Problem 1. *Let $c : [0, L] \to \mathbb{R}^2$ be a unit-speed plane curve. Write $c'(t) = (\cos\theta(t), \sin\theta(t))$, where $\theta : [0, L] \to \mathbb{R}$. Show that differentiability of c implies differentiability of θ.* [1] [2]

Solution. Suppose $c(t) = (x(t), y(t))$, therefore, we can get $c'(t) = (x'(t), y'(t))$. Since we know c is differentiable, x, y should also be differentiable.

We get the following equations from above

$$x'(t) = \cos\theta(t) \qquad\qquad y'(t) = \sin\theta(t).$$

Therefore, we can get

$$\theta(t) = \cos^{-1} x'(t) \qquad\qquad \theta(t) = \sin^{-1} y'(t)$$

However, we should be aware that inverse trigonometric functions have branch issues when $\cos\theta = 0$ or $\sin\theta = 0$. To avoid this, we define θ piecewise:

$$\theta(t) = \begin{cases} \arcsin(y'(t)), & x'(t) > 0, \\ \pi - \arcsin(y'(t)), & x'(t) < 0, \\ \arccos(x'(t)), & y'(t) > 0, \\ 2\pi - \arccos(x'(t)), & y'(t) < 0. \end{cases}$$

These agree on overlaps and give a continuous θ with $\cos\theta = x'$, $\sin\theta = y'$.

Differentiating on $\{x' \neq 0\}$ we get $\theta' = \frac{y''}{x'}$, and on $\{y' \neq 0\}$ we get $\theta' = -\frac{x''}{y'}$. Unit speed implies $x'x'' + y'y'' = 0$, so these formulas coincide on overlaps. Thus θ is differentiable. □

[1] Recall: for us (and do Carmo) differentiable means "infinitely differentiable".

[2] Remark: we used this problem in our proof of the fundamental theorem of plane curves.

Problem 2. *In this problem you work out a formula for curvature of a space curve that's not necessarily unit speed. Let* $f : [a, b] \to \mathbb{R}^3$ *be a curve (not necessarily unit speed), and let* $r(t) = \int_a^t |f'(u)|\, du$ *be its arclength function (in particular* $r'(t) = |f'(t)|$*). From class, we can define a unit speed curve* c *so that* $c \circ r = f$*. The curvature* $\kappa(t)$ *of* f *at time* $t \in [a, b]$*, is defined as the curvature of* c *at time* $r(t)$*, which we defined in class as* $|c''(r(t))|$*.*

(a) *Derive from this setup that the curvature of* f *is give by the formula*

$$\kappa(t) = \frac{|T'(t)|}{r'(t)},$$

where $T(t)$ *is defined as* $f'(t)/|f'(t)| = f'(t)/r'(t)$*.* [3]

(b) *Derive the formula*

$$\kappa(t) = \frac{|f'(t) \times f''(t)|}{|f'(t)|^3}$$

[4]

Solution. (a) First, we know that $f(t) = c(r(t))$ or $c \circ r$. Let us first begin by differentiating f:

$$f'(t) = \frac{d}{dt} c(r(t)) = r'(t)c'(r(t)) \tag{6}$$

Then, knowing $f' = r'T$, we can set the above equation equal to $r'T$:

$$r'(t)c'(r(t)) = r'(t)T \implies c'(r(t)) = T \tag{7}$$

Next let's differentiate for T':

$$T'(t) = r'(t)c''(r(t)) \implies c''(r(t)) = \frac{T'(t)}{r'(t)} \tag{8}$$

As we now have a relation for c"(r(t)), and recalling curvature is defined by this, we can now relate curvature as such:

$$\kappa(t) = |c''(r(t))| = \frac{|T'(t)|}{|r'(t)|} \tag{9}$$

Finally, as we expect r'(t) to be strictly positive by definition ($r'(t) = |f'(t)|$ which $|f(t)|$ always be greater than 0), we can drop the absolute value bars to conclude:

$$\kappa(t) = \frac{|T'(t)|}{r'(t)} \tag{10}$$

(b) First, we begin by differentiating the provided relation, $f' = r'T$:

$$f'' = \frac{d}{dt} r'T = r''T + r'T' \tag{11}$$

Now, let us find the cross product of f' and f'':

$$f' \times f'' = (r'T) \times (r''T + r'T') \tag{12}$$

[3] Hint: differentiate! and again!

[4] Hint: first differentiate $f' = r'T$ to get a formula for f''.

