Problem 1. *Let v, w be vector fields along a curve $c : I \to S$. Prove*[1] *that*

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \langle \nabla_\alpha v, w\rangle + \langle v, \nabla_\alpha w\rangle.$$

Solution. $\frac{d}{dt}\langle v(t), w(t)\rangle = \langle v'(t), w(t)\rangle + \langle v(t), w'(t)\rangle$.

Each v' and w' can be broken down into their parts which are in the tangent space and the part parallel to the normal vector.

$\langle v', w\rangle + \langle v, w'\rangle = \langle \nabla_c v + \langle v', N\rangle N, w\rangle + \langle v, \nabla_c w + \langle w', N\rangle N\rangle$

$= \langle \nabla_c v, w\rangle + \langle v, \nabla_c w\rangle + \langle \langle v', N\rangle N, w\rangle + \langle v, \langle w', N\rangle N\rangle$

$= \langle \nabla_c v, w\rangle + \langle v, \nabla_c w\rangle + \langle v', N\rangle\langle w, N\rangle + \langle w', N\rangle\langle v, N\rangle$

If you assume v and w are tangent vector fields then $\langle v, N\rangle = \langle w, N\rangle = 0$

$\Rightarrow \frac{d}{dt}\langle v(t), w(t)\rangle = \langle \nabla_c v, w\rangle + \langle v, \nabla_c w\rangle$ □

[1] Hint: sometimes it's good to work in coordinates, and sometimes not..

Problem 2. *Let S be the cylinder $x^2 + y^2 = 1$ and let C be the curve obtained by intersecting S with the plane $x - z = 0$. Compute the geodesic curvature of C at the point $(1, 0, 1)$.*

Solution. First, we fix a chart for the cylinder to be $\phi(u, v) = (\cos u, \sin u, v)$. The unit normal is $(\cos u, \sin u, 0)$.

Now, note that the curve obtained by intersecting S with $x - z = 0$ is $f(u) = (\cos u, \sin u, \cos u)$. To determine its geodesic curvature, we derive it from its curvature κ at $(1, 0, 1)$ (that is, when $u = 0, v = 1$.)

Using our formula for curvature of non-unit-speed curves, we have

$$f'(u) = (-\sin u, \cos u, -\sin u) \qquad f''(u) = (-\cos u, -\sin u, -\cos u)$$
$$|f'(0)| = \sqrt{1 + \sin^2(0)} = 1 \qquad (f' \times f'')(0) = (-1, 0, 1) \qquad |f' \times f''| = \sqrt{2}$$

Hence $\kappa = \sqrt{2}$.

Now, we can use the fact that $\kappa_g = \kappa \sin\theta$ where θ is the angle of f'' with N. This works even though f is not unit speed, since f'' only has an additional tangential component, which disappears when we take the dot product, and thus has no bearing on the result.

Since $f''(0) = (-1, 0, -1)$ and $N(0) = (1, 0, 0)$, we have that $\theta = \frac{5\pi}{4}$, and thus

$$\kappa_g = \kappa \sin\left(\frac{5\pi}{4}\right) = \sqrt{2} \cdot \frac{-\sqrt{2}}{2} = -1$$

$\square$

Problem 3.

Let S be a surface, and suppose $\phi : U \to S$ is a coordinate chart whose first fundamental form satisfies $F = 0$ and $E = \lambda = G$ for some function λ.

(a) Prove that $\phi_{uu} + \phi_{vv}$ is orthogonal to ϕ_u and ϕ_v.

Proof:

By the coefficients of the FFF, we have

$$\begin{aligned} E &= \langle \phi_u, \phi_u \rangle = \lambda \\ F &= \langle \phi_u, \phi_v \rangle = 0 \\ G &= \langle \phi_v, \phi_v \rangle = \lambda \end{aligned}$$

By $\langle \phi_u, \phi_u \rangle = \lambda$, we have

$$\begin{aligned} \frac{\partial}{\partial u}\langle \phi_u, \phi_u \rangle &= 2\langle \phi_{uu}, \phi_u \rangle = \lambda_u \\ \frac{\partial}{\partial v}\langle \phi_u, \phi_u \rangle &= 2\langle \phi_{uv}, \phi_u \rangle = \lambda_v \end{aligned}$$

By $\langle \phi_v, \phi_v \rangle = \lambda$, we have

$$\begin{aligned} \frac{\partial}{\partial u}\langle \phi_v, \phi_v \rangle &= 2\langle \phi_{vu}, \phi_v \rangle = \lambda_u \\ \frac{\partial}{\partial v}\langle \phi_v, \phi_v \rangle &= 2\langle \phi_{vv}, \phi_v \rangle = \lambda_v \end{aligned}$$

Finally, by $\langle \phi_u, \phi_v \rangle = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial u}\langle \phi_u, \phi_v \rangle &= 0 \\ \langle \phi_{uu}, \phi_v \rangle + \langle \phi_u, \phi_{vu} \rangle &= 0 \\ \langle \phi_{uu}, \phi_v \rangle &= -\langle \phi_u, \phi_{uv} \rangle \\ \langle \phi_{uu}, \phi_v \rangle &= -\frac{1}{2}\lambda_v \end{aligned}$$

