Problem 1.

For the monkey saddle $z = x^3 - 3xy^2$, compute the plane-curve curvature of the curve on S lying over each line through the origin in the xy-plane. Conclude that the monkey saddle has Gaussian curvature $K = 0$ at the origin.

Solution:

A line through the origin in the xy-plane can be parametrized by fixing an angle θ and letting a parameter t vary. We can write this as:

$$\begin{cases} x(t) = t\cos(\theta) \\ y(t) = t\sin(\theta) \end{cases}$$

Substituting this into the equation for the surface S, we find the z-component for the curve on the surface:

$$\begin{aligned} z(t) &= (t\cos(\theta))^3 - 3(t\cos(\theta))(t\sin(\theta))^2 \\ &= t^3\cos^3(\theta) - 3t^3\cos(\theta)\sin^2(\theta) \\ &= t^3(\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)) \\ &= t^3\cos(3\theta) \end{aligned}$$

Thus, for any fixed angle θ, the curve $\phi_{\theta(t)}$ on the surface is given by the parametrization:

$$\phi(t) = (t\cos(\theta), t\sin(\theta), t^3\cos(3\theta))$$

Then, the curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{\|\phi'(t) \times \phi''(t)\|}{\|\phi'(t)\|^3}$$

First, we have

$$\phi'(t) = (\cos(\theta), \sin(\theta), 3t^2\cos(3\theta))$$

$$\phi''(t) = (0, 0, 6t\cos(3\theta))$$

We are interested in the curvature at $(0,0)$, which corresponds to the point $t = 0$. At $t = 0$, we have

$$\phi'(0) = (\cos(\theta), \sin(\theta), 0)$$

$$\phi''(0) = (0, 0, 0)$$

$$\phi'(0) \times \phi''(0) = (\cos(\theta), \sin(\theta), 0) \times (0,0,0) = (0,0,0)$$

$$\|\phi'(0) \times \phi''(0)\| = 0$$

$$\|\phi'(0)\| = \sqrt{\cos^2(\theta) + \sin^2(\theta) + 0^2} = \sqrt{1} = 1$$

Substituting these results into the curvature formula at $t = 0$:

Problem 2. *Let S be a surface, and fix $q \in \mathbb{R}^3$. Define $f : S \to \mathbb{R}$ by $f(p) = |p - q|^2$. Compute Df_p.*[2] *Describe geometrically when is p a critical point*[3] *of f.*

Solution. We compute alone curve $c(t) \subset S$ with $c(0) = p, c'(0) = w$. Therefore

$$\begin{aligned} (f \circ c)(t) &= |c(t) - q|^2 \\ \implies (f \circ c)'(0) &= 2(c(0) - q) \cdot c'(0) = 2(p - q) \cdot w \\ \implies Df_p &= \langle 2(p - q), w \rangle \end{aligned}$$

When p is a critical point of f, we get that $Df_p = 0$, therefore $p - q$ should be orthogonal to every $w \in T_pS$. Then we know that $p - q$ should be the normal vector of S at p, which implies q lies on the normal line to the surface at p. □

[2] Here (in our usual notation) $Df_p : T_pS \to \mathbb{R}$ is defined on $w = c'(0)$ by $Df_p(w) = (f \circ c)'(0)$.

[3] We say that $p \in S$ is a critical point of $f : S \to \mathbb{R}$ if $Df_p = 0$.

Problem 4. *Let $S \subset \mathbb{R}^3$ be a surface. Suppose that there exists a point $q \in \mathbb{R}^3$ such that the normal line through $p \in S$ passes through q for each $p \in S$. Prove that S is contained in a sphere.* [5]

Solution. Consider the function $f : S \to \mathbb{R}$ that was defined in a Problem 2. In fact, $Df_p(w) = 2\langle w, p-q\rangle w$. Note that because the normal line through p passes through q, the vector $p-q$ is parallel to $N(p)$. Hence, $p - q = \lambda N(p)$, which implies $Df_p = 2\langle w, \lambda N(p)\rangle w = 0$ because $w \in T_pS$ and every such w is orthogonal to $N(p)$. Hence, f is constant, which implies that $\mid p-q \mid$ is constant with respect to p; hence, S is in a sphere. □

[5] Hint: Define an appropriate function on S whose derivative can be computed as zero...

PROBLEM 5

This statement is true.

Fix a surface S and a point $p \in S$. Suppose $\mathrm{K}(p) < 0$. This means that κ_1 and κ_2 have different signs because $\mathrm{K} = \kappa_1\kappa_2$. If κ_1 is the maximum of the second fundamental form at p and κ_2 is the minimum, this means that $\kappa_1 > 0 > \kappa_2$. The second fundamental form is continuous because it is the composition of continuous functions, so by the intermediate value theorem, there is a $w \in T_pS$ so that $\mathbb{I}_p(w) = 0$. Let c be a unit speed curve with $c(0) = p$ and $c'(0) = w$. Then, $\kappa_{c,s}(0) = \mathbb{I}_{c(0)}(c'(0)) = \mathbb{I}_p(w) = 0$. Thus, we have found a curve through p with normal curvature 0 at p. $\square$