Homework 3

Math 2420

Due Friday, Feb 16 by 5pm

Your Name: Bena

Collaborator names:

Topics covered: relative homotopy groups, LES of a pair, fibrations

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (encouraged!), please list your collaborators above.
- If you are stuck, please ask for help (from me or a classmate). Use Campuswire!
- You may freely use any fact proved in class. Usually you should be able to solve the problems without outside knowledge. You should provide proof for facts that you use that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Show $\mathbb{R}^n$ is not a union of finitely many k-dimensional planes when $k < n$.* [1] [2] [3]

Proof 1: induction on number of planes. We show that $\mathbb{R}^n$ is not a union of proper subspaces, inducting on the number of subspaces. The case of 1 subspace is true because $\mathbb{R}^n$ doesn't have a basis of size $< n$. Now suppose W is a union of ℓ proper subspaces and suppose V is another proper subspace. If either $V \subset W$ or $W \subset V$, then $W \cup V \neq \mathbb{R}^n$ by the induction hypothesis and the base case. Assume $V \not\subset W$ and $W \not\subset V$, choose $w \in W \setminus V$ and $v \in V \setminus W$, and consider the line L spanned by $\{v, w\}$. Since L is not contained in either V or W, it intersects each in a finite set. Hence there is $x \in L$ so that $x \notin W \cup V$, which proves $\mathbb{R}^n \neq W \cup V$. This completes the induction. $\square$

Proof 2: induction on ambient dimension. First observe that it suffices to assume $k = n-1$, since given a collection of k-planes, we can extend to a collection of $n-1$ planes and this only makes the union bigger. (There are many ways to do this, but this is irrelevant.)

Now we prove that $\mathbb{R}^n$ is not a union of $n-1$-planes by induction on n. For the base case $n = 1$, the statement is that $\mathbb{R}^1$ is not a union of finitely many points, which is true because $\mathbb{R}^1$ is uncountable. For the induction step, given a collection of $(n-1)$-planes in $\mathbb{R}^n$, consider the collection of normal vectors to these (hyper)-planes. This is a finite subset of S^{n-1}, so there is some direction $v \in \mathbb{S}^{n-1}$ so that $v^\perp$ is not parallel to any of the given $(n-1)$-planes. Then each of these planes intersects $v^\perp \cong \mathbb{R}^{n-1}$ in an $(n-2)$-plane. By the induction hypothesis, $\mathbb{R}^{n-1}$ is not a union of finitely many $(n-2)$-planes, which shows that the original collection of $(n-1)$-planes is a proper subset of $\mathbb{R}^n$. $\square$

[1] Hint: it's possible to argue by induction on the number of planes.

[2] Remark: this fact was used in the proof that $\pi_k(S^n) = 0$ for $k < n$.

[3] Remark: this result is false for vector spaces over finite fields.

Problem 2. *Say/compute as much as you can about the homotopy groups of $\mathbb{C}P^n$.* [4]

Solution. There's a fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. Since $\pi_k(S^{2n+1}) = 0$ for $k \leq 2n$, we conclude that $\pi_k(\mathbb{C}P^n) \cong \pi_{k-1}(S^1)$ in this range. Furthermore, since $\pi_k(S^1) = 0$ for $k \geq 2$, we conclude that $\pi_k(\mathbb{C}P^n) \cong \pi_k(S^{2n-1})$ for $k \geq 2$. Combining all of this with our current knowledge of $\pi_k(S^n)$, we conclude

$$\pi_k(\mathbb{C}P^n) = \begin{cases} 0 & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & 3 \leq k \leq 2n \\ \mathbb{Z} & k = 2n+1 \\ \pi_k(S^{2n-1}) & k \geq 2n+2 \end{cases}$$

□

[4] Hint: Define a fibration over $\mathbb{C}P^n$ that generalizes the Hopf fibration.

