Problem 1. *Write down a linear system of differential equations in functions $f_1, \dots, f_6$ that is satisfied by $f_1 = T \cdot N$, $f_2 = T \cdot B$, $f_3 = N \cdot B$, $f_4 = T \cdot T$, $f_5 = N \cdot N$, $f_6 = B \cdot B$ when T, N, B satisfy the Frenet equations.*[1] *Verify that the functions $f_1 = f_2 = f_3 = 0$ and $f_4 = f_5 = f_6 = 1$ is a solution to your system of differential equations.*

Solution. Utilizing the property that $f = T \cdot B \implies f' = T' \cdot B + T \cdot B'$, we can differentiate each of the 6 dot product relations, and plug in our known Fernet Equations:

$$f_1' = T' \cdot N + T \cdot N' = \kappa(N \cdot N) - \kappa(T \cdot T) - \tau(T \cdot B) = \kappa f_5 - \kappa f_4 - \tau f_2 \tag{1}$$

$$f_2' = T' \cdot B + T \cdot B' = \kappa(N \cdot B) + \tau(T \cdot N) = \kappa f_3 + \tau f_1 \tag{2}$$

$$f_3' = N' \cdot B + N \cdot B' = -\kappa(T \cdot B) - \tau(B \cdot B) + \tau(N \cdot N) = -\kappa f_2 - \tau f_6 + \tau f_5 \tag{3}$$

$$f_4' = T' \cdot T + T \cdot T' = \kappa(N \cdot T) + \kappa(N \cdot T) = 2\kappa f_1 \tag{4}$$

$$f_5' = N' \cdot N + N \cdot N' = -\kappa(T \cdot N) - \tau(B \cdot N) - \kappa(N \cdot T) - \tau(N \cdot B) = -2\kappa f_1 - 2\tau f_3 \tag{5}$$

$$f_6' = B' \cdot B + B \cdot B' = \tau(N \cdot B) + \tau(B \cdot N) = 2\tau f_3 \tag{6}$$

So our system of differential equations are:

$$\boxed{\begin{aligned} &f_1' = \kappa f_5 - \kappa f_4 - \tau f_2 \\ &f_2' = \kappa f_3 + \tau f_1 \\ &f_3' = -\kappa f_2 - \tau f_6 + \tau f_5 \\ &f_4' = 2\kappa f_1 \\ &f_5' = -2\kappa f_1 - 2\tau f_3 \\ &f_6' = 2\tau f_3 \end{aligned}}$$

Finally to verify the solution (0,0,0,1,1,1) satisfies the found equations, we plug in below:

$$f_1' = \kappa(1) - \kappa(1) - \tau(0) = 0 \tag{7}$$

$$f_2' = \kappa(0) + \tau(0) = 0 \tag{8}$$

$$f_3' = -\kappa(0) - \tau(1) + \tau(1) = 0 \tag{9}$$

$$f_4' = 2\kappa(0) = 0 \tag{10}$$

$$f_5' = -2\kappa(0) - 2\tau(0) = 0 \tag{11}$$

$$f_6' = 2\tau(0) = 0 \tag{12}$$

We see that the derivatives of all 6 equations are 0, which implies the equations are all equal to constants. Thus the solution (0,0,0,1,1,1) satisfies the found equations. □

[1] Hint: differentiate these dot product functions, and express the answer back in terms of these functions. The final answer should be a system of differential equations involving $f_1, \dots, f_6$, and *not(!)* involving T, N, B.

Problem 2. *Use the previous problem to give a careful proof of the fundamental theorem of space curves, finishing the argument from class. (Specifically, show for every $\kappa : [0, L] \to [0, \infty)$ and $\tau : [0, L] \to \mathbb{R}$, there is a unit speed curve with curvature κ and torsion τ.)*

Solution. The fundamental theorem of space curves states that given $\kappa : [0, L] \to (0, \infty)$ and $\tau : [0, L] \to \mathbb{R}$, there exists a unit speed curve $c : [0, L] \to \mathbb{R}^3$ with curvature κ and torsion τ, unique up to isometry.

To construct c, we first start with constructing a basis T, N, B from κ and τ.

