Problem 1. *Let $S \subset \mathbb{R}^3$ be the torus of revolution from class (revolving a circle of radius 1 centered at $(2, 0)$ in the xz-plane about the z-axis). It's parameterized by*

$$\phi(t, \theta) = (\cos\theta(2 + \cos t), \sin\theta(2 + \cos t), \sin t).$$

(a) Compute using the parameterization all the points whose tangent space is the xy-plane.

(b) Repeat (a) for the yz-plane.

(c) Similarly, determine all the points whose tangent space contains the z-axis.

Finally, draw each of the sets you found above on the torus.

Solution. For the torus, we know the tangent vectors should be

$$\phi_t = (-\sin t\cos\theta, -\sin t\sin\theta, \cos t), \phi_\theta = (-2\sin\theta - \cos t\sin\theta, 2\cos\theta + \cos\theta\cos t, 0).$$

And their normal should be

$$N = \phi_t \times \phi_\theta = -(\cos t + 2)(\cos\theta\cos t, \sin\theta\cos t, \sin t)$$

(a) For tangent places of all points to be xy-plane, their normal vector shall all be all parellel to $(0, 0, 1)$. Therefore, we need $\cos\theta\cos t = 0$ and $\sin\theta\cos t = 0$. In which we know $\cos t = 0$. Therefore, we need for all θ, $t = \frac{(1+2k)\pi}{2}, \forall k \in \{0, 1\}$. To parameterize it, we have then the set

$$\{(2\cos\theta, 2\sin\theta, \pm 1), \forall\theta \in [0, 2\pi)\}$$

(b) Similarly, for tangent places of all points to be yz-plane, their normal vector shall all be parellel to $(1, 0, 0)$. Therefore, we need $\sin\theta\cos t = 0$ and $\sin t = 0$. From $\sin t = 0$, we get $t = 0$ or $t = \pi$. Since the x component is proportional to $\cos\theta\cos t$, and shall not be 0, we know that $\cos t \neq 0$. From above, since $\sin\theta\cos t = 0$, we know $\sin\theta = 0$, in which implies $\theta = 0$ or $\theta = \pi$. To parameterize it, we have the set

$$\{(3, 0, 0), (1, 0, 0), (-1, 0, 0), (-3, 0, 0)\}.$$

(c) For all points whose tangent space contains the z-axis, is equivalent to say z-axis is orthogonal to the normal. Which is to say, $N \cdot (0, 0, k) = 0 = N_z, \forall k \in \mathbb{R}$. We then have $\sin t = 0 \implies t = 0$ or $t = \pi$. To parameterize it, we have the set

$$\{(3\cos\theta, 3\sin\theta, 0), \forall\theta \in [0, 2\pi)\} \cup \{(cos\theta, \sin\theta, 0), \forall\theta \in [0, 2\pi)\}$$

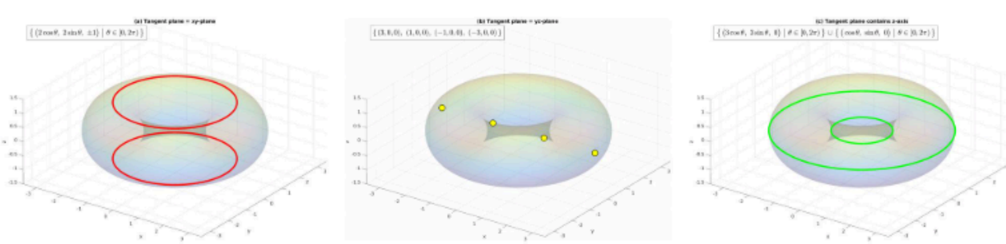


Figure 1: Torus of revolution with the sets from parts (a)–(c).

□

PROBLEM 2

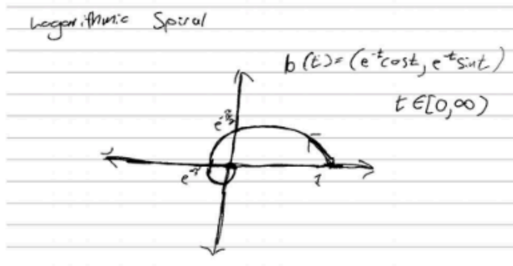
Let $v = (v_1, v_2, v_3)$ be some unit vector. Let $c(t)$ be some curve written as $c(t) = (x(t), y(t), z(t))$. Let $F(t) = \langle c(t), v\rangle = v_1 x(t) + v_2 y(t) + v_3 z(t)$. Observe that $F'(t) = v_1 x'(t) + v_2 y'(t) + v_3 z'(t) = \langle c'(t), v\rangle$. So by the fundamental theorem of calculus:

$$\int_a^b \langle c'(t), v\rangle \; dt = \int_a^b F'(t) \; dt = F(b) - F(a) = \langle c(b) - c(a), v\rangle$$

