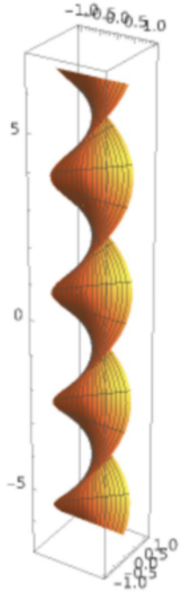


Problem 1. *The helicoid is the surface given by the chart*

$$\phi(u,v) = (v\cos u, v\sin u, u), \quad u,v \in \mathbb{R}.$$

Use a mathematica ***ParametricPlot3D*** *(or similar) to plot this surface. Compute (by hand) the first and second fundamental forms* I, II *and mean curvature* H *of this surface.*[1]

Solution. $\phi_u = (-v\sin u, v\cos u, 1)$ $\phi_v = (\cos u, \sin u, 0)$
$\phi_{uu} = (-v\cos u, -v\sin u, 0)$ $\phi_{vv} = (0,0,0)$ $\phi_{uv} = (-\sin u, \cos u, 0)$

$$\mathrm{I} = \begin{pmatrix} <\phi_u,\phi_u> & <\phi_u,\phi_v> \\ <\phi_v,\phi_u> & <\phi_v,\phi_v> \end{pmatrix} = \begin{pmatrix} v^2+1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\phi_u \times \phi_v = (-\sin u, \cos u, -v^2)$, $|\phi_u \times \phi_v| = \sqrt{v^2+1}$

$$N \circ \phi = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{1}{\sqrt{v^2+1}}(-\sin u, \cos u, -v^2)$$

$$\mathrm{II} = \begin{pmatrix} <N\circ\phi,\phi_{uu}> & <N\circ\phi,\phi_{uv}> \\ <N\circ\phi,\phi_{uv}> & <N\circ\phi,\phi_{vv}> \end{pmatrix} = \frac{1}{\sqrt{v^2+1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$H = -\frac{1}{2}\left(\frac{eG+gE-2fF}{EG-F^2}\right) = 0$$

□

[1] Hint: Your answer for the mean curvature, if correct, will be exceedingly simple.

Problem 2. *Consider the curve*[2]

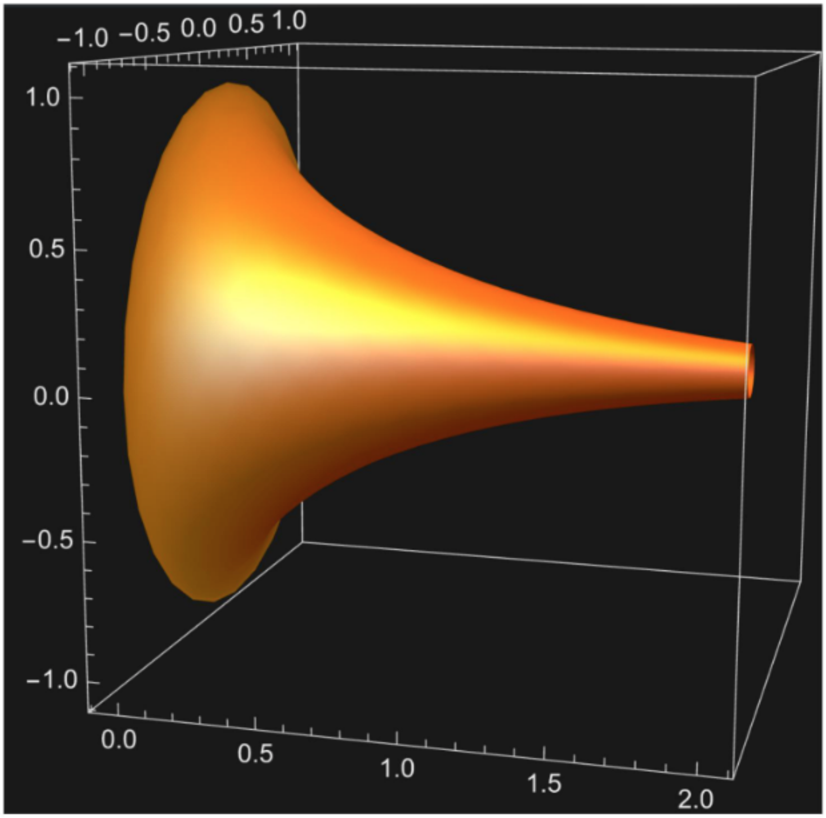
$$\alpha(t) = (t - \tanh t, secht, 0), \quad t > 0.$$