First, we want T, N, B to fulfill the Frenet equations:

$$T' = \kappa N \qquad N' = -\kappa T + \tau B \qquad B' = -\tau N$$

This is a system of differential equations, so by the ODEs blackbox, we know there exists a unique solution for T, N, B up to initial conditions. We can simply choose the initial conditions to be $T(0) = (1, 0, 0)$, $N(0) = (0, 1, 0)$, and $B = (0, 0, 1)$, since the goal is for them to be orthonormal.

Now, we show that T, N, B are orthonormal for all t. From question 1, we know that since T, N, B satisfy the Frenet equations, that they also satisfy the system of differential equations given by $f_1', f_2', \ldots, f_6'$. We also chose T, N, B so that at $t = 0$, $f_1(0) = f)2(0) = f_3(0) = 0$ and $f_4(0) = f_5(0) = f_6(0) = 1$.

However, recall that $f_1 = f_2 = f_3 = 0$ and $f_4 = f_5 = f_6 = 1$ are also solutions to the system of differential equations, with the same initial conditions as given by T, N, B. Thus, by uniqueness of ODE solutions up to initial conditions, we have that for all t,

$$\begin{aligned} \langle T, N\rangle &= f_1 = 0 & \langle T, B\rangle &= f_2 = 0 & \langle N, B\rangle &= f_3 = 0 \\ \langle T, T\rangle &= f_4 = 1 & \langle N, N\rangle &= f_5 = 1 & \langle B, B\rangle &= f_6 = 1 \end{aligned}$$

Therefore T, N, B are an orthonormal basis that satisfy the Frenet equations.

Now, we define the curve $c(t) = \int_0^t T(s)ds$. First, note that this is unit speed, as $|c'(t)| = |T'(t)| = 1$ for all t. Now, we show that its Frenet frame T_c, N_c, B_c is exactly equal to T, N, B, and that its curvature and torsion κ_c, τ_c are exactly the κ, τ we were given.

First, note that $T_c = c' = T$. Then, it follows that

$$\kappa_c = |T_c'| = |T'| = |\kappa N| = \kappa$$

From there, we have that

$$N_c = \frac{T_c'}{\kappa_c} = \frac{T'}{\kappa} = \frac{\kappa N}{\kappa} = N$$

It follows that

$$B_c = T_c \times N_c = T \times N = B$$

Finally, we have that

$$\tau_c = -\langle N_c, B_c'\rangle = -\langle N, B'\rangle = -\langle N, -\tau N\rangle = \tau\langle N, N\rangle = \tau$$

To conclude, we have found that if we fix κ and τ and fix initial conditions, we can derive a unique orthonormal basis T, N, B which is exactly the Frenet frame for a unit speed curve c. It follows that the curvature and torsion of c are exactly the κ and τ we were given. Hence, c can be completely determined by κ and τ, and is unique up to isometry. $\square$

Problem 3. *Let $c : [0, L] \to \mathbb{R}^2$ be a unit-speed plane curve, and assume $c(0) = c(L)$ and $c'(0) = c'(L)$. Write $c'(t) = (\cos\theta(t), \sin\theta(t))$, where $\theta : [0, L] \to \mathbb{R}$. Then $\frac{1}{2\pi}\big(\theta(L) - \theta(0)\big)$ is an integer, called the turning number of α. Compute the turning number of the following curves.*[2]

Solution. The way I did this was just to follow the tangent vector c' with my eyes, and count how many times it points straight up. A positive turn happens when the tangent rotates counter-clockwise, meaning when its derivative $|c''|$ is positive (i.e. curvature is positive). A negative turn occurs when it rotates clockwise (i.e. curvature is negative). So, for a positive count, c'' should be pointing to the left each time c' is pointing up. For a negative count, c'' will be pointing right.

Following these rules, I got the following turning numbers:

Left curve: turning number $= +2$

Middle curve: turning number $= +1$

Right curve: turning number $= 0$ □

[2]Hint: Fix a direction, e.g. $(1, 0)$ and find all the points on the curve where the tangent vector points in the direction of $(1, 0)$ (not its negative). Then count these points with sign...

