Problem 1. *Assume X is locally compact, and let Y, T be spaces. Prove that if $H : T \to Y^X$ is continuous, then the map $h : T \times X \to Y$ defined by $h(t, x) = H(t)(x)$ is also continuous.*

Solution. We proved in class that the evaluation map $Y^X \times X \xrightarrow{e} Y$ is continuous. Then h is the composition of continuous maps $T \times X \xrightarrow{H \times \mathrm{id}} Y^X \times X \xrightarrow{e} Y$. Thus h is continuous. $\square$

Problem 2. *Finish the proof of the H-group theorem. Show that the multiplication μ defined by $[\mu] = [p_1] \cdot [p_2] \in [Y \times Y, Y]$ is associative up to homotopy and has inverses up to homotopy.*

Solution. Let (Y, y_0) be a based space. We assume that the functor $(X, x_0) \mapsto [X, Y]$ is group valued. So, we have that $[Y, Y]$ has a group structure.

identity

We checked in class that $[\mu \circ (id_Y \times c)] = [id_Y] \in [Y, Y]$, where $c : Y \to Y$ is the constant map $c(y) = y_0 \ \forall y \in Y$. Similarly we have $[(id_Y \times c) \circ \mu] = [id_Y] \in [Y, Y]$.

associativity

Let $\pi_1, \pi_2, \pi_3 : Y \times Y \times Y \to Y$ be the projection maps of the 1st, 2nd, 3rd coordinates respectively. Since $[Y \times Y \times Y, Y]$ has a group structure, its group operation is bijective, hence $([\pi_1] \cdot [\pi_2]) \cdot [\pi_3] = [\pi_1] \cdot ([\pi_2] \cdot [\pi_3])$.

In $[Y \times Y \times Y, Y]$, we have that $[\mu \circ (\mu \times id_Y)] = (\mu \times id_Y)^*([\mu]) = (\mu \times id_Y)^*([p_1] \cdot [p_2]) = (\mu \times id_Y)^*([p_1]) \cdot (\mu \times id_Y)^*([p_2]) = [p_1 \circ (\mu \times id_Y)] \cdot [p_2 \circ (\mu \times id_Y)] = ([\pi_1] \cdot [\pi_2]) \cdot [\pi_3]$

Likewise, $[\mu \circ (id_Y \times \mu)] = [p_1 \circ (id_Y \times \mu)] \cdot [p_2 \circ (id_Y \times \mu)] = [\pi_1] \cdot ([\pi_2] \cdot [\pi_3])$

And so we have $[\mu \circ (\mu \times id_Y)] = [\mu \circ (id_Y \times \mu)]$.

inverses

Since $[Y, Y]$ has a group structure, for the element $[id_Y] \in [Y, Y]$ there is a unique inverse, say $[\tau] \in [Y, Y]$, such that $[id_Y] \cdot [\tau] = [\tau] \cdot [id_Y] = [c]$, where $\cdot$ denotes the group operation in $[Y, Y]$. Then, $[\mu \circ (id_Y \times \tau)] = [id_Y] \cdot [\tau] = [c] \in [Y, Y]$ (by the "key formula" we proved in class, $[id_Y] \cdot [\tau] = (id_y \times \tau)^*([\mu]) = [\mu \circ (id_Y \times \tau)]$) and likewise $[\mu \circ (\tau \times id_Y)] = [\tau] \cdot [id_Y] = [c] \in [Y, Y]$.

Exercise 2.3. *Explain why the correct version of the homeomorphism $Y^{T\times X} \cong (Y^T)^X$ for based spaces involves the smash product $T \wedge X$ rather than the "ordinary" product $(T, t_0) \times (X, x_0) = (T \times X, (t_0, x_0))$.*

Solution. Let x_0, y_0, and t_0 denote the base points of X, Y, and T respectively. First, notice that Y^X is a based space with basepoint the constant map $c : Y \to X$ given by $y \mapsto x_0$ for all y.