If we differentiate w.r.t v,

$$\begin{aligned} \frac{\partial}{\partial v}\langle \phi_u, \phi_v \rangle &= 0 \\ \langle \phi_{uv}, \phi_v \rangle + \langle \phi_u, \phi_{vv} \rangle &= 0 \\ \langle \phi_u, \phi_{vv} \rangle &= -\langle \phi_{uv}, \phi_v \rangle \\ \langle \phi_u, \phi_{vv} \rangle &= -\frac{1}{2}\lambda_u \end{aligned}$$

Finally, note that

$$\langle \phi_{uu} + \phi_{vv}, \phi_u \rangle = \langle \phi_{uu}, \phi_u \rangle + \langle \phi_{vv}, \phi_u \rangle = \frac{1}{2}\lambda_u + \left(-\frac{1}{2}\lambda_u\right) = 0$$

This tells us that $\phi_{uu} + \phi_{vv}$ is orthogonal to ϕ_u.

$$\langle \phi_{uu} + \phi_{vv}, \phi_v \rangle = \langle \phi_{uu}, \phi_v \rangle + \langle \phi_{vv}, \phi_v \rangle = \left(-\frac{1}{2}\lambda_v\right) + \frac{1}{2}\lambda_v = 0$$

This tells us that $\phi_{uu} + \phi_{vv}$ is orthogonal to ϕ_v.

■

(b) By (a), $\phi_{uu} + \phi(vv) = \mu N$ for some μ. Compute μ.

Proof:

Since $\phi_{uu} + \phi(vv)$ is orthogonal to both ϕ_u and ϕ_v, it's orthogonal to the surface. Then, by the property of dot products, we know

$$\mu = \langle \phi_{uu} + \phi(vv), N \rangle = \langle \phi_{uu}, N \rangle + \langle \phi_{vv}, N \rangle$$

But then, note that the coefficients of SFF are defined by $e = \langle \phi_{uu}, N \rangle$ and $g = \langle \phi_{vv}, N \rangle$, which tells us $\mu = e + g$.

The Mean Curvature is defined by

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)} = \frac{\lambda g - 0 + \lambda e}{2(\lambda \cdot \lambda - 0)} = \frac{e+g}{2\lambda}$$
$$e + g = 2\lambda H$$
$$\boxed{\mu = 2\lambda H}$$

(c) Show that if S is a minimal surface, then ϕ is harmonic, i.e. $\phi_{uu} + \phi_{vv} = 0$. Note: This is called an isothermal chart.

Proof:

Assume S is a minimal surface. Then, $H = 0$ on all points on the surface.

From (b), we have

$$\begin{aligned} \phi_{uu} + \phi_{vv} &= 2\lambda H N \\ \phi_{uu} + \phi_{vv} &= 2\lambda(0) N \\ \phi_{uu} + \phi_{vv} &= 0 \\ \Delta\phi &= 0 \end{aligned}$$

Therefore, by definition, ϕ is harmonic. ■

Problem 4. *Let $\phi : U \to S$ be an isothermal chart. Prove that*

$$K = \frac{-1}{2\lambda}\Delta(\log \lambda),$$

where $\Delta f = f_{uu} + f_{vv}$ is the Laplacian. [4]

Solution. We begin by calculating the Christoffel symbols in isothermal coordinates. They end up being nice and simple.

$$\Gamma_{11}^1 = \frac{\lambda_u}{2\lambda}, \quad \Gamma_{12}^1 = \frac{\lambda_v}{2\lambda}, \quad \Gamma_{22}^1 = -\frac{\lambda_u}{2\lambda}, \quad \Gamma_{11}^2 = -\frac{\lambda_v}{2\lambda}, \quad \Gamma_{12}^2 = \frac{\lambda_u}{2\lambda}, \quad \Gamma_{22}^2 = \frac{\lambda_v}{2\lambda}$$

We should also compute two partial derivatives because they appear in the equation we will use.

$$(\Gamma_{12}^2)_u = \frac{\lambda_{uu}(2\lambda) - \lambda_u(2\lambda_u)}{4\lambda^2} = \frac{\lambda_{uu}\lambda - \lambda_u^2}{2\lambda^2}$$

$$(\Gamma_{11}^2)_v = \frac{\lambda_{vv}(-2\lambda) - \lambda_v(-2\lambda_v)}{4\lambda^2} = \frac{\lambda_v^2 - \lambda_{vv}\lambda}{2\lambda^2}$$

Then we can compute the Gaussian curvature starting from the equation of the Theorema Egregium.

$$\begin{aligned} K &= \frac{-(\Gamma_{12}^2)_u + (\Gamma_{11}^2)_v + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2}{E} \\ &= \frac{1}{\lambda}\left(-\frac{\lambda_{uu}\lambda - \lambda_u^2}{2\lambda^2} + \frac{\lambda_v^2 - \lambda_{vv}\lambda}{2\lambda^2} + \frac{\lambda_u^2}{4\lambda^2} + \frac{\lambda_u\lambda_v}{4\lambda^2} - \frac{-\lambda_v^2}{4\lambda^2} - \frac{\lambda_u^2}{4\lambda^2}\right) \\ &= \frac{1}{2\lambda}\left(\frac{\lambda_u^2 + \lambda_v^2 - \lambda_{uu}\lambda - \lambda_{vv}\lambda}{\lambda^2}\right) \end{aligned}$$