Problem 4. *For the hyperboloid $\phi(u,v) = (u, v, v^2 - u^2)$ with unit normal $N = \phi_u \times \phi_v / |\phi_u \times \phi_v|$, compute DN_p at $p = (0,0,0)$.*

Solution. We begin by determining the tangent plane for $p = (0,0,0)$. The tangent space is spanned by the partial derivatives of ϕ, which we can determine as

$$\frac{\partial}{\partial u}(u, v, v^2 - u^2) = (1, 0, -2u)$$
$$\frac{\partial}{\partial v}(u, v, v^2 - u^2) = (0, 1, 2v)$$

At $p = (0,0,0)$, we have $x_u = (1,0,0)$ and $x_v = (0,1,0)$, so that $T_pS = \text{span}\{x_u, x_v\}$. Now, we consider that the action of dN_p on the basis vectors x_u and x_v is given by the partial derivatives of the normal:

$$dN_p(x_u) = N_u$$
$$dN_p(x_v) = N_v$$

Our goal now is to calculate the partial derivatives N_u and N_v. We have the normal as $N = \phi_u \times \phi_v / |\phi_u \times \phi_v|$, while we know that

$$\phi_u = (1, 0, -2u)$$
$$\phi_v = (0, 1, 2v)$$

Such that we have the following

$$N_p = \frac{(2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}$$

Let $R = \sqrt{1 + 4u^2 + 4v^2}$. Applying the product rule, we obtain:

$$\begin{aligned} N_u &= (-2, 0, 0)R^{-1} - (-2u, 2v, 1)4R^{-\frac{5}{2}}u \\ N_u(0,0,0) &= (-2, 0, 0)(1) + (0, 0, 1)(0) \\ N_u(p) &= (-2, 0, 0) \end{aligned}$$

We calculate N_v through applying the product rule as well:

$$\begin{aligned} N &= (-2u, 2v, 1)R^{-1} \\ N_v &= (0, 2, 0)R^{-1} + -(-2u, 2v, 1)R^{-2}(\frac{\partial}{\partial v}R) \\ N_v &= (0, 2, 0)R^{-1} + (2u, 2v, 1)4R^{-\frac{5}{2}}v \\ N_v(0,0,0) &= (0, 2, 0)(1) + (0, 2, 1)(0) \\ N_v(p) &= (0, 2, 0) \end{aligned}$$

As our basis vectors are simply $x_u = (1,0,0)$ and $x_v = (0,1,0)$, we conclude that DN_p is $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

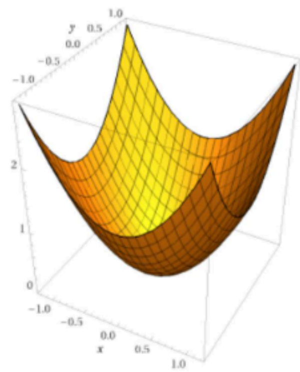
□

Problem 5. *Describe the image of the Gauss map of the following surfaces. Do* not *compute using a chart. Instead, reason geometrically.*

(a) Paraboloid $z = x^2 + y^2$.

(b) Hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution. (a) We can get the figure as the following.

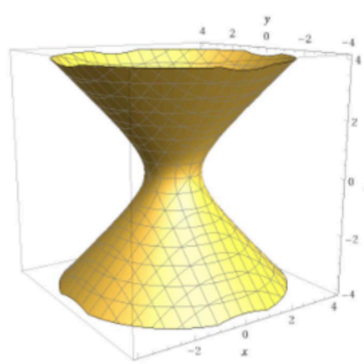


From there, we observe the normal at $(0,0,0)$ is $(0,0,-1)$, and with x, y increase, the normal is becoming closer and closer horizontal, but never downward. So the Image is a open bottom hemisphere

$$\{Z < 0\} \subset S^2$$

for S^2 the unit 2-sphere.

