Homework 5

Math 2420

Due Friday, March 1 by 5pm

Your Name:

Collaborator names:

Topics covered: homotopy excision, cellular maps

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (encouraged!), please list your collaborators above.
- If you are stuck, please ask for help (from me or a classmate). Use Campuswire!
- You may freely use any fact proved in class. Usually you should be able to solve the problems without outside knowledge. You should provide proof for facts that you use that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Fix a pair (X, A). Assume that (X, A) is n-connected and A is m-connected. Prove that $\pi_k(X, A) \cong \pi_k(X/A)$ for $k \le n + m$.* [1]

Solution. Since CA is contractible

$$X \cup CA \simeq (X \cup CA)/CA \cong X/A.$$

Combining this with the LES for $(X \cup CA, CA)$ we obtain (again using CA contractible)

$$\pi_k(X/A) \cong \pi_k(X \cup CA) \cong \pi_k(X \cup CA, CA).$$

Now we use homotopy excision. We have a space $X \cup_A CA$. The pair (X, A) is n-connected and (CA, A) is $(m+1)$-connected by assumption, so by homotopy excision

$$\pi_k(X \cup CA, CA) \cong \pi_k(X, A)$$

for $k \le m + n$. □

[1] Hint: replace X/A with $X \cup CA$ (where CA is the cone), and apply the LES of a pair and excision. Your solution should be pretty short.

Problem 2. *For* $A = S^1 \vee S^1$ *embedded in* S^2*, compute* $\pi_2(X/A)$*. Compare with* $\pi_2(X, A)$*.* [2]

Solution. The quotient is $S^2 \vee S^2 \vee S^2$ and has $\pi_2 = \mathbb{Z}^3$. This is different from $\pi_2(X, A)$ computed on a previous HW. $\square$

[2] Remark: this example shows the failure of excision for homotopy groups.

Problem 3. *Let X, Y be cell complexes and assume they are homotopy equivalent. Show that the n-skeleta $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent if X, Y don't have $(n+1)$-dimensional cells. Give an example to show this assumption on $(n+1)$-cells is necessary.*

Solution. By assumption, there exists a homotopy equivalence $f : X \to Y$ with a homotopy inverse $g : Y \to X$. By cellular approximation, we can assume that f, g are cellular and still homotopy inverses. In particular by restriction, we obtain maps $f' : X^{(n)} \to Y^{(n)}$ and $g' : Y^{(n)} \to X^{(n)}$. We claim that f', g' are homotopy inverses. By assumption, there is a homotopy $H : X \times I \to X$ between $g \circ f$ and the identity. Since $g \circ f$ and the id_X are cellular, we can use the relative version of cellular approximation to homotope H to a cellular map H' without changing H on $X \times \{0, 1\}$. Since X has no cells of dimension $(n+1)$ the image of H' is contained in the n-skeleton, so it restricts to a homotopy $X^{(n)} \times I \to X^{(n)}$ between $g' \circ f'$ and the identity. By a similar argument we show that $f' \circ g'$ is homotopic to the identity on $Y^{(n)}$.

For the counterexample, consider two cell structures on S^1, where X has two 0-cells and two 1-cells, and Y has one 0-cell and one 1-cell. Here X and Y are even homeomorphic, but the 0-skeleta are not homotopy equivalent (one is connected and the other is not). □

Problem 4. *Generally, for a path $\gamma : I \to X$, there is a change of basepoint homomorphism $\pi_k(X, x_0) \to \pi_k(X, x_1)$, where $x_i = \gamma(i)$ for $i = 0, 1$. See Hatcher pg. 341. Taking γ to be a loop based at x_0 defines an action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$ for $k \geq 1$.*

(a) *Let $i : S^2 \hookrightarrow S^2 \vee S^1$ be the inclusion, viewed as an element of $\pi_2(S^2 \vee S^1)$. Describe the orbit of i under the action of $\pi_1(S^2 \vee S^1) \cong \mathbb{Z}$.* [3] [4]

(b) *Compute the action of $\pi_1(\mathbb{R}P^n)$ on $\pi_n(\mathbb{R}P^n) = \mathbb{Z}$.* [5]

Solution. (a) As a π_1 module, π_2 is isomorphic to $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$. One can draw a picture for how the generator of π_1 acts on the inclusion $S^2 \to S^1 \vee S^2$.

(b) To identify the action it helps to lift to the universal cover. A key part of the computation is to show $\widetilde{\gamma.f} = \widetilde{\gamma} * (\delta_\gamma \circ \widetilde{f})$, where $\widetilde{\gamma}, \widetilde{f}$ are lifts based at the basepoint of S^n and δ_γ is the deck transformation. Recall that δ_γ is the antipodal map which has degree $(-1)^{n+1}$. Consequently (using Hopf degree) we see that $\pi_1(\mathbb{R}P^n)$ acts on $\pi_n(\mathbb{R}P^n) \cong \pi_n(S^n)$ by multiplication by $(-1)^{n+1}$. (Hopefully the sign is correct.) $\square$

[3] Remark: Here "describe" means in explicit geometric terms, e.g. in terms of representative maps.

[4] Remark: It is also informative to think about how the action looks on π_2 of the universal cover.

[5] Hint: the answer depends on whether n is even or odd.