any nonplanar curve will work.

- A curve on the sphere with zero curvature.

  This does not exist. Recall the formula for normal curvature of a curve on a surface $k_{c,S} = \kappa \cos\theta$. Here κ is the space curve curvature of the (unit-speed) curve c and θ is the angle between c'' and the surface normal N. For the unit sphere $k_{c,S} = 1$ for every curve (because $k_1 = k_2 = 1$), so the equation makes it impossible for $\kappa = 0$.

- A curve on the sphere of radius 1 with curvature $1/2$.

  This is ruled out by the same argument as the previous problem.

- A curve on the sphere of radius 1 with curvature 10.

  This is fine. Take a small latitude.

- A surface with first and second fundamental form equal to identity.

  No example exists. Suppose it did: 2nd FF equal to identity implies that Gauss curvature is identically 1. 1st FF equal to identity implies locally isometric to the plane. But the Gauss curvature is intrinsic (Theorem Egregium) and the plane has Gauss curvature identically 0, so this is a contradiction.

Problem Compute the curvature of

- the logarithmic spiral $c(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$ for $t \in [0, \infty)$.
- the curve $c(t) = (t, t^2, t^3)$.

The main thing you were supposed to realize is that these curves are not unit speed (e.g. on homework, you found a unit speed parameterization of the logarithmic spiral, and we have also discussed before that $t \mapsto (t, t^2)$ is not unit speed). So you need to use the formula $\kappa(t) = \frac{|c'(t) \times c''(t)|}{|c'(t)|^3}$.

Problem Compute first and second fundamental form for the surface

- $z = x^2 - 3xy + y^2$.

This is just a computation.

Problem Consider the curve $c(t) = (t, f(t), 0)$, where $f(t) > 0$ for all t. Let S be the surface of revolution obtained by revolving c about the x-axis. Compute the first and second fundamental form, the Gauss curvature, and explain when K is positive/negative in terms of f.

The surface is parameterized by

$$\phi(t, \theta) = \big(t, f(t)\cos(\theta), f(t)\sin(\theta)\big).$$

After computation, one finds that K is a function of f, f', f'' and the sign of K is opposite the sign of f'' (e.g. $K > 0$ when $f'' < 0$). This makes geometric sense. E.g. at a local min of f, $f'' > 0$ and the surface of revolution looks like a saddle.

Problem

- Show that a curve with zero torsion is contained in a plane.
- Show that a (connected) surface with second fundamental form $= 0$ is contained in a plane.

Both of these examples were from class (and I even told you that the latter would be a good midterm question).

Suppose c is a unit speed curve with zero torsion. Since $B' = \tau N$, this means $B' = 0$ and so B is constant. Consider the function $f(t) = \langle c(t) - c(0), B\rangle$. If we show $f = 0$, then every point on the curve c is contained in the plane containing $c(0)$ and orthogonal to B. Now one computes $f' = 0$ and evaluates $f(0) = 0$ to conclude.

The argument for a surface with 2nd FF equal to 0 is similar. The assumption implies that the surface normal is constant, and it suffices to show the function $f(t) = \langle c(t) - c(0), B\rangle$ is zero for every curve on the surface.