(b) We can get the figure



Normals are horizontal at $z = 0$; as $|z| \to \infty$, the shape is asymptotic to the cone $x^2 + y^2 = z^2$, so normals approach $\frac{\pi}{4}$

$$\text{Image}(N) = \{(X, Y, Z) \in S^2 : \ |Z| < 1/\sqrt{2}\}.$$

$\square$

Problem 6. *In this problem you prove the spectral theorem for self-adjoint linear operators $A : \mathbb{R}^2 \to \mathbb{R}^2$ (using differential geometry!).* [3]

Consider the composition $f \circ c$, where $c : [0, 2\pi] \to \mathbb{R}^2$ is the standard parameterization of the unit circle, and $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(p) = \langle A(p), p \rangle$.

(a) Compute $(f \circ c)'(t)$ and prove that t is a critical point of $f \circ c$ if and only if $c(t)$ is an eigenvector of A. Relate the corresponding eigenvalue to f.

(b) Use facts from calculus to deduce that either A is a scalar matrix or A has two distinct eigenvalues.

(c) Prove that there exist a pair of eigenvectors for A that form an orthonormal basis for $\mathbb{R}^2$. [4]

Solution. 1.

$$c(t) = (\cos(t), \sin(t))$$
$$(f \circ c)(t) = \langle A(c(t)), c(t) \rangle$$
$$(f \circ c)'(t) = \langle A(c(t)), c'(t) \rangle + \langle A'(c(t))c'(t), c(t) \rangle = \langle A(c(t)), c'(t) \rangle + \langle A(c'(t)), c(t) \rangle$$
$$= \langle c'(t), A(c(t)) \rangle + \langle A(c'(t)), c(t) \rangle$$

Since A is self-adjoint, we have

$$\langle c'(t), A(c(t)) \rangle + \langle A(c'(t)), c(t) \rangle = 2\langle c'(t), A(c(t)) \rangle$$

By definition, t is a critical point of $f \circ c$ if and only if $(f \circ c)'(t) = 0$. We also know that $(f \circ c)'(t) = 2\langle c'(t), A(c(t)) \rangle$. So, t is a critical point if and only if $c'(t)$ is orthogonal to $A(c(t))$. In $\mathbb{R}^2$, if $A(c(t))$ is orthogonal to $c'(t)$, it is parallel to $c(t)$. So, there must be some $\lambda \in \mathbb{R}$ such that $A(c(t)) = \lambda c(t)$. Therefore, $c(t)$ is an eigenvector of A if and only if t is a critical point of $f \circ c$.

Since $|c(t)| = 1$, we have $f(c(t)) = \langle \lambda c(t), c(t) \rangle = \lambda \langle c(t), c(t) \rangle = \lambda$. So, when $c(t)$ is an eigenvector of A, the corresponding eigenvalue is $f(c(t))$.

2. Since $c : [0, 2\pi] \to \mathbb{R}^2$, we know that $f \circ c$ must have maximum and minimum values. The maximum and minimum values are considered to be critical points. Let t_1 be the max and t_2 be the min. Then, $c(t_1)$ and $c(t_2)$ are eigenvectors of A. So, $f(c(t_1))$ and $f(c(t_2))$ are eigenvalues. If these are distinct values, then we have two distinct eigenvalues of A.

   Otherwise, since the max and min are the same, the function $f \circ c$ must be constant. Then, $\langle A(p), p \rangle = \lambda \langle p, p \rangle$ for any p. Therefore, A is a scalar matrix.

3. If A is a scalar matrix, any vector p is an eigenvector. So, we can take any two orthonormal vectors to create an orthonormal basis. Otherwise, Let A have two distinct eigenvalues such that $A(v) = \lambda_1 v$ and $A(u) = \lambda_2 u$. Then, since A is self-adjoint,

$$\lambda_1 \langle v, u \rangle = \langle \lambda_1 v, u \rangle = \langle A(v), u \rangle = \langle v, A(u) \rangle = \langle v, \lambda_2 u \rangle = \lambda_2 \langle v, u \rangle$$

   Since λ_1 and λ_2 are distinct eigenvalues, then $\langle v, u \rangle = 0$ so v and u are orthonormal. Therefore, v and u are a pair of eigenvectors for A that form an orthonormal basis for $\mathbb{R}^2$.

□

[3] Remark: This special case of the theorem is the most important case for us in the course.

[4] Hint: this is just a linear algebra exercise. Make sure to use the hypothesis...