Now, notice that the statement $Y^{T\times X} \cong (Y^T)^X$ resembles the tensor hom-adjunction, namely it states that

$$\text{Hom}(T \times X, Y) \cong \text{Hom}(T, \text{Hom}(X, Y)).$$

However, if we use the regular product $T \times X$ instead of $T \wedge X$ then the natural maps are not well-defined. Specifically, the natural map

$$\text{Hom}(T \times X, Y) \to \text{Hom}(T, \text{Hom}(X, Y)) \quad \text{defined by} \quad f : T \times X \to Y \mapsto \tilde{f}(t)(x) = f(t, x)$$

However, notice that $\tilde{f}(t_0)$ is not, in general, the constant map, but $\tilde{f}$ is a based map so it must send the base point t_0 of T to the basepoint $c : X \to Y$ of Y^X. Therefore, this mapping is not well-defined. To remedy this, we need to only consider maps $f \in \text{Hom}(T \times X, Y)$ such that $f(t_0, x) = y_0$ for all x. Additionally, note that $\tilde{f}(t)$ is not, in general, a based map so we need to further only consider maps $f \in \text{Hom}(T \times X, Y)$ satisfying $f(t, x_0) = y_0$ for all t. Therefore, we're only considering maps $f \in \text{Hom}(T \times X, Y)$ such that f is constant on $\{t_0\} \times X \cup T \times \{x_0\} \subseteq T \times X$, i.e., it's constant on $T \vee X$, sending each element of $T \vee X$ to y_0.

Thus, we've described a natural bijection between based maps $f \in \text{Hom}(T \times X, Y)$ satisfying $f(T \vee X) = y_0$ and based maps $T \to Y^X$. By the universal property of quotient spaces, we have a natural bijection between based maps $f \in \text{Hom}(T \times X/T \vee X, Y)$ and based maps $T \to Y^X$. Notice that by definition $T \times X/T \vee X = T \wedge X$. This is why we must consider the smash product.

Moreover, I don't think the problem asks for this so I won't do it but to rigorously show this we can define a homeomorphism $Y^{T\wedge X} \xrightarrow{\sim} (Y^X)^T$ by $f \mapsto \tilde{f}(t)(x) = f(t, x)$ with inverse mapping $g \mapsto \tilde{g}(t, x) = g(t)(x)$, as described above, and check that both mappings are continuous.

■

Problem 4. *Prove that there is no multiplication on $\mathbb{R}^3$ that makes it into a field.* [1] [2]

Solution. We first claim that if $e \in \mathbb{R}^3$ is the multiplicative identity, then $(ce) \cdot v = cv$ for every $v \in \mathbb{R}^3$ and scalar $c \in \mathbb{R}$. By field properties, this already holds for $c \in \mathbb{Q}$: if $c = p/q$, then $(pe) \cdot v = (e + e + \cdots + e) \cdot v = pv$, and $((1/q)e \cdot v + (1/q)e \cdot v + \ldots (1/q)e \cdot v)$ (written q times) is equal to v, so $(1/q)e \cdot v = (1/q)v$. By continuity of the multiplication (and since $\mathbb{Q}$ is dense in $\mathbb{R}$), we have $(ce) \cdot v = cv$ for all $c \in \mathbb{R}$. Now, pick $u \notin \operatorname{span}\{e\}$, and define $f \colon S^2 \to \mathbb{R}^3$ by $f(s) = (us) \times s$. Since us is not parallel to s by construction, $f(s)$ is perpendicular to s, and hence f is continuous since the cross product is continuous and the multiplication is continuous. Thus, we have a nowhere vanishing vector field on S^2 (this is nowhere vanishing since $u \neq 0$), which contradicts the hairy ball theorem. $\square$

[1] Hint: proceed by contradiction and construct a nowhere vanishing vector field on S^2.

[2] Further hint: try fixing $u \in \mathbb{R}^3$ and defining vector field $F(x) = ux$. This won't quite work – how can you fix it?

Problem 5. *Recall the special orthogonal group $SO(n)$ is the group of $n \times n$ matrices that satisfy $A^t A = I$ and $\det(A) = 1$.*[3] *Let $p : SO(n+1) \to S^n$ denote the map $A \mapsto Ae_{n+1}$, where $e_{n+1} = (0, \ldots, 0, 1)$. Construct a section of p over $S^n \setminus \{-e_{n+1}\}$, i.e. construct a continuous map*

$$s : S^n \setminus \{-e_{n+1}\} \to SO(n+1)$$

such that $p \circ s = id$. [4] [5]