Now, consider $F'(t) = \langle c'(t), v\rangle = |c'(t)||v| \cos\theta$. v is a unit vector, so $|v| = 1$. $\cos\theta \in [-1, 1]$, so $|c'(t)||v| \cos\theta = |c'(t)| \cos\theta \leq |c'(t)|$. This completes the inequality:

$$(c(a) - c(b)) \cdot v = \int_a^b c'(t) \cdot v \; dt \leq \int_a^b |c'(t)| \; dt$$

If we let $v = (c(b) - c(a))/|c(b) - c(a)|$, then we can see that $|c(b) - c(a)| \leq \int_a^b |c'(t)| \; dt$ so the norm of the line between $c(a)$ and $c(b)$ is less than or equal to any curve. Thus, the shortest path between two points is a straight line. □



Problem 3. *The curve $b(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$ for $t \in [0, \infty)$ is called the logarithmic spiral. Give a rough plot of this curve by hand, and compute its length.*

Solution. The length of the curve is given by $\int_0^\infty |b'(t)|\, dt$.

$b'(t) = (-e^{-t}(\sin t + \cos t), -e^{-t}(\sin t - \cos t))$

$|b'(t)|^2 = e^{-2t}(\sin^2 t + \cos^2 t + 2\sin t\cos t) + e^{-2t}(\sin^2 t + \cos^2 t - 2\sin t \cos t) = 2e^{-2t}$

$|b'(t)| = \sqrt{2}e^{-t}$

$\int_0^\infty |b'(t)|\, dt = \int_0^\infty \sqrt{2}e^{-t}\, dt = -\sqrt{2}(e^{-\infty} - e^0) = \sqrt{2} =$ length of the logarithmic spiral. $\square$

Problem 4. *Give an explicit unit-speed parameterization for the logarithmic spiral*

Solution. First, I want to solve for $\lambda : [0, \infty) \to [0, \sqrt{2}]$

$\lambda(t) = \int_0^t |b'(s)| \, ds = \sqrt{2} \int_0^t e^{-s} \, ds = -\sqrt{2}(e^{-t} - 1) = \sqrt{2}(1 - e^{-t})$

$e^{-t} = 1 - \frac{\lambda(t)}{\sqrt{2}} \Rightarrow t = -\ln(1 - \frac{\lambda(t)}{\sqrt{2}})$

So, $r : [0, \sqrt{2}] \to [0, \infty)$ is $r(t) = -\ln(1 - \frac{t}{\sqrt{2}})$

The unit speed curve should then be

$(b \circ r)(t) = ((1 - \frac{t}{\sqrt{2}}) \cos[\ln(1 - \frac{t}{\sqrt{2}})], -(1 - \frac{t}{\sqrt{2}}) \sin[\ln(1 - \frac{t}{\sqrt{2}})])$

I can check for unit speed by finding that

$(b \circ r)'(t) = (\frac{1}{\sqrt{2}} \cos[\ln(1 - \frac{t}{\sqrt{2}})] - \frac{1}{\sqrt{2}} \sin[\ln(1 - \frac{t}{\sqrt{2}})], -\frac{1}{\sqrt{2}} \sin[\ln(1 - \frac{t}{\sqrt{2}})] - \frac{1}{\sqrt{2}} \cos[\ln(1 - \frac{t}{\sqrt{2}})])$

Then using the trig identity $\cos^2 \theta + \sin^2 \theta = 1$ to find that

$|(b \circ r)'(t)|^2 = 2(\frac{1}{\sqrt{2}})^2 = 1$ □

Problem 5. *Derive a general formula for the area of a surface of revolution (say, of a curve $c(t) = (x(t), 0, z(t)$ in the xz-plane, revolving about the z-axis) by finding a chart and computing the first fundamental form.*[1]