Problem 3. *Let (B, b_0) be any based space. Let $PB = (B, b_0)^{(I,0)}$ denote the path space. Show that the map $p : PB \to B$ given by evaluation $p(f) = f(1)$ is a fibration. Do this by solving the lifting problem explicitly.*[5]

Solution. Consider a lifting problem

$$\begin{tikzcd} D^n \times 0 \arrow[r, "H_0"] \arrow[d, hook] & PB \arrow[d, "p"] \\ D^n \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

For each $x \in D^k$, we are given a path $\gamma_x : t \mapsto h(x, t)$ and an element $H_0(x)$, which is a path from b_0 to $h(x, 0)$. We want to define $H(x, t)$ to be a path b_0 to $h(x, t)$ for each t (this is what it means for H to be a lift). The obvious thing to do is to concatenate $H_0(x)$ with the path γ_x restricted (or truncated) to the interval $[0, t]$. Combining formulas for concatenation and truncation, we obtain the formula

$$H(x,t)(s) = \begin{cases} H_0(x)(2s) & s \leq 1/2 \\ h(x, t(2s-1)) & s \geq 1/2. \end{cases}$$

For continuity of $H : D^k \times I \to P^B$, we use the adjunction and show that the associated map $H : I \times D^k \times I \to B$ is continuous. This is true because the restriction to $s < 1/2$ and $s > 1/2$ is continuous (by continuity of H_0 and h) and true at $s = 1/2$ because the maps agree here and hence glue continuously. $\square$

[5] Hint: concatenate.

Problem 4. *Fix an embedding of $A = S^1 \vee S^1$ in $X = S^2$. Compute $\pi_2(X, A)$, and describe a generating set for this group.*

Solution. From the long exact sequence, we obtain

$$1 \to \pi_2(X) \to \pi_2(X, A) \mapsto \pi_1(A) \to 1.$$

We know $\pi_2(X) = \mathbb{Z}$ and $\pi_1(A) \cong F_2$. Any extension like this splits because $\pi_1(A)$ is free, so $\pi_2(X, A)$ is a semi-direct product $\pi_2(X, A) \cong \mathbb{Z} \rtimes F_2$. In fact, $\pi_2(X, A) \cong \mathbb{Z} \times F_2$. To see this, use an argument similar to the one that shows $\pi_2(X)$ is abelian to show that the image of $\pi_2(X)$ in $\pi_2(X, A)$ is central.

The generator of $\pi_2(S^2)$ is the identity map. The loops in $A = S^1 \vee S^1$ bound disks in S^2 and choosing a disk for each loop gives generators for $F_2 < \pi_2(X, A)$. $\square$

Problem 5. *A section of $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = 1_B$. We say p has local sections if for each $b \in B$, there is section of p defined on an open neighborhood of b.*

Let G be a topological group, let $H < G$ be a subgroup, and consider the quotient map $p : G \to G/H$. Prove that if p has local sections, then p is locally trivial.[6] [7]

Solution. Fix $gH \in G/H$ and a section $s : U \to G$ defined on a neighborhood of gH. Note that p is defined as $p(g) = gH$, and by definition of s, $gH = p(s(gH)) = s(gH)H$ for $gH \in G/H$. To show p is locally trivial, we define $\phi : p^{-1}(U) \to U \times H$ by

$$\phi(g) = \big(p(g), g^{-1}s(p(g))\big)$$

and $\psi : U \times H \to p^{-1}(U)$ by

$$\psi(u, h) = s(u)h^{-1}.$$

Then

$$\phi(\psi(u, h)) = \phi(s(u)h^{-1}) = (u, hs(u)^{-1}s(u)) = (u, h)$$

and

$$\psi(\phi(g)) = \psi(p(g), g^{-1}s(p(g))) = s(p(g))s(p(g))^{-1}g = g.$$

These maps are continuous since they're defined using the group multiplication and inversion. Since they're inverses, this shows that p is locally trivial. $\square$

[6] Hint: the key thing to observe is that the action of H on G by right multiplication preserves each fiber of p.

[7] Remark: on HW2 you showed that $SO(n + 1) \to S^n$ has local sections.