Problem 1. *Assume $f : S_1 \to S_2$ is a differentiable bijection and is area-preserving*[1]. *Let $\phi : U \to S_1$ be a chart. Let E_i, F_i, G_i be the corresponding first fundamental form of S_i for $i = 1, 2$. Prove that $E_1G_1 - F_1^2 = E_2G_2 - F_2^2$.*

Solution. Let $\phi : U \subset \mathbb{R}^2 \to S_1$ be a local chart of S_1, and define

$$X_1 = \phi : U \to S_1, \qquad X_2 = f \circ \phi : U \to S_2.$$

Then X_1 and X_2 give parameterizations of S_1 and S_2 over the same domain U. Denote the coefficients of the first fundamental forms by

$$E_i = \langle (X_i)_u, (X_i)_u \rangle, \qquad F_i = \langle (X_i)_u, (X_i)_v \rangle, \qquad G_i = \langle (X_i)_v, (X_i)_v \rangle, \quad i = 1, 2.$$

For any region $V \subset U$, the area formula for a parametrized surface gives

$$\mathrm{Area}(X_i(V)) = \iint_V \sqrt{E_iG_i - F_i^2}\, du\, dv.$$

Now choose an arbitrary region $R \subset S_1$ and let $V = \phi^{-1}(R) \subset U$. Since f is area-preserving, we have

$$\mathrm{Area}(X_1(V)) = \mathrm{Area}(X_2(V)).$$

Thus,

$$\iint_V \sqrt{E_1G_1 - F_1^2}\, du\, dv = \iint_V \sqrt{E_2G_2 - F_2^2}\, du\, dv \qquad \text{for all } V \subset U.$$

Define

$$a(u, v) = \sqrt{E_1G_1 - F_1^2}, \qquad b(u, v) = \sqrt{E_2G_2 - F_2^2}.$$

The above identity shows that

$$\iint_V (a - b)\, du\, dv = 0 \quad \text{for every region } V \subset U.$$

By continuity, this implies $a \equiv b$ on U; otherwise we could choose a small V on which $a - b$ has a fixed sign, contradicting the integral equality.

Therefore,

$$\sqrt{E_1G_1 - F_1^2} = \sqrt{E_2G_2 - F_2^2},$$

and squaring both sides yields

$$E_1G_1 - F_1^2 = E_2G_2 - F_2^2.$$

This completes the proof. □

[1] i.e. area of $f(R)$ equals area of R for (reasonable) $R \subset S_1$

Problem 2. *True or false: the Mobius band from the first homework can be made out of paper. Explain your answer.*

Solution. This is false. It is true that you can take a flat strip of paper, twist it, and glue the edges together. However, if this was a true Mŏbius strip, it would imply that the Mŏbius strip is isometric to the plane, since bending and gluing the plane does not change lengths or distances on it. If these two surfaces were isometric, then they would both have a Gaussian curvature of 0 everywhere, since Gaussian curvature is preserved under isometry.

Let's take a look at the Gaussian curvature on the Mŏbius strip. First, we fix a chart for it. Define $\phi(t,\theta) : (-1,1) \times (0,\pi) \to S$ as

$$\phi(t,\theta) = (\cos 2\theta(2 + t\sin\theta), \sin 2\theta(2 + t\sin\theta), t\cos\theta)$$

Let's compute the Gaussian curvature at $p = (0, \frac{\pi}{2})$. We compute the partials:

$$\begin{aligned}
\phi_\theta(0,\pi/2) &= \frac{d}{d\theta}(2\cos 2\theta, 2\sin 2\theta, 0)\Big|_{\theta=\frac{\pi}{2}} \\
&= (-4\sin 2\theta, 4\cos 2\theta, 0)\Big|_{\theta=\frac{\pi}{2}} \\
&= (0,-4,0) \\
\phi_t(0,\pi/2) &= \frac{d}{dt}(\cos\pi(2 + t\sin(\pi/2)), \sin\pi(2 + t\sin(\pi/2)), t\cos(\pi/2))\Big|_{t=0} \\
&= (\sin(\pi/2)\cos\pi, \sin(\pi/2)\sin\pi, \cos(\pi/2))\Big|_{t=0} \\
&= (-1,0,0)
\end{aligned}$$

Thus, for the first fundamental form, we have $E = 4, F = 0, G = 1$. Computing the unit normal at p gives us $N_p = (0, 0 - 1)$. Now, we compute the second partials:

$$\begin{aligned}
\phi_{\theta\theta}(0,\pi/2) &= \frac{d}{dt}(-4\sin 2\theta, 4\cos 2\theta, 0)\Big|_{\theta=\frac{\pi}{2}} \\
&= (-8\cos\pi, -8\sin\pi, 0) \\
&= (-8,0,0) \\
\phi_{t\theta}(0,\pi/2) &= (\cos\theta\cos 2\theta + \sin\theta(-2\sin 2\theta), \cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta, -\sin\theta)\Big|_{t=0,\theta=\frac{\pi}{2}} \\
&= (0,-2,-1) \\
\phi_{tt}(0,\pi/2) &= (0,0,0)
\end{aligned}$$

Thus, computing the second fundamental form gives us

$$e = N \cdot \phi_{\theta\theta} = 0 \qquad f = N \cdot \phi t\theta = 1 \qquad g = N \cdot \phi_{tt} = 0$$

Therefore, the Gaussian curvature at this point is $K = \frac{eg-f^2}{EG-F^2} = -\frac{1}{16}$. We conclude that the Mŏbius strip cannot be isometric to the plane, so it cannot be made faithfully from paper.