Let $c(t) = (x(t), 0, z(t))$, $t \in [a, b]$, be a regular curve in the xz-plane with $x(t) \geq 0$. Rotating c about the z-axis gives the parametrization

$$\Phi(t, \theta) = \big(x(t)\cos\theta,\ x(t)\sin\theta,\ z(t)\big), \qquad (t, \theta) \in [a, b] \times [0, 2\pi].$$

First fundamental form. Compute the partials:

$$\Phi_t = \big(x'(t)\cos\theta,\ x'(t)\sin\theta,\ z'(t)\big), \qquad \Phi_\theta = \big(-x(t)\sin\theta,\ x(t)\cos\theta,\ 0\big).$$

Hence

$$E = \langle \Phi_t, \Phi_t \rangle = x'(t)^2 + z'(t)^2, \qquad F = \langle \Phi_t, \Phi_\theta \rangle = 0, \qquad G = \langle \Phi_\theta, \Phi_\theta \rangle = x(t)^2.$$

Therefore

$$\sqrt{EG - F^2} = \sqrt{(x'^2 + z'^2)\, x^2} = x(t)\sqrt{x'(t)^2 + z'(t)^2}.$$

Area. The area of the surface is

$$\boxed{A = \iint \sqrt{EG - F^2}\, dt\, d\theta = \int_0^{2\pi} \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2}\, dt\, d\theta = 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2}\, dt.}$$

Solution. □

[1] Perhaps you did this some other way in MVC.

Problem 6. *Consider the surface obtained by revolving the curve $c(t) = (1/t, 0, t)$ for $t \geq 1$ about the z-axis. Compute the area of this surface using the previous problem*[2]. *Compute the volume bounded by this surface and the plane $z = 1$.*[3] *Explain why these computations are paradoxical (how long would it take to paint the surface versus fill it with paint).*

Solution. Using the formula we computed in the previous problem, we have that the surface area of this surface is

$$\begin{aligned}&\lim_{t\to\infty}\int_1^t 2\pi\sqrt{\frac{1}{s^2}\left(\frac{1}{s^4}+1\right)}\,ds\\ &= 2\pi\lim_{t\to\infty}\int_1^t \frac{1}{s}\sqrt{\frac{1}{s^4}+1}\,ds\end{aligned}$$

However, note that $\sqrt{\frac{1}{s^4}+1} > 1$, so we have that $\frac{1}{s}\sqrt{\frac{1}{s^4}+1} > \frac{1}{s}$ and hence that

$$\begin{aligned}2\pi\lim_{t\to\infty}\int_1^t \frac{1}{s}\sqrt{\frac{1}{s^4}+1}\,ds &> 2\pi\lim_{t\to\infty}\int_1^t \frac{1}{s}\,ds\\ &= 2\pi\lim_{t\to\infty}(\ln t - \ln 1)\\ &= 2\pi\lim_{t\to\infty}\ln t\end{aligned}$$

However, $\ln t$ diverges as you approach infinity, and since the surface area is bounded below by it, the surface area diverges as well. So, this surface has infinite surface area.

Now we compute the volume of the surface by summing the areas of intersection of the solid with the plane $z = t$, where t varies. First, we find a parameterization of the surface:

$$\phi(t,\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{t}\\ 0\\ t\end{pmatrix} = \left(\frac{\cos\theta}{t}, \frac{\sin\theta}{t}, t\right)$$

Now, let $z = s \in \mathbb{R}$. Then the equation for the intersection is $(\frac{\cos\theta}{s}, \frac{\sin\theta}{s}, s)$, which is a circle with radius $\frac{1}{s}$. So, the area of the intersection of the solid is $\frac{\pi}{s^2}$.

Now we sum all of these areas:

$$\begin{aligned}\lim_{t\to\infty}\int_1^t \frac{\pi}{s^2}\,ds &= \pi\lim_{t\to\infty}\int_1^t \frac{1}{t^2}\,ds\\ &= \pi\lim_{t\to\infty}(-\frac{1}{t}+1)\\ &= \pi(0+1) = \pi\end{aligned}$$

So, the surface area of the surface is infinite, but the volume, which is π, is clearly finite. This is paradoxical because it would take infinite paint to cover the whole surface, but finite paint to fill it.

□

[2] Hint: reduce to the computation of $\int_1^\infty \frac{1}{a}\,da$, which you may look up if you don't remember.

[3] Thinking in terms of Riemann sums, the volume is given by $\int_1^\infty A(t)\,dt$, where $A(t)$ is the area of the intersection of the solid with the plane $x = t$.