Problem 1. *Let* $\alpha : I \to \mathbb{R}^3$ *and* $\beta : I \to \mathbb{R}^3$ *be two curves. Let* $\langle \alpha, \beta \rangle : I \to \mathbb{R}$ *be the function defined by* $\langle \alpha, \beta \rangle(t) = \langle \alpha(t), \beta(t) \rangle$*, where* $\langle \cdot, \cdot \rangle$ *denotes the standard inner product on* $\mathbb{R}^3$*. Prove that*

$$\langle \alpha, \beta \rangle'(t) = \langle \alpha'(t), \beta(t) \rangle + \langle \alpha(t), \beta'(t) \rangle.$$

Solution. By the definition of the standard inner product, we have

$$\begin{aligned}
\langle \alpha, \beta \rangle'(t) &= \frac{d}{dt} \sum_{i=1}^{3} \alpha_i(t)\beta_i(t) && (1) \\
&= \sum_{i=1}^{3} \frac{d}{dt} \alpha_i(t)\beta_i(t) && (2) \\
&= \sum_{i=1}^{3} \alpha_i'(t)\beta_i + \alpha_i(t)\beta_i'(t) && (3) \\
&= \sum_{i=1}^{3} \alpha_i'(t)\beta_i + \sum_{i=1}^{3} \alpha_i(t)\beta_i'(t) && (4) \\
&= \langle \alpha'(t), \beta(t) \rangle + \langle \alpha(t), \beta'(t) \rangle && (5)
\end{aligned}$$

Where (1) follows from the definition of the inner product, (2) from the linearity of the derivative, (3) from the product rule, (4) from the linearity of finite sums, and (5) from the definition of the inner product. □

Problem 2. *Show that the cylinder $\{(x,y,z) : x^2+y^2=1\}$ is a surface by covering it with two coordinate charts.*

Solution. Let $U \subset \mathbb{R}^2$ with $U = (-2,0)\times(-\infty,\infty) \cup (0,2)\times(-\infty,\infty)$. We define two charts $\phi_1 : U \to \mathbb{R}^3$, $\phi_2 : U \to \mathbb{R}^3$ as follows:

$$\phi_1 = \begin{cases} (\sqrt{1-(u-1)^2}, u-1, v), & u \in (0,2) \\ (-\sqrt{1-(u+1)^2}, u+1, v) & u \in (-2,0) \end{cases}$$

$$\phi_2 = \begin{cases} (u-1, \sqrt{1-(u-1)^2}, v) & u \in (0,2) \\ (u+1, -\sqrt{1-(u+1)^2}, v) & u \in (-2,0) \end{cases}$$

Now we show that these are charts.

For injectivity, suppose $\phi_1(u,v) = \phi_1(x,y)$. Then there are two possibilities: $(u,v),(x,y)$ are both in $(0,2)\times(-\infty,\infty)$ or both in $(-2,0)\times(\infty,-\infty)$, or they are in different intervals.

However, note that the latter case is not possible, as that would imply that $\sqrt{1-(u-1)^2} = -\sqrt{1-(x+1)^2}$, which is only possible when $\sqrt{1-(u-1)^2} = -\sqrt{1-(x+1)^2} = 0$. Then it would have to be that $u=2$ and $x=-2$, but then (u,v) and (x,y) are not in our domain U. So, we can disregard this case.

So, we know that if $\phi_1(u,v) = \phi_1(x,y)$, then (u,v) and (x,y) are in the same interval, so it must be that $u-1 = x-1$ or $u+1 = x+1$, and $v = y$. It is easy to see that in both cases, we have that $(u,v) = (x,y)$. Hence ϕ_1 is injective, and by similar reasoning, it follows that ϕ_2 is injective (the only difference is that the coordinates are rearranged, but there is no difference in the computations).