Let S be the surface obtained by revolving α about the x-axis. Use a mathematica `ParametricPlot3D` *(or similar) to plot this surface. Compute (by hand) the first and second fundamental forms* I, II *and Gauss curvature K of this surface.*[3]

Solution. We are given a curve of the form $(x(t), y(t), 0)$. To express a surface of revolution on the x-axis, we rotate along the x-axis by an angle θ: we write $(x(t), y(t)\cos(\theta), y(t)\sin(\theta))$. We then have R produced by α as

$$R(t) = (t - \tanh(t), \operatorname{sech}(t)\cos(\theta), \operatorname{sech}(t)\sin(\theta))$$

This produces the following graph in Mathematica:



We compute the first partials of the chart with respect to spanning vectors u and v. Letting

[2]Recall the hyperbolic trig functions are defined by $\cosh(t) = \frac{e^t + e^{-t}}{2}$, $\sinh(t) = \frac{e^t - e^{-t}}{2}$, etc. I suggest you derive or look up formulas for the derivatives and identities satisfied by these functions.

[3]Hint: Your answer for the Gauss curvature, if correct, will be exceedingly simple.

$s = \text{sech}(t)\tanh(t)$,

$$R_t = (\tanh^2(t), -s\cos(\theta), -s\sin(\theta))$$
$$R_\theta = (0, -\text{sech}(t)\sin(\theta), \text{sech}(t)\cos(\theta))$$

We can next compute the inner products for R_t and R_θ which are the coefficients for I:

$$\begin{aligned} E = & \langle R_t, R_t \rangle \\ = & (1-\text{sech}^2(t))^2 + s^2\cos^2(\theta) + s^2\sin^2(\theta) \\ = & (1 - 2\text{sech}^2(t) + (\text{sech}^2(t))^2 + \text{sech}(t)^2\tanh(t)^2 \\ = & 1 + (\text{sech}^2(t))(-2 + \text{sech}^2(t) + \tanh^2(t) \\ = & 1 + (\text{sech}^2(t))(-2+1) = 1 - \text{sech}^2(t) \end{aligned}$$

$$\begin{aligned} F = & \langle R_t, R_\theta \rangle \\ = & (-s\cos(\theta))(-\text{sech}(t)\sin(\theta)) + (-s\sin(\theta))(\sin(t)\cos(\theta)) \\ = & s\text{sech}(t)\cos(\theta)\sin(\theta) - s\text{sech}(t)\cos(\theta)\sin(\theta) \\ = & 0 \end{aligned}$$

$$\begin{aligned} G = & \text{sech}^2(t)\sin^2(\theta) + \text{sech}^2(t)\cos^2(\theta) \\ = & \text{sech}^2(t) \end{aligned}$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1-\text{sech}^2(t) & 0 \\ 0 & \text{sech}^2(t) \end{pmatrix}$$

We can calculate N:

$$\begin{aligned} N|R_t \times R_\theta| = & R_t \times R_\theta \\ = & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \tanh^2(t) & -s\cos(\theta) & -s\sin(\theta) \\ 0 & -\text{sech}(t)\sin(\theta) & \text{sech}(t)\cos(\theta) \end{vmatrix} \end{aligned}$$

Problem 3. *Let T be a torus of revolution. Use the Gauss map to argue that the average Gaussian curvature over T is zero.*[4]

Solution. A torus of revolution can be obtained by revolving a circle of radius r centered at $(R, 0)$ in the x–z plane about the z-axis. Its parametrization is

$$\phi(u, v) = \big((R + r\cos v)\cos u,\ (R + r\cos v)\sin u,\ r\sin v\big), \qquad 0 \le u, v < 2\pi.$$