Separately, we should compute $\Delta \log(\lambda)$.

$$\nabla \log(\lambda) = \frac{\lambda_u + \lambda_v}{\lambda}$$

$$\Delta \ln(\lambda) = \frac{\lambda_{uu}\lambda - \lambda_u^2 + \lambda_{vv}\lambda - \lambda_v^2}{\lambda^2}$$

This turns out to be the negative of the parenthetical we already calculated, so by pulling out the -1, we obtain what we wanted to show.

$$K = \frac{-1}{2\lambda}\Delta(\log \lambda)$$

□

[4] Hint: Use the formula for K in terms of E and the Christoffel symbols. Then write the Christoffel symbols in terms of E, F, G and their derivatives.

Problem 5.

Define the third fundamental form on T_pS by $\mathrm{III}_p = \langle DN_p(x),\, DN_p(y)\rangle$. Prove with an explicit formula that the third fundamental form can be expressed in terms of the first and second fundamental forms.

Proof

By the definitions of the fundamental forms,

$$\begin{aligned}\mathrm{I}_p(x,y) &= \langle x,y\rangle\\ \mathrm{II}_p(x,y) &= -\langle DN_p(x),y\rangle\\ \mathrm{III}_p(x,y) &= \langle DN_p(x),DN_p(y)\rangle\end{aligned}$$

Since DN_p is self-adjoint,

$$\mathrm{III}_p(x,y) = \langle DN_p(x),\, DN_p(y)\rangle = \langle (DN_p \circ DN_p)(x),\, y\rangle = \langle (DN_p)^2(x),\, y\rangle$$

The Cayley-Hamilton theorem states that a linear operator satisfies its own characteristic equation. Let $A = DN_p$. By the definition of DN_p, A is a 2×2 linear operator. Its characteristic polynomial $P(\lambda)$ is given by:

$$P(\lambda) = \det(A-\lambda I) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$$

(Notation note: I is FFF, I is the identity matrix)

By the Cayley-Hamilton theorem, $P(A) = 0$ (where 0 is the zero operator). Substituting the definitions of $K = \det(A)$ and $H = -\frac{1}{2}\,\mathrm{tr}(A)$ gives us

$$\begin{aligned}(DN_p)^2 - (-2H)DN_p + KI &= 0\\ (DN_p)^2 + 2HDN_p + KI &= 0\end{aligned}$$

Applying the operator equation above to a vector x and taking the inner product with y gives us

$$\begin{aligned}\langle \big((DN_p)^2 + 2HDN_p + KI\big)(x),\, y\rangle &= 0\\ \langle (DN_p)^2(x),\, y\rangle + 2H\langle DN_p(x),\, y\rangle + K\langle x,\, y\rangle &= 0\end{aligned}$$

Substituting the definitions of the fundamental forms,

$$\begin{aligned}\mathrm{III}_p(x,y) + 2H\big(-\mathrm{II}_p(x,y)\big) + K\mathrm{I}_p(x,y) &= 0\\ \mathrm{III}_p(x,y) &= 2H\mathrm{II}_p(x,y) - K\mathrm{I}_p(x,y)\end{aligned}$$

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Problem 6. *Let S be the hyperboloid $x^2 + y^2 = z^2 + 1$. Find a geodesic on S that is a straight line. Prove that S is a union of straight lines. Use this to give a method for boiling pasta that prevents the noodles from sticking to each other.*

Solution. To find a straight line geodesic, we intersect S with the plane $y = 1$, which yields $x^2 = z^2$. The curve $c(t) = (t, 1, t)$ is a straight line lying on S. Since $c''(t) = 0$, the acceleration is zero everywhere, trivially satisfying the geodesic equation.

To prove S is a union of straight lines, we define a family of lines $L_\theta$as

$$L_\theta(t) = (\cos\theta - t\sin\theta, \sin\theta + t\cos\theta, t).$$

Substituting these components into the equation $x^2 + y^2 - z^2 = 1$, we get that every such line lies entirely on S.

To show every point (x_0, y_0, z_0) on S belongs to a unique line, we set $t = z_0$ and solve the system $x_0 = \cos\theta - z_0 \sin\theta$ and $y_0 = \sin\theta + z_0 \cos\theta$. This can be viewed as a linear system for the variables $\cos\theta$ and $\sin\theta$ with determinant $1 + z_0^2$. Since the determinant is non-zero, there is a unique solution for θ, proving that S is the union of these disjoint lines.

This geometry provides a method for boiling pasta. Stirring the water creates a vortex shaped like a hyperboloid. Spaghetti noodles naturally align with the straight lines L_θ that generate the surface. Since we proved these lines are disjoint, the noodles remain separated while rotating, preventing them from sticking together. $\square$