Now we show that they are differentiable. Note that $u-1, u+1, v$ are linear and thus differentiable everywhere. As for $\sqrt{1-(u-1)^2}$ and $\sqrt{1-(u+1)^2}$, note that they are defined on U, and that they are the composition of functions $\sqrt{t}$ and $1-(t-1)^2, t-(t+1)^2$, which are both differentiable on the intervals $(0,2)$ and $(-2,0)$. Hence the composition is also differentiable on those intervals. Since each of the component functions of ϕ_1 and ϕ_2 are differentiable, they are differentiable.

Now, we compute $D\phi_1$ and $D\phi_2$:

$$D\phi_1 = \begin{cases} \begin{pmatrix} -\frac{u-1}{\sqrt{1-(u-1)^2}} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & u \in (0,2) \\ \begin{pmatrix} \frac{u+1}{\sqrt{1-(u+1)^2}} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & u \in (-2,0) \end{cases} \qquad D\phi_2 = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{u-1}{\sqrt{1-(u-1)^2}} & 0 \\ 0 & 1 \end{pmatrix} & u \in (0,2) \\ \begin{pmatrix} 1 & 0 \\ \frac{u+1}{\sqrt{1-(u+1)^2}} & 0 \\ 0 & 1 \end{pmatrix} & u \in (-2,0) \end{cases}$$

Note that in all these matrices, the columns are linearly independent, as the first column can never be a scalar multiple of the second column. Then, since $D\phi_1$ and $D\phi_2$ are linear maps, this shows that they are injective.

Finally, we show that these charts cover all of the cylinder. First, note that $\text{Im}\,\phi_1, \text{Im}\,\phi_2 \subset \{(x,y,z) : x^2+y^2=1\}$. Next, observe that ϕ_1 almost fully covers the cylinder, as it covers two halves of it, but not the seams of the halves. However, those seam points are covered by ϕ_2. So, for every point on the cylinder, there exists a chart mapping onto it. Thus the cylinder is a surface. □

Problem 3. *Let $\alpha : I \to \mathbb{R}^3$ be a curve. Prove that $|\alpha(t)|$ is constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.*

Solution. Suppose $|\alpha(t)|$ is constant, i.e., $|\alpha(t)| = c$ for all $t \in I$ and some c. Equivalently, we have $\langle \alpha(t), \alpha(t) \rangle = c^2$. We have by **Problem 1** that $\langle \alpha, \beta \rangle'(t) = \langle \alpha'(t), \beta(t) \rangle + \langle \alpha(t), \beta'(t) \rangle$. Differentiating both sides of our original expression, we obtain

$$\langle \alpha, \alpha \rangle'(t) = \langle \alpha(t)', \alpha(t) \rangle + \langle \alpha(t), \alpha(t)' \rangle = 0$$

As the derivative of a constant vanishes. But the inner product on $\mathbb{R}$ is symmetric, so we swamp combine terms to yield

$$2\langle \alpha(t), \alpha(t)' \rangle = 0 \implies \langle \alpha(t), \alpha(t)' \rangle = 0$$

And so $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Suppose $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$, so that $\langle \alpha(t), \alpha'(t) \rangle = 0$ for all t. We then have that

$$2\langle \alpha(t), \alpha'(t) \rangle = 0$$

And so by symmetry

$$\langle \alpha(t), \alpha'(t) \rangle + \langle \alpha'(t), \alpha(t) \rangle = \langle \alpha(t), \alpha(t) \rangle' = 0$$

This implies

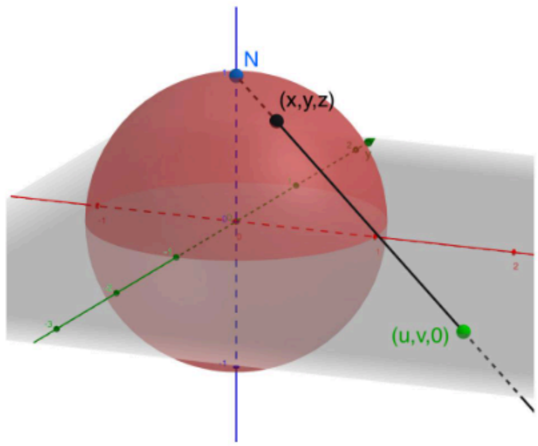
$$(|\alpha(t)|^2)' = 0$$

For all t, and so $|\alpha(t)|^2$ is constant for all t, since the derivative of a differentiable function vanishes everywhere if and only if it is constant. This implies that $|\alpha(t)|$ is constant as well, as desired.

