Homework 4

Math 2420

Due Friday, Feb 23 by 5pm

Your Name: Bena

Collaborator names:

Topics covered: fibrations, fiber bundles, $\pi_n(S^n)$

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (encouraged!), please list your collaborators above.
- If you are stuck, please ask for help (from me or a classmate). Use Campuswire!
- You may freely use any fact proved in class. Usually you should be able to solve the problems without outside knowledge. You should provide proof for facts that you use that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Use the connecting homomorphism in the long exact sequence of the Hopf fibration $p : S^3 \to S^2$ to argue that $\pi_2(S^2)$ is generated by the identity map.*

Solution. We want to prove that the identity of $\mathbb{C}P^1$ is mapped to the identity of S^1 under the connecting map $\delta : \pi_2(S^2) \to \pi_1(S^1)$ for the Hopf fibration

$$S^1 \to S^3 \xrightarrow{p} S^2.$$

First note that we can translate between maps $\mathbb{C}P^1 \to \mathbb{C}P^1$ and maps $(D^2, S^1) \to (\mathbb{C}P^1, [0:1])$ by precomposing with the quotient map $q : D^2 \to \mathbb{C}P^1$ defined by $q(z) = [z : \sqrt{1-|z|}]$. Then instead of computing the image of id_{S^2} under the connecting homomorphism, it suffices to show that the image of $[q]$ under the connecting homomorphism is id_{S^1}. Recall that δ is a composition of isomorphisms

$$\pi_2(S^2) \xrightarrow{(p_*)^{-1}} \pi_2(S^3, S^1) \xrightarrow{\partial} \pi_1(S^1)$$

To compute $(p_*)^{-1}([q])$ we want to find a map $f : (D^2, S^1) \to (S^3, S^1)$ so that $p_*([f]) = [q]$. Observe that $f : D^2 \to \mathbb{S}^3$ defined by $f(z) = (z, \sqrt{1-|z|})$ satisfies $p \circ f = q$. Note that on ∂D^2, we have $f(e^{i\theta}) = (e^{i\theta}, 0)$, so $\partial([f]) = [id_{S^1}]$, as desired. □

Problem 2. *Let $p : E \to B$ be a Serre fibration and assume B is connected. Fix $b, b' \in B$ with fibers F, F', respectively. Define a homomorphism $\pi_k(F) \to \pi_k(F')$, and show that it's a bijection.* [1]

Solution. Fix basepoints $e \in F$. Fix a path $\gamma : I \to B$ from b to b', and let $\tilde{\gamma} : I \to E$ be a lift with $\tilde{\gamma}(0) = e$. Set $e' = \tilde{\gamma}(1) \in F'$.

Defining a homomorphism. First we define a map $\Phi : \pi_k(F, e) \to \pi_k(F', e')$. Given $f : (D^k, \partial D^k) \to (F, e_0)$, consider maps

$$h : D^k \times I \to I \xrightarrow{\gamma} B$$

where the first map is projection, and define

$$H_0 : D^k \times 0 \cup \partial D^k \times I \to E$$

to be the map that restricts to f on $D^k \times 0$ and restricts to $\partial D^k \times I$ as the composition $\partial D^k \times I \to I \xrightarrow{\tilde{\gamma}} E$. Applying the HLP (note that the pair $(D^k \times I, D^k \times 0 \cup \partial D^k \times I)$ is homeomorphic to the pair $(D^k \times I, D^k \times 0)$), we obtain $f' : (D^k, \partial D^k) \to (F', e')$ as the time one map of a homotopy lifting $H_f : D^k \times I \to E$, i.e. $f' = H_f \mid_{D^k \times 1}$. We define $\Phi([f]) = [f']$.

