Homework 1

Math 2420

Due Friday, Feb 2 by 5pm

Your Name: Bena

Collaborator names:

Topics covered: Homotopy groups, H-groups, mapping spaces

Instructions:

- This assignment must be submitted on Gradescope by the due date. Gradescope Entry Code: GPB45Y.
- If you collaborate with other students (which is encouraged!), please list your collaborators above.
- If you are stuck, please ask for help (from me or a classmate). Use Campuswire!
- You may freely use any fact proved in class. Usually you should be able to solve the problems without outside knowledge. You should provide proof for facts that you use that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Version 1: Show that the set $[X, Y]$ of based homotopy classes of maps does not depend on the basepoints if X, Y are path-connected.* [1]

Version 2: Assume given $x_0 \in X$ and $y_0, y_1 \in Y$. Show the two definitions of $[X, Y]$ are the same.

Proof of Version 2. For this argument we need (X, x_0) to satisfy the HEP. In this case people call (X, x_0) is a "well-pointed space". For example, this is true if X is a cell complex with x_0 in the 0-skeleton.

Defining a map. First we define a map $[(X, x_0), (Y, y_0)] \to [(X, x_0), (Y, y_1)]$. Fix a path $\gamma : I \to Y$ from y_0 to y_1. For each $f : (X, x_0) \to (Y, y_0)$, consider the homotopy extension problem $h : X \times \{0\} \cup \{x_0\} \times I \to Y$ where $h = f$ on $X \times 0$ and $h = \gamma$ on $x_0 \times I$. By HEP, there exists $H : X \times I \to Y$, which is a homotopy from f to a map f_1 such that $f_1(x_0) = y_1$. We want to define $[f] \mapsto [f_1]$. We need to check several things.

Well-defined. There are two choices we made in defining the map. We chose a representative f of $[f]$ and we chose the homotopy extension H. We should check that the map on homotopy classes is independent of these choices.

First suppose we chose a different solution H' to the homotopy extension problem. Any two solutions are homotopic. A homotopy $X \times I \times I \to Y$ can be constructed by setting up a homotopy extension problem where $h = H$ on $X \times I \times 0$, $h = H'$ on $X \times I \times 0$, $h(x, 0, t) = f(x)$ on $X \times 0 \times I$ and $h(x_0, t, s) = \gamma(t)$ on $\{x_0\} \times I \times I$. This is equivalent to a homotopy extension problem for $(X \times I \times I, x_0 \times I \times I)$, and the HEP gives then a homotopy between H and H' that restricts to a homotopy between f_1 and f_1'.

Suppose $f, g : (X, x_0) \to (Y, y_0)$ are homotopic (rel basepoints), i.e. there exists $G : X \times I \to Y$ homotopy. We will homotope G to a homotopy between f_1, g_1 (rel basepoints). Consider a homotopy extension problem $h : (X \times I) \times 0 \cup \{x_0\} \times I \times I$, where $h = G$ on $X \times I \times 0$ and $h(x_0, t, s) = \gamma(s)$. Note that $h(x_0, t, 0) = G(x_0, t) = x_0 = \gamma(0) = h(x_0, t, 0)$ for all t, so these two maps glue continuously. Our assumption implies $(X \times I, x_0 \times I)$ has HEP, so there is a homotopy H, which induces a homotopy between f_1, g_1, or at least a homotopy between maps that are homotopy equivalent (rel basepoint) to f_1, g_1 respectively (by the previous paragraph).

Inverse. We can construct an inverse in an obvious way using the reverse path $\bar{\gamma}$. In this way we obtain a map $f \mapsto f_1 \mapsto f_2$. By construction there is a homotopy $G : X \times I \to Y$ between f and f_2, but such that G restricts to $\gamma * \bar{\gamma}$ on x_0. Nevertheless, again we can use HEP to homotope G to a homotopy that's constant on x_0. To do this, we define $h : X \times I \times 0 \cup \{x_0\} \times I \times I \to Y$ by $h = G$ on $X \times I \times 0$ and by a homotopy between $\gamma * \bar{\gamma}$ to the constant x_0 on $\{x_0\} \times I \times I$. Now apply HEP to get the desired homotopy, which shows that $[f] = [f_2]$ in $[(X, x_0), (Y, y_0)]$. This shows that the composite $[X, (Y, y_0)] \to [X, (Y, y_1)] \to [X, (Y, y_0)]$ is the identity. Arguing similarly shows the other composite is the identity two, so we conclude that $[f] \mapsto [f_1]$ is a bijection $[X, (Y, y_0)] \cong [X, (Y, y_1)]$. □

[1] Hint: use the homotopy extension property. Applying this probably requires some additional mild assumption on X, Y (I will let you think about this).

Problem 2. *Let $p : \tilde{X} \to X$ be the universal cover of a path connected space X. Show that p induces an isomorphism on homotopy groups π_k for $k \geq 2$.*

Solution.

Defining a map. Fix basepoints $\tilde{x}_0 \in \tilde{X}$ and $x_0 \in X$. Consider the map $\pi_k(\tilde{X}, \tilde{x}_0) \to \pi_k(X, x_0)$ defined by $[f] \mapsto [p \circ f]$. It's easy to see that this is well-defined since a homotopy f_t induces a homotopy $p \circ f_t$.

Surjectivity. A given $[g] \in \pi_k(X, x_0)$ is represented by a map $g : (S^k, s_0) \to (X, x_0)$. By the homotopy lifting criterion, since $\pi_1(S^k) = 0$, there is a lift $\tilde{g} : (S^k, s_0) \to (\tilde{X}, \tilde{x}_0)$ (there are many lifts, depending on a basepoint and here we choose the (unique) lift that sends s_0 to $\tilde{x}_0$).