□

Problem 4. *Let $N = (0,0,1) \in S^2$ be the north pole. Define stereographic projection $\pi : S^2 \backslash \{N\} \to \mathbb{R}^2$ as follows. Given $p = (x,y,z)$, define $\pi(p)$ as the intersection of the line between N and p with the xy-plane.*[1] *Derive a formula for the inverse of π, and check that it is a chart.* [2]

Solution. Suppose we have $(u,v,0)$ in the xy-plane. We want to find the point $p = (x,y,z)$ such that $\pi(p) = (u,v,0)$.



Nice picture!

We do this by noting that there are two restrictions for p: it needs to lie on S_2, and it needs to lie on the line between N and $(u,v,0)$, which can be parameterized as $(0,0,1) + t(u,v,-1)$ for $t \in \mathbb{R}$. The first condition means that $x^2 + y^2 + z^2 = 1$, and the second condition means that $(x,y,z) = (0,0,1) + t(u,v,-1)$ for some t. Hence, we have that $x = tu, y = tv, z = 1-t$, and plugging this into our first equation gives us that

$$\begin{aligned}(tu)^2 + (tv)^2 + (1-t)^2 &= 1\\ t^2u^2 + t^2v^2 + 1 - 2t + t^2 &= 1\\ t(tu^2 + tv^2 + t - 2) &= 0\end{aligned}$$

Now we can have $t = 0$ or $(tu^2 + tv^2 + t - 2) = 0$; note that $t = 0$ just gives us the point $(0,0,1)$, which is not the point we are looking for, so we consider the other option:

$$\begin{aligned}tu^2 + tv^2 + t - 2 &= 0\\ t(u^2 + v^2 + 1) &= 2\\ t &= \frac{2}{u^2+v^2+1}\end{aligned}$$

Hence $p = \left(\dfrac{2u}{u^2+v^2+1}, \dfrac{2v}{u^2+v^2+1}, \dfrac{u^2+v^2-1}{u^2+v^2+1}\right)$.

Thus we can define an inverse map $\pi^{-1} : \mathbb{R}^2 \to S^2$ defined as $\pi^{-1}(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$. We show this is a chart.

[1] Draw a picture.

[2] This problem gives yet another way to cover the sphere by coordinate charts.

For injectivity, suppose that $\pi^{-1}(u,v) = \pi^{-1}(x,y)$. Then we have

$$\begin{aligned}
\frac{u^2+v^2-1}{u^2+v^2+1} &= \frac{x^2+y^2-1}{x^2+y^2+1} \\
(u^2+v^2-1)(x^2+y^2+1) &= (u^2+v^2+1)(x^2+y^2-1) \\
u^2+v^2-x^2-y^2 &= -u^2-v^2+x^2+y^2 \\
2u^2+2v^2 &= 2x^2+2y^2 \\
u^2+v^2 &= x^2+y^2
\end{aligned}$$

We also have that

$$\begin{aligned}
\frac{2u}{u^2+v^2+1} &= \frac{2x}{x^2+y^2+1} \\
\frac{2u}{x^2+y^2+1} &= \frac{2x}{x^2+y^2+1} \quad \text{from what we showed above} \\
2u &= 2x \\
u &= x
\end{aligned}$$

Similarly, we can obtain that $v = y$, and hence π^{-1} is injective.