Well defined: Suppose $[f_0] = [f_1]$, and let $f_t : D^k \to F$ be a based homotopy between f_0, f_1. Now set up a homotopy lifting problem with

$$(D^k \times I) \times I \to I \xrightarrow{\gamma} B$$

defined in the obvious way, and

$$(D^k \times I \times 0) \cup (D^k \times 0 \times I) \cup (D^k \times 1 \times I) \cup (\partial D^k \times I \times I) \to E$$

defined on $D^k \times I \times 0$ to be the homotopy f_t; for $i = 0, 1$, it's defined on $D^k \times \{i\} \times I$ to be the homotopy lifting H_{f_i} defined above; on $\partial D^k \times I \times I$ the map factors through $\tilde{\gamma}$. By the HLP, we obtain a lifting $(D^k \times I) \times I \to E$, which by construction restricts to $D^k \times I \times 1$ as a homotopy between f' and g'. This shows Φ is well defined.

Homomorphism: We want to show $(f * g)'$ is homotopic to $(f') * (g')$. Recall from above that we have defined $(f * g)' = H_{f*g} \mid_{D^k \times 1}$ and $f' = H_f \mid_{D^k \times 1}$ and $g' = H_g \mid_{D^k \times 1}$. The key observation is that $(f') * (g')$ is the time-1 map of the concatenation $H_f * H_g$ of the homotopies H_f, H_g (concatenation happening in one of the D^k directions) and that $H_f * H_g$ is a solution to the homotopy lifting problem for $f * g$. Since solutions to HLP are unique up to homotopy, this implies that H_{f*g} and $H_f * H_g$ are homotopic (it's slightly better than this – there's is a "fiber preserving" homotopy). Any such homotopy restricts to a homotopy between $(f * g)'$ and $(f') * (g')$.

Bijection: Using the reverse $\overline{\gamma}$, we define a homomorphism $\Psi : \pi_k(F', e') \to \pi_k(F, e)$. We need to check these are inverses. We show this using the following claim.

Claim: If γ_0, γ_1 are homotopic rel endpoints b, b', then for any f, the time-1 maps of the homotopies $H_{\gamma_0, f}$ and $H_{\gamma_1, f}$ are homotopic and consequently the time-1 maps of these homotopies are homotopic (rel endpoints).

[1] Hint/suggestion: revisit the similar argument from class. Be thorough and give details for parts of the argument that were skimmed over in lecture.

First we use the claim to finish the problem. Observe that for $[f] \in \pi_k(F, e)$, the map $(f')'$ is the time-1 map of the concatenation of homotopies $H_f * H_{f'}$ (concatenation in the I direction of $D^k \times I$), and this is a solution to the HLP for f over $\gamma * \bar{\gamma}$. By the claim, this is homotopic to a solution for the HLP for f over the constant map. But a solution to the latter is the constant homotopy at f. This shows that $(f')'$ is homotopic to f.

To prove the claim, we once again set up a homotopy lifting problem. Fix a homotopy $\Gamma : I \times I \to B$ between γ_0 and γ_1. And as above, fix lifts $\tilde{\gamma}_i$ that are paths e to e'. It is easy to argue as before that Γ can be lifted to a homotopy rel endpoints $\tilde{\Gamma}$ between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. (For this it may help to view Γ as a null homotopy of $\gamma_0 * \bar{\gamma}_1$.)

Now consider maps

$$D^k \times I \times I \to I \times I \xrightarrow{\Gamma} B$$

and

$$(D^k \times 0 \times I) \cup (D^k \times 1 \times I) \cup (\partial D^k \times I \times I) \to E$$

defined by $H_{\gamma_i, f}$ on $D^k \times \{i\} \times I$ and defined on $\partial D^k \times I \times I$ by the composition $\partial D^k \times I \times I \to I \times I \xrightarrow{\tilde{\Gamma}} E$. Now the HLP gives a homotopy $D^k \times I \times I \to E$ between $H_{\gamma_0, f}$ and $H_{\gamma_1, f}$, which restricts to a homotopy (rel endpoints even) between the time-1 maps. $\square$

Problem 3. *Show every map $f : S^n \to S^n$ is homotopic to a multiple of the identity map by the following steps.*

(a) Use local PL lemma to reduce to the case that there exists $q \in S^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible linear map near each p_i.

