Homework 6

Math 2420

Due Friday, March 8 by 5pm

Your Name:

Collaborator names:

Topics covered: cellular approximation, Whitehead's theorem

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (encouraged!), please list your collaborators above.
- If you are stuck, please ask for help (from me or a classmate). Use Campuswire!
- You may freely use any fact proved in class. Usually you should be able to solve the problems without outside knowledge. You should provide proof for facts that you use that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Show the double comb space is not contractible.*[1]

Solution. Denote the double comb space by X. If X is contractible, then there is a homotopy from the identity to some constant map. Since X is connected, we can assume that constant map has value the wedge point x_0.

First show there no homotopy that fixes x_0 for all time, i.e. no deformation retract. For a deformation retract x_0 is fix for all time. Let U be a small neighborhood of x_0 that is not connected. By continuity, there exists a neighborhood $V \subset U$ of x_0 with the property that $f_t(V) \subset U$ for all t. Consider a point y in V in a different component of V than x_0. On the one hand, the path $f_t(y)$ stays in U by construction. On the other hand, $f_t(y)$ is a path from y to x_0 so it cannot stay in U since y and x_0 are not in the same component of U.

Next we show the general case. Let $t_0 \in [0, 1]$ be the largest time so that $f_t(x_0) = x_0$ for $t \leq t_0$. I.e. this is the first time when x_0 starts to move. Wlog, say x_0 moves "up" a small amount at time $t_0 + \epsilon$. Now choose a neighborhood U of $f_{t_0+\epsilon}(x_0)$ that is small enough to be disjoint from points to the "right" of x_0. By continuity (and compactness of I), there exists a neighborhood V of x_0 so that $f_t(V) \subset U$ for $t \leq t_0 + \epsilon$. Fix $y \in V$ to the "right" of x_0 in a different component. On the one hand, the path $f_t(y)$ stays in U by construction, but this is already a contradiction since $f_0(y) = y \notin U$. □

[1]Hint: Let x_0 be the wedge point. It may help to start by showing that there is no deformation retract to x_0. Use continuity to argue that there are small neighborhoods $x_0 \in V \subset U$ so that $f_t(V) \subset U$ for all t. Use this to reach a contradiction. The general argument should work similarly.

Problem 2 (Hatcher 4.1)**.** *Show that a cell complex X is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic.* [2] [3]

Solution. By Whitehead's theorem, it suffices to show that $\pi_k(X) = 0$ for each k. Fix $f : S^k \to X$. Since $f(S^k)$ is compact and the $X = \bigcup X_i$, we find that $f(S^k)$ is contained in X_k for some k. By assumption, f the composition $f : S^k \to X_k \to X_{k+1} \to X$ is nullhomotopic. This means that $f : S^k \to X^{k+1}$ extends to a map $\hat{f} : D^{k+1} \to X^{k+1}$. Using this disk (and a deformation retract of D^{k+1} to the basepoint on S^k), we conclude that f is homotopic through based maps to a constant, which shows $[f] = 0$ in $\pi_k(X)$. $\square$

[2] Hint: your solution should be short.

[3] Remark: note that this implies that S^∞ is contractible, as similarly the union of iterated suspensions $\Sigma^n X$ for any space X.

Problem 3. *Show that any two $K(A,n)$ spaces are homotopy equivalent.*[4] *Where does this argument break down in trying to show that $\mathbb{R}P^2$ and $\mathbb{S}^2 \times \mathbb{R}P^\infty$ are homotopy equivalent?*

Solution. By cellular approximation for space, there is cell complex X that is a $K(A,n)$ and only has cells in dimension $\geq n$. Let Y be any other $K(A,n)$ space. We define a homotopy equivalence $f : X \to Y$ inductively over skeleta.

For the base of the induction, note (wlog) $X^{(n)} = \bigvee S^n_\alpha$ where α ranges over a generating set of A. Choose representatives $\phi_\alpha : S^n \to Y$ for $\pi_n(Y)$, and define $f : X^{(n)} \to Y^{(n)}$ as the map $f = \bigvee \phi_\alpha$.

Each $(n+1)$-cell e of X gives a relation on $\pi_n(X)$, and since $\pi_n(X) \cong \pi_n(Y)$, $f\mid_{\partial e}$ is nullhomotopic in Y, so f extends over the attachment of e. At this stage, we can observe that $f : X^{(n+1)} \to Y^{(n+1)}$ is an isomorphism on π_n.

Now inductively, we extend $f : X^{(m)} \to Y^{(m)}$ to a map $f : X^{(m+1)} \to Y^{(m+1)}$. The key point is that since $\pi_m(Y) = 0$, so for any cell attachment $\phi_\alpha : \partial e^{m+1} \to X$, the composition $f \circ \phi_\alpha$ is nullhomotopic, so it extends over e. This does not change the induced map on π_n, and since the other homotopy groups are 0, the induced maps are automatically isomorphisms.

In the above argument we use that the higher homotopy group (beyond n) vanish to extend the map. This is the step that doesn't work in showing $\mathbb{R}P^2$ and $\mathbb{S}^2 \times \mathbb{R}P^\infty$ are homotopy equivalent (and indeed they are not). $\square$

[4] Hint: Construct a weak homotopy equivalence inductively. You can use cellular approximation of spaces, to make your life a little simpler.

Problem 4. *Associated to a fiber bundle $F \to E \to S^1$, there is a homeomorphism $\phi : F \to F$ call the monodromy, which is well-defined up to isotopy. It's defined as the "first return map" of a sequence of local trivializations around the circle.*[5] *For the trefoil knot complement fibering $F \to S^3 \setminus K \to S^1$, compute the action of the monodromy on $H_1(F)$.*[6]

Solution. $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. This matrix has order 6. $\square$

[5] For example, the monodromy of the n-fold covering map $S^1 \to S^1$ is a cyclic permutation. Understand this!

[6] Hint: start by fixing a basis, so that the answer is a 2×2 matrix.