Now we compute $D\pi^{-1}$:

$$D\pi^{-1} = \begin{pmatrix} \frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2} & \frac{-4uv}{(u^2+v^2+1)^2} \\ \frac{-4uv}{(u^2+v^2+1)^2} & \frac{2u^2-2v^2+2}{(u^2+v^2+1)^2} \\ \frac{4u}{(u^2+v^2+1)^2} & \frac{4v}{(u^2+v^2+1)^2} \end{pmatrix}$$

Note that since $(u^2+v^2+1)^2 > 0$, the derivative is always defined. Additionally, each component function of π^{-1} is the product of functions that are differentiable on $\mathbb{R}^2$, and so each component is differentiable. Thus, so is π^{-1}.

Now, we note that the two columns of $D\pi^{-1}$ are linearly independent: suppose, for sake of contradiction, that the left column is a scalar multiple of the right column. Then, in particular, there exists a such that $a \cdot \frac{4u}{(u^2+v^2+1)} = \frac{4v}{(u^2+v^2+1)}$. It is evident that $a = \frac{v}{u}$, but note that if we multiply $\frac{-4uv}{(u^2+v^2+1)^2}$ by $\frac{v}{u}$, we do not get $\frac{2u^2-2v^2+2}{(u^2+v^2+1)^2}$. Hence the columns cannot be scalar multiples of each other, and $D\pi^{-1}$ is injective.

We can conclude that π^{-1} is a chart.

□

Problem 5. *Show that the tangent plane of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at $p = (u, v, f(u, v))$ is the graph of the differential $Df_{(u,v)}$.*

Solution. Consider the canonical coordinate chart $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $(u, v) \mapsto (u, v, f(u, v))$. Then, $(D\phi)_{(u,v)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_u(u,v) & f_v(u,v) \end{bmatrix}$, so the tangent space is spanned by $\begin{bmatrix} 1 \\ 0 \\ f_u(u,v) \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ f_v(u,v) \end{bmatrix}$, which is $\begin{bmatrix} a \\ b \\ af_u(u,v) + bf_v(u,v) \end{bmatrix}$ for real a, b. Note that the z-coordinate is precisely the differential $Df_{(u,v)} = af_u(u,v) + bf_v(u,v)$, so $\begin{bmatrix} a \\ b \\ af_u(u,v) + bf_v(u,v) \end{bmatrix}$, is the graph of this differential.

□

Problem 6. *Derive a formula for a differentiable map from a rectangle in $\mathbb{R}^2$ whose image is a Möbius strip in $\mathbb{R}^3$. Do the same for an annulus with two twists. Include an image of your surfaces, plotted using* `ParametricPlot3D` *in Mathematica, or similar.*

Solution. We derive our formula by considering the Möbius strip as *almost* a surface of revolution—almost in that the curve in the xz-plane rotates as the surface is rotated about the z-axis. This is to say we will parameterize the rotation in the vertical plane by the same parameter used for the rotation in the horizontal plane. We begin by parameterizing a line through the origin $\mathbf{c} = (0, t, 0)$ for $t \in (-1/2, 1/2)$. To make this line complete exactly half of a period of rotation during the full revolution about the z-axis (as defines the Möbius strip), we apply the rotation matrices

$$R_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{s}{2} & -\sin\frac{s}{2} \\ 0 & \sin\frac{s}{2} & \cos\frac{s}{2} \end{pmatrix} \qquad R_h = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Where R_v provides the rotation in the vertical plane and R_h in the horizontal plane, presuming $s \in (0, 2\pi)$. That is to say that our surface $\mathbf{S}(s,t) = R_h(s)(R_v(s)\mathbf{c}(t) + (0,1,0))$, where the $(0,1,0)$ provides an offset from the z-axis so as to avoid overlap. Applying these transformations yields the formula

$$\mathbf{S}(s,t) = \begin{pmatrix} t\sin s \sin\frac{s}{2} - \sin s \\ -t\cos s\sin\frac{s}{2} + \cos s \\ t\cos\frac{s}{2} \end{pmatrix}$$

To convert this formula into that of a double-twisted annulus, we need only double the speed at which the curve rotates in the vertical plane, which is to say change the $s/2$ terms in R_v to s. This yields the analogous formula

$$\mathbf{S}(s,t) = \begin{pmatrix} t\sin^2 s - \sin s \\ -t\cos s\sin s + \cos s \\ t\cos s \end{pmatrix}$$

Where s and t have the same bounds as before. A Mathematica-generated plot of each surface is included below.