(b) For f as in (a), consider the composition $g \circ f$ where $g : S^n \to S^n$ collapses the complement of a small ball about q to the basepoint. Use this to reduce (a) further to the case $k = 1$. [2]

(c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or a reflection.

Solution. We work with maps $f : (S^n, \infty) \to (S^n, \infty)$.

(a) By local PL, given f, there exists a polyhedron $L \subset \mathbb{R}^n \subset S^n$ (the domain) and an open set $U \subset \mathbb{R}^n$ (the codomain) such that f is PL on L and $f^{-1}(U) \subset L$.

Next we choose $q \in U$. Write $L = L_1 \cup L_2$, where L_1 is a union of polyhedra such that f is injective on each, and f is not injective on the L_2. Choose q disjoint from the image $f(L_2)$ (possible since $f(L_2)$ is a union of k-planes with $k < n$). Since an injective linear map $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, the pre-image $f^{-1}(q)$ is a finite set $\{p_1, \dots, p_k\}$.

(b) The composition of $g \circ f$ is a map that is constant away from small balls around $p_1, \dots, p_k$. From this we quickly deduce that $g \circ f$ is the concatenation of k local maps. Since g is homotopic to the identity, $g \circ f \sim f$, so we see that $[f] = [f_1] * \cdots * [f_k]$, where f_i is the map defined in a neighborhood of p_i, hence is a linear map.

(c) First we recall that $GL_n(\mathbb{R})$ has two path components determined by the determinant. One way to see this is to use Gram–Schmidt to connect any matrix by a path to an orthogonal matrix; then use that $SO(n)$ is connected (because of the fibration $SO(n) \to S^{n-1}$). (Since a reflection has determinant -1, composing with a fixed reflection gives an element in $SO(n)$, and we can take a path to the identity. This construction gives a path from any $\det = -1$ matrix to a fixed reflection.)

Given this, we see that each $[f_i]$ in (b) is either homotopic to I (identity) or R (reflection). To finish the proof, it remains to show that $I * R$ is homotopic to a constant (so these are inverses in $\pi_n(S^n)$). This is easy to see using I^k coordinates. The reflection in the first coordinate concatenated with the identity is the map

$$t \mapsto \begin{cases} 2t & t \leq 1/2 \\ 2 - 2t & t \geq 1/2 \end{cases}$$

and this map is easily seen to be homotopic to a constant. Using this we can homotope $I * R$ to the constant. □

[2] Hint: Express f as a concatenation.

Problem 4. *Give a decomposition of the trefoil knot complement $S^3 \setminus K$ into a union of copies of a punctured torus, index by points in the circle.*[3] [4] [5]

Solution. No one solved this. I'm sad by the effort. I know it's not easy, but you can and should ask questions if you're stuck and discuss with me whether your'e on the right track.

There will be a related problem on a future assignment. □

[3] Remark: in fact there is a fiber bundle $S^3 \setminus K \to S^1$, where the fibers are punctured tori. This fiber bundle structure gives a decomposition of $S^3 \setminus K$ into a union of fibers. I'm not asking you to prove there's a fiber bundle, only to explain the decomposition into a union of fibers.

[4] Hint: it's helpful to recall that S^3 is a union of solid tori $S^1 \times D^2$ and $D^2 \times S^1$ (an "inside" and an "outside" solid torus), glued along a torus $S^1 \times S^1$. Put the trefoil knot on the surface of the torus. The fibers can be described in terms of their intersections with the solid tori. You can do this so that each fiber intersects the outside solid torus in two disjoint (meridinal) disks. This should help to pin down how to choose the intersection with the inside solid torus.

[5] Hint: it's likely that some combination of pictures and explanation are most effective for expressing a solution.