Solution. First note that S^n is an n-dimensional manifold embedded in $\mathbb{R}^{n+1}$. Further, let φ denote stereographic projection of S^n through $-e_{n+1}$ to $\mathbb{R}^n$. Then $(S^n \setminus \{-e_{n+1}, \varphi)$ is a smooth chart on S^n. For each $u \in S^n \setminus \{-e_{n+1}\}$ we can use this chart to find a basis of the tangent space $T_u S^2$ since it is isomorphic to $T_{\varphi(u)}\mathbb{R}^n$. We get that $\{\frac{\partial}{\partial x^i}|_{\varphi(u)}\}_{i=1}^n$ is a basis for $T_{\varphi(u)}\mathbb{R}^n$ and given this basis, we can apply the Gram–Schmidt process to get an orthonormal basis, $\{v_i'\}_{i=1}^n$ in $\mathbb{R}^n$. For all $1 \leq i \leq n$, let $v_i = (v_i', 0)$. Note that each v_i is orthogonal to u by construction and so $\{v_1, \cdots, v_n, u\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$. Further, we can order these vectors so that the vector space they span is positively oriented and place these vectors as the column vectors, in this order into an $(n+1) \times (n+1)$ matrix A_u such that u is the last column. Since the columns of A_u form an orthonormal basis for $\mathbb{R}^{n+1}$, $A_u^t A_u = I$ and by construction, $\det A_u = 1$. Therefore, $A_u \in SO(n+1)$. Notice further that this process is a composition of continuous functions so is itself a continuous function $s : S^n \setminus \{-e_{n+1}\} \to SO(n+1)$. Further, for $u \in S^n \setminus \{-e_{n+1}\}$, $p \circ s(u) = p(A_u) = A_u e_{n+1} = u$. So, $p \circ s = \text{id}$.

□

[3] Note: the condition $A^t A = I$ means that the columns of A form an orthonormal basis for $\mathbb{R}^n$.
[4] Remark: the significance of this example will be explained later.
[5] Suggestion: do the case $n = 2$ first. Then generalize.

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Solution. Rephrasing our goal, we want to continuously assign to each vector $x \in S^n$ a positively oriented ortonormal basis with x as last vector. Let us first give the intuition behind the process for $n = 2$.

For any point $x \in S^2$, there exists a geodesic in S^2 from e_3 to x, and it is unique unless $x = -e_3$. Such geodesic is trivial if $x = e_3$, otherwise it is obtained by intersecting the sphere with the plane through the origin containing x and e_3 and then taking the smallest arc on the resulting great circle. Let us assume $x \neq e_3$. Now, let us call H_x the plane containing $0, x, e_3$ and let n_x be the line orthogonal to H_x. If θ_x is the smallest angle in H_x between x and e_3 and centered at the origin, then we rotate S^2 by θ_x around n_x. The rotation maps the standard (orthonormal, positively oriented) basis $\mathbf{e} = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ to a new orthonormal and positively oriented basis $\mathbf{e_x}$. By construction, the rotation maps e_3 to x, and then x is the last vector of $\mathbf{e_x}$. Since the geodesics vary continuously on $S^2 \setminus \{-e_3\}$ and rotations vary continuously with the angle, the assignment $x \mapsto \mathbf{e_x}$ is a continuous function. Let A_x be the matrix whose columns are the vectors of $\mathbf{e_x}$. Hence, we can define $s(x) := A_x$ if $x \neq e_3$ and $s(e_3) = I_3$.

In general, if $x \in S^n \setminus \{-e_{n+1}\}$ and $x \neq e_{n+1}$, we can consider the unique geodesic between e_{n+1} and x. Again, such geodesic is obtained by taking the smallest arc in the intersection of S^n with the plane H_x through $0, x, e_{n+1}$. Let θ_x be the smallest angle between x and e_{n+1} in H_x and centered at the origin. If we denote by $H_x^\perp$ the orthogonal of H_x, then since $\mathbb{R}^{n+1} = H_x \oplus H_x^\perp$, we can define a linear transformation $L\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that rotates of θ_x the vectors in H_x and fixes the vectors in $H_x^\perp$. Then, called A_x the matrix representing L, we get that $A_x \in SO(n+1)$, since it is congruent (via a base change matrix) to the matrix

$$\left(\begin{array}{c|c} R_{\theta_x} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{n-1} \end{array}\right)$$

and moreover $A_x e_{n+1} = L(e_{n+1}) = x$. Therefore, we can define $s(x) = A_x$ if $x \neq e_{n+1}$ and $s(e_{n+1}) = I_{n+1}$. The resulting map is continuous because essentially it rotates the vectors of the standard basis $\{e_1, \dots, e_{n+1}\}$ by an angle that depends continuously on x. □

[3] Note: the condition $A^tA = I$ means that the columns of A form an orthonormal basis for $\mathbb{R}^n$.
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