Because the cross product is linear, we have:

$$(r'T) \times (r''T + r'T') = (r'T \times r''T) + (r'T \times r'T') \tag{13}$$

We also know the first cross product evaluates to 0 as through bilinearity it is equivalent to $(r'r''(T \times T))$ where $T \times T = 0$. Using bilinearity on the second term, we see:

$$f' \times f'' = r'(t)^2(T \times T') \tag{14}$$

Now, taking the norm on both sides we see:

$$|f' \times f''| = r'(t)^2|T \times T'| \tag{15}$$

We know $|T| = 1$, implying $|T|^2 = 1$. Next, if we differentiate this term we see $2T \cdot T' = 0$, implying orthogonality of T and T'. Finally, let's analyze the cross product $|T \times T'|$ with this in mind. For two orthogonal cross products, the norm can be simplified to $|T||T'|$. And knowing that $|T| = 1$ implies that $|T \times T'| = |T'|$. Plugging back in we see:

$$|f' \times f''| = r'(t)^2|T'| \implies |T'| = \frac{|f' \times f''|}{r'(t)^2} \tag{16}$$

Finally, plugging this new equation for —T'— into our curvature equation for part a), and recalling that $r'(t) = |f'(t)|$, we see:

$$\kappa(t) = \frac{|T'|}{r'(t)} = \frac{|f' \times f''|}{r'(t)^3} = \frac{|f' \times f''|}{|f'(t)|^3} \tag{17}$$

□

Problem 3. *Let $\phi : I \to \mathbb{R}$ be a smooth function, and consider $f(t) = (t, \phi(t))$ (a parameterization of the graph of ϕ). Compute the curvature of f.*

Solution. We find the curvature using the formula derived in the previous question. First, we compute some derivatives:

$$f'(t) = (1, \phi'(t)) \qquad\qquad f''(t) = (0, \phi''(t))$$
$$|f'(t)| = \sqrt{1 + \phi'(t)^2}$$

Now, we compute $f'(t) \times f''(t)$:

$$\begin{vmatrix} 1 & \phi'(t) \\ 0 & \phi''(t) \end{vmatrix} = \phi''(t)$$

Hence, using our formula, we have that

$$\kappa(t) = \frac{|\phi''(t)|}{(\sqrt{1 + \phi'(t)^2})^3}$$

□

Problem 4. *The tangent line is the line that best approximates a curve at a point. Similarly, the osculating circle is the circle that best approximates a plane curve at a point. Recall that for points a, b, c in the plane, not on a line, there is a unique circle passing through these points. Write $C(a, b, c)$ for the center of this circle. The osculating circle at $\alpha(t)$ is defined as the circle through $\alpha(t)$ with center*

$$C = \lim_{s \to 0} C(\alpha(t-s), \alpha(t), \alpha(t+s)).$$

(i) Fix $\lambda > 0$ and define $\beta(s) = (s, \lambda s^2)$. For $s \neq 0$, compute the center of the circle that passes through $\beta(s), \beta(0)$, and $\beta(-s)$. [5]

(ii) Assume α satisfies $\alpha(0) = (0,0)$ and $\alpha'(0) = (1,0)$. Use the preceding part and the Taylor expansion of $\alpha(t)$ to show the radius of the osculating circle at $\alpha(0)$ is $1/\kappa$, where $\kappa = \kappa(0)$ is the curvature. [6] [7]

Solution. (i) We have $\beta(s) = (s, \lambda s^2)$, $\beta(0) = (0, 0)$, and $\beta(-s) = (-s, \lambda s^2)$. If they are all on a circle, then

$$(s-a)^2 + (\lambda s^2 - b)^2 = (-a)^2 + (-b)^2 = (-s-a)^2 + (\lambda s^2 - b)^2$$
$$(s^2 - 2sa + a^2) + (\lambda^2 s^4 - 2\lambda s^2 b + b^2) = a^2 + b^2 = (s^2 + 2sa + a^2) + (\lambda^2 s^4 - 2\lambda s^2 b + b^2)$$

The right and left equation cancel to $-sa = sa$, so $a = 0$ since $s \neq 0$ for those equations. Then, using the left and middle equations, we get

$$0 = s^2 + \lambda^2 s^4 - 2\lambda s^2 b$$
$$b = \frac{s^2 + \lambda^2 s^4}{2\lambda s^2} = \frac{1 + \lambda^2 s^2}{2\lambda}$$

So, the center of the circle that passes through $\beta(s)$, $\beta(0)$, and $\beta(-s)$ is $(0, \frac{1+\lambda^2 s^2}{2\lambda})$.