Injectivity. Suppose $[f_0], [f_1] \in \pi_k(\tilde{X}, \tilde{x}_0)$ and $[p \circ f_0] = [p \circ f_1]$. This means there is a (based!) homotopy $h : S^k \times I \to X$ such that the restriction of h to $S^k \times \{i\}$ is $p \circ f_i$ for $i = 0, 1$. By the homotopy lifting property, there is a map $\tilde{h} : S^k \times I \to \tilde{X}$ such that $\tilde{h}$ restricts to f_0 on $S^k \times 0$. Since h is a based homotopy (i.e. $s_0 \times I$ maps to x_0), the lift $\tilde{h}$ is also based (because the preimage of x_0 in $\tilde{X}$ is discrete). Therefore, $\tilde{h}$ is a homotopy between f_0 and a map that lifts $p \circ f_1$. Since f_1 is such a lift and these two lifts agree at s_0, they are equal. Hence $[f_0] = [f_1]$, proving injectivity. □

Problem 3. *For a based space X, ΩX denotes the loop space, and $c \in \Omega X$ denotes the constant map. Show that the map $\Omega X \to \Omega X$ defined by $\gamma \mapsto \gamma * c$ is homotopic to the identity, i.e. there is a homotopy $I \times \Omega X \to \Omega X$.* [2] [3]

Solution. Consider the map $I \times \Omega X \to \Omega X$ defined by $(s, \gamma) \mapsto [t \mapsto \gamma((1-s)t + s\delta(t))]$ with $\delta(t) = 2t$ for $t \leq 1/2$ and $\delta(t) = 1$ for $t \geq 1/2$. We want to show this map is continuous. Doing this directly is tedious. But by a lemma from class, this map is equivalent to a map $I \times I \times \Omega X \to X$. We'll show this map is continuous. It factors

$$I \times I \times \Omega X \xrightarrow{\phi \times 1} I \times \Omega X \xrightarrow{\epsilon} X$$

where $\phi(s,t) = (1-s)t + s\delta(t)$ is the re-parameterization, and ϵ is the evaluation map. Both of these maps are continuous (the latter from class), so the composition is continuous, as desired. $\square$

[2] Remark: last semester you (probably) showed that $\gamma \simeq \gamma * c$ for each fixed γ. This is weaker than what is asked for here.

[3] Remark: arguing similarly for associativity and inverses shows that ΩX is an H-group.

Problem 4. *Let A, B_1, B_2 be based spaces.*

(a) Prove that $(B_1 \times B_2)^A \cong B_1^A \times B_2^A$ (homeomorphism).

(b) Prove that $[A, B_1 \times B_2] \cong [A, B_1] \times [A, B_2]$ (bijection of sets).

Solution. Let $p_i : B_1 \times B_2 \to B_i$ be the projection. Define $(B_1 \times B_2)^A \to B_1^A \times B_2^A$ by $f \mapsto (p_1 \circ f, p_2 \circ f)$. This map has inverse given by $(f_1, f_2) \mapsto f_1 \times f_2$. These maps are continuous. We check this on open sets generating the topology. The pre-image of $N(K, U_1 \times U_2)$ is $N(K, U_1) \times N(K, U_2)$, which is open. Similarly, the preimage of $N(K_1, U_1) \times N(K_2, U_2)$ is $N(K_1, U_1 \times B_2) \cap N(K_2, B_1 \cap U_2)$, which is also open. By definition of the compact open topology, this implies both maps are continuous.

(b) follows from (a) together with a few general facts. Recall from class that $[X, Y]$ is the same as components $\pi_0(Y^X)$ (suppressing basepoints). Then

$$[A, B_1 \times B_2] = \pi_0((B_1 \times B_2)^A) \cong \pi_0(B_1^A \times B_2^A) \cong \pi_0(B_1^A) \times \pi_0(B_2^A) \cong [A, B_1] \times [A, B_2].$$

Here we use general facts $\pi_0(U \times V) \cong \pi_0(U) \times \pi_0(V)$ and that a homeomorphism $\Omega_1 \cong \Omega_2$ induces a bijection on π_0. □

Problem 5. *Identify $X = \mathbb{R}P^\infty$ with the projectivization of the space of polynomials with coefficients in $\mathbb{R}$, and use this to define a monoid structure $m : X \times X \to X$. Show that the map $X \ni f \mapsto m(f, f) \in X$ is homotopic to a constant.*[4] *Conclude that $\mathbb{R}P^\infty$ is an H-group.*

Solution. Write $\phi : X \to X$ for the map $f \mapsto m(f, f)$. Consider the induced map $\phi_* : \pi_1(X) \to \pi_1(X)$.

Main Claim. ϕ_* is the trivial map.

First we use the claim to solve the problem. By the lifting criterion, ϕ lifts to a map $\tilde{\phi} : X \to S^\infty$. In other words, ϕ factors as $X \xrightarrow{\tilde{\phi}} S^\infty \xrightarrow{p} X$, where $p : S^\infty \to X$ is the covering map. Since S^∞ is contractible, it follows that $\tilde{\phi}$ is homotopic to a constant, which implies ϕ is homotopic to a constant, as desired.

Proof of claim. The map $\phi(f) = m(f, f)$ can be expressed as a composition

$$\mathbb{R}P^\infty \xrightarrow{id \times id} \mathbb{R}P^\infty \times \mathbb{R}P^\infty \xrightarrow{m} \mathbb{R}P^\infty.$$

Look at these maps on π_1. The first map is the diagonal map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, i.e. $1 \mapsto (1, 1)$. The second map $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ sends $(1, 0)$ and $(0, 1)$ to 1 (by straightforward computation). Combining these computations, we see that $\phi_*(1) = 0$, as desired. □

[4] Hint: use the fundamental group, covering spaces.