We compute

$$\phi_u = (-(R + r\cos v)\sin u,\ (R + r\cos v)\cos u,\ 0), \quad \phi_v = (-r\sin v\cos u,\ -r\sin v\sin u,\ r\cos v).$$

Then

$$E = \phi_u\cdot\phi_u = (R + r\cos v)^2, \qquad F = \phi_u\cdot\phi_v = 0, \qquad G = \phi_v\cdot\phi_v = r^2.$$

Next, the second derivatives are

$$\begin{aligned}
\phi_{uu} &= (-(R + r\cos v)\cos u,\ -(R + r\cos v)\sin u,\ 0),\\
\phi_{uv} &= (r\sin v\sin u,\ -r\sin v\cos u,\ 0),\\
\phi_{vv} &= (-r\cos v\cos u,\ -r\cos v\sin u,\ -r\sin v).
\end{aligned}$$

The normal vector is obtained from

$$\phi_u \times \phi_v = r(R + r\cos v)(\cos u\cos v,\ \sin u\cos v,\ \sin v),$$

so the unit normal (Gauss map) is

$$N(u, v) = (\cos u\cos v,\ \sin u\cos v,\ \sin v).$$

Using this, we compute

$$\begin{aligned}
e &= \phi_{uu}\cdot N = -(R + r\cos v)\cos v,\\
f &= \phi_{uv}\cdot N = 0,\\
g &= \phi_{vv}\cdot N = -r.
\end{aligned}$$

The Gaussian curvature is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(-(R + r\cos v)\cos v)(-r)}{(R + r\cos v)^2 r^2} = \frac{\cos v}{r(R + r\cos v)}.$$

The area element is

$$dA = \|\phi_u \times \phi_v\|\, du\, dv = r(R + r\cos v)\, du\, dv.$$

Hence

$$\int_T K\, dA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos v}{r(R + r\cos v)}\, r(R + r\cos v)\, dv\, du = \int_0^{2\pi}\int_0^{2\pi} \cos v\, dv\, du.$$

Integrating,

$$\int_0^{2\pi} \cos v\, dv = \sin v\Big|_0^{2\pi} = 0,$$

so

$$\int_T K\, dA = 0.$$

□

[4]Hint: use some symmetry. Try not to do much computation.

Problem 4. *Let A be a symmetric 2×2 matrix. Write $v(\theta) = (\cos\theta, \sin\theta)$ for a point on the unit circle. Show that the average value of $\langle Av(\theta), v(\theta)\rangle$ is equal to $\frac{1}{2}\mathrm{tr}(A)$. Use this to explain the terminology "mean curvature".*[5]

Solution. We define A as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{, where } a_{12} = a_{21}.$$

Therefore, we can compute

$$\begin{aligned}\langle Av(\theta), v(\theta)\rangle &= \langle \begin{pmatrix} a_{11}\cos\theta + a_{12}\sin\theta \\ a_{21}\cos\theta + a_{22}\sin\theta \end{pmatrix}, v(\theta)\rangle \\ &= a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\cos\theta\sin\theta\end{aligned}$$

To find the average, we do the sum over the range, which is

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\cos\theta\sin\theta\, d\theta \\ &= \frac{a_{11}\pi + a_{22}\pi}{2\pi} \\ &= \frac{1}{2}(a_{11} + a_{22}) =: \frac{1}{2}\mathrm{tr}(A)\end{aligned}$$

Geometrically, if A is the shape operator of a surface, then for any unit tangent vector $v(\theta)$, the quantity

$$\kappa_n(\theta) = \langle Av(\theta), v(\theta)\rangle$$

is the normal curvature in that direction. The computation above shows that the average of $\kappa_n(\theta)$ over all directions satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta = \frac{1}{2}\mathrm{tr}(A).$$

Since the eigenvalues of A are the principal curvatures κ_1, κ_2, we have

$$\frac{1}{2}\mathrm{tr}(A) = \frac{\kappa_1 + \kappa_2}{2} =: H.$$

Therefore, the mean curvature H is literally the average of the normal curvatures in all tangent directions, which explains the term "mean curvature."

□

[5] There is only one completely correct answer here, so please think carefully.