(ii) Since $|a'(0)| = |(1, 0)| = 1$, the curve is unit speed at $t = 0$. Then, the curvature at 0 is $\kappa(0) = |a''(0)|$. Since $a'(0)$ and $a''(0)$ must be orthogonal, we can use the orientation where $a''(0) = (0, \kappa)$. The degree 2 Taylor approximation of $\alpha(t)$ at 0 is

$$\alpha(0) + \alpha'(0)(t-0) + \frac{\alpha''(0)}{2}(t-0)^2 = (0,0) + (1,0)t + \frac{(0,\kappa)}{2}t^2 = (t, \frac{\kappa t^2}{2})$$

Then, if we set $\lambda = \frac{\kappa}{2}$, we have the form $(t, \lambda t^2)$ like in the previous part. By definition, the osculating circle at $\alpha(0)$ is

$$\lim_{s \to 0} C(\alpha(-s), \alpha(0), \alpha(s))$$

We can then use the formula we found from the previous part to find the center. Therefore, the center of the osculating circle is

$$\lim_{s \to 0} (0, \frac{1 + \lambda^2 s^2}{2\lambda}) = \frac{1}{2\lambda} = \frac{1}{\kappa}$$

□

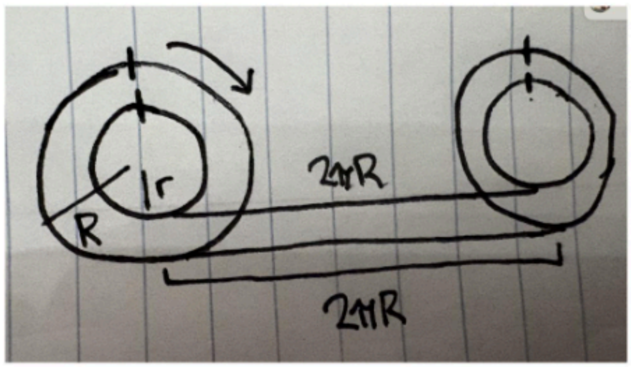
[5] Hint: You can solve this by finding a, b, r so that $\beta(\pm s), \beta(0)$ satisfy the equation $(x-a)^2 + (y-b)^2 = r^2$.

[6] Hint: use specifically the degree-2 Taylor approximation.

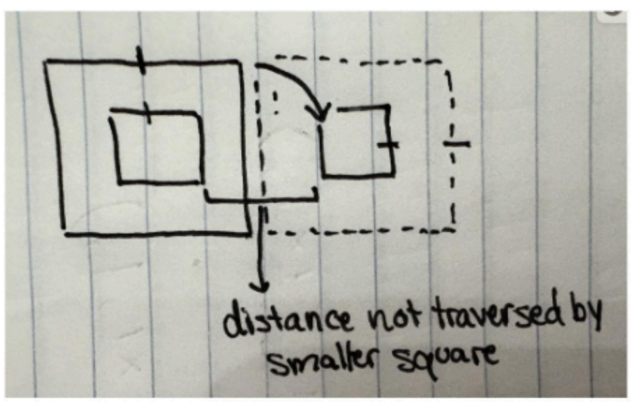
[7] Remark: this problem gives a geometric interpretation for the curvature.

Problem 5. *Resolve the cycloid paradox. Suggestion: first solve the paradox for a square wheel (consider a smaller concentric square and keep track of the times when the smaller square has its sides parallel to the x, y-axes). This should give a good clue for what is going on (viewing the circle as the limit of regular n-gon as n goes to infinity). To explain what's going on in the circle case, it may help to draw the path traced by a point on a smaller concentric circle and compare it to the cycloid.*

Solution. We begin first by stating the paradox, for two concentric circles of differing radius that roll simultaneously, we expect the smaller circle to traverse a distance of $2\pi r$ while the larger circle traverses a distance of $2\pi R$. However, this is not what happens, and both circles roll a distance of 2π times the larger circle's radius. This looks like:



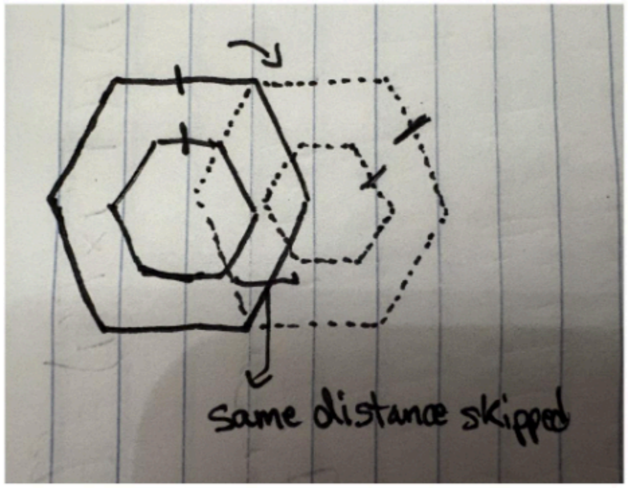
Let's first consider two concentric squares that are undergoing the same effect. We notice for a 90 degree rotation, the larger square would span the whole distance traveled. However, for the smaller square, there is a distance that was not covered by it rotating over it. This looks like: The conclusion is that for a full 360 degree rotation, the distance traveled will be the perimeter of