□

Problem 3 (dC, 4.4.3). *Show that the surfaces $\phi(u,v) = (u\cos v, u\sin v, \log u)$ and $\psi(u,v) = (u\cos v, u\sin v, v)$ have the same Gauss curvature, but $\psi \circ \phi^{-1}$ is not an isometry. How does this relate to the Theorem Egregium?* [2]

Solution. For ϕ's Gaussian curvature:

$$\phi_u = \left(cosv, sinv, \frac{1}{u}\right) \qquad \text{and} \qquad \phi_v = (-usinv, ucosv, 0) \tag{7}$$

Meaning we can now compute the 2nd partials, and construct a normal vector:

$$\phi_{uu} = \left(0, 0, -\frac{1}{u^2}\right) \qquad \phi_{vv} = (-ucosv, -usinv, 0) \qquad \phi_{uv} = (-sinv, cosv, 0) \tag{8}$$

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-cosv, -sinv, u)}{\sqrt{u^2+1}} \tag{9}$$

From this information, we calculate the First and Second Fundamental Forms:

$$\boxed{\begin{aligned} E &= \langle \phi_u, \phi_u \rangle = 1 + \frac{1}{u^2} \\ F &= \langle \phi_u, \phi_v \rangle = 0 \\ G &= \langle \phi_v, \phi_v \rangle = u^2 \\ e &= \langle \phi_{uu}, N \rangle = -\frac{1}{u\sqrt{u^2+1}} \\ f &= \langle \phi_{uv}, N \rangle = 0 \\ g &= \langle \phi_{vv}, N \rangle = \frac{u}{\sqrt{u^2+1}} \end{aligned}}$$

From this, we find the Gaussian Curvature K to be:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\frac{u}{u(u^2+1)}}{u^2+1} = -\frac{1}{(u^2+1)^2} \tag{10}$$

For ψ's Gaussian curvature:

$$\psi_u = (cosv, sinv, 0) \qquad \text{and} \qquad \psi_v = (-usinv, ucosv, 1) \tag{11}$$

Meaning we can now compute the 2nd partials, and construct a normal vector:

$$\psi_{uu} = (0, 0, 0) \qquad \psi_{vv} = (-ucosv, -usinv, 0) \qquad \psi_{uv} = (-sinv, cosv, 0) \tag{12}$$

$$N = \frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|} = \frac{(sinv, -cosv, u)}{\sqrt{u^2+1}} \tag{13}$$

From this information, we calculate the First and Second Fundamental Forms:

[2] There is only one answer that is completely correct, so please think carefully.

Problem 4. *Let S be the torus of revolution. Determine which of the meridians and which of the longitudes on S are geodesics.* [3]

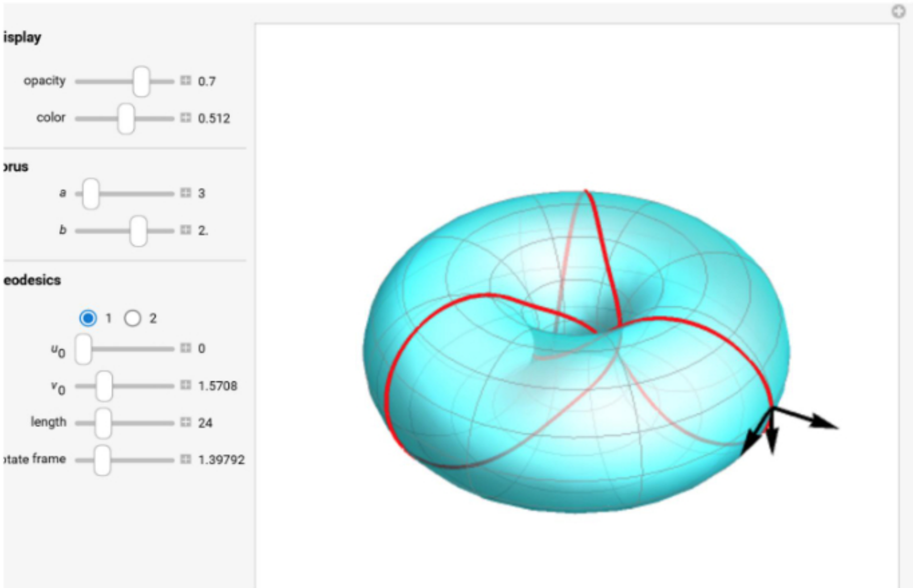
Solution. For any longitude or meridian, we can easily determine the direction of c'' because c'' will point towards the center of the circle (aka the longitude or meridian we are looking at).

The only longitudes that are geodesics are the very outer and inner circles of the torus (the two meridians with the same z value as the center of the circle that was revolved). On these circles, we can see that c'' points towards the center of the torus. On the outer circle, the normal also points towards the center. On the inner, it points away from the center. Thus, c'' is proportional to the normal on these circles, so they are geodesics. Any other longitude will not be geodesic because c'' will have a z component of 0, while the normal will have a non-zero z component.

Every meridian is a geodesic. In a meridian, we can see that c'' points towards the center of the meridian. Similarly, the normal is pointing towards the center of the meridian. Thus, c'' is proportional to the normal so they are geodesics. □

[3] If the revolution happens around the z-axis, longitudes are the circles on the torus with a fixed height (i.e. z-coordinate). Meridians are the circles that are being revolved.

Problem 5. *Show that the trefoil knot can be realized as a geodesic on a torus of revolution. Use* `https://demonstrations.wolfram.com/GeodesicsOfATorusSolvedWithAMethodOfLagrange/`*. Your solution should be a picture including all the parameters. Explain how you found your answer.* [4]



Solution.

I calculated that we would like a geodesic to go two times along the longitude circle of a torus and go 3 times along the meridian circle in order to form a trefoil. The easiest spot to start was on the outer circle ($1.5708 = \frac{\pi}{2}$), and from that point, I chose an appropriate angle.

[4]Hint: it may be helpful to think about how to find this systematically.