□

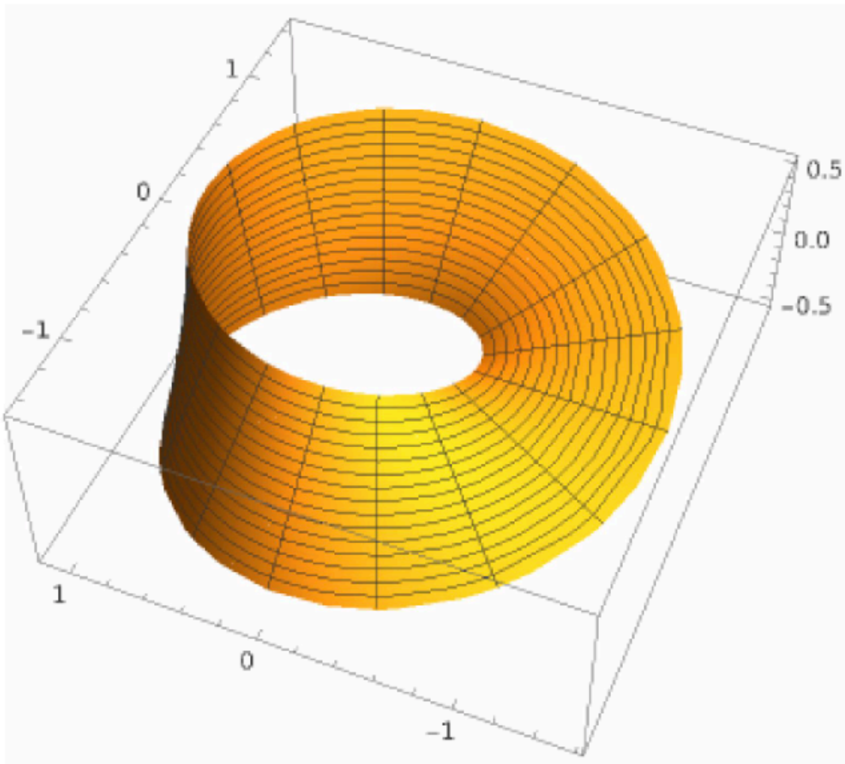


Figure 1: A Möbius strip plotted with `ParametricPlot3D`.

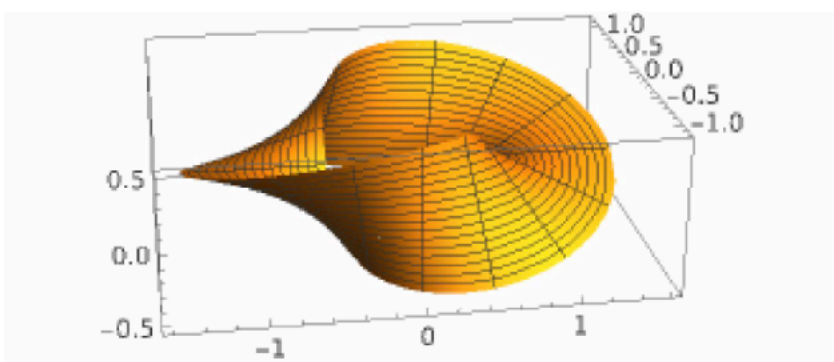


Figure 2: A double-twisted annulus plotted with `ParametricPlot3D`.