the larger square, however, this distance cannot equal the perimeter of the smaller square. This is because of the slipped or missing distance, which when added to the perimeter of the smaller square would equal the perimeter of the larger square.

It is also worth pointing out that the smaller this center square is, the greater distance it skips when rotating with the larger square.

Extrapolating this behavior to larger and larger n-gons, we would expect that every rotation from one face to another to result in the smaller n-gon skipping some fixed distance. Although for larger and larger n-gons, this skipped distance gets smaller from one face to another, the increasing amount of faces makes it an important factor to consider each time. Here is an example of what I am talking about in a hexagon:



Finally, we conclude that taking $n \to \infty$, we have a circle, which would still be subject to the same behavior. Thus for each minute rotation, the larger circle 'drags' on the smaller circle, pulling it an extra distance. Adding up all these minute distances over a full rotation of 360 degrees we see:

$$2\pi R = 2\pi r + d_{tot} \tag{33}$$

Showing that in fact $2\pi R \neq 2\pi r$, resolving the cycloid paradox.

□

Problem 6. *The url below takes you to an image of a curve. Let T, N, B denote the Frenet frame at the specified point. Rotate the image so that you are looking down at the plane spanned by T, N. Do the same with T, B, and with N, B. Submit screenshots of your answer, and draw and label the T, N, B axes. Make sure to explain your answer.*

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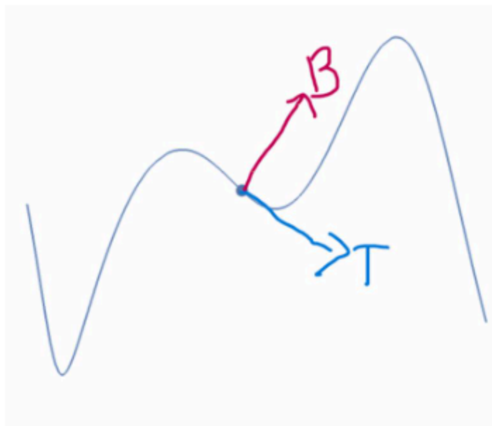
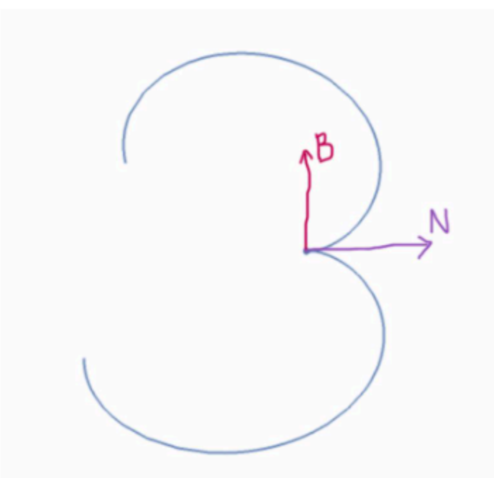
Solution.





The curve looks locally quadratic if B vanishes, cubic if N vanishes, and as $x^{\frac{3}{2}}$ if T vanishes - this is the consequence of so-called "stick theorem" proven in the class.

□

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Solution. As seen in class and in textbook, for the space curve $\alpha(t)$ and the third-order Taylor approximation:

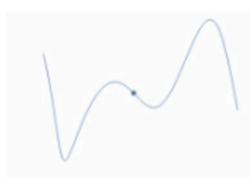
$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}a''cd'(0)$$

we have the following parameterizations for $x(s), y(s)$, and $z(s)$ on (T, N, B):

$$x(s) = s - \frac{k^2 s^3}{6}$$
$$y(s) = \frac{k}{2}s^2 + \frac{k' s^3}{6}$$
$$z(s) = \frac{-k\tau}{6}s^3$$

Taking T, N as the axes with B set to 0, we will obtain a parabola:

Taking T, B as the axes with N set to 0, we will obtain a cubic:



Taking B, N as the axes with T set to 0, we will obtain a parabolic cusp:

□