1 Problem 1

Let G be a connected, bipartite graph. Compute $\chi(G, 2)$ with proof.

1.1 Solution

First, we note that G has a spanning tree T. Because T is a connected subgraph of G with the same vertices, we can see that $\chi(G, 2) \leq \chi(T, 2)$. We can see this more clearly by noting that T is G after removing some edges. Because T is a tree we know that $\chi(T, 2) = 2 \cdot (2-1)^{n-1} = 2$, where n is the number of vertices. Thus, $\chi(G, 2) \leq 2$.
Now we show that $\chi(G, 2) \geq 2$. We know that G is bipartite, so let X, Y be the partition. Then, we can color all of X one color because there are no edges within X, and all of Y another color because there are no edges within Y either. We can also switch the colors between that of X and that of Y. This gives us two 2-colorings of G. Therefore, $\chi(G, 2) \geq 2$.
Because $2 \geq \chi(G, 2) \geq 2$, $\chi(G, 2) = 2$. ∎

Problem 1. *Let G be a connected, bipartite graph. Compute $\chi(G, 2)$ with proof.*

Solution. We claim that $\chi(G, 2) = 2$ for any connected, bipartite graph.

To see this, first partition $V(G)$ into two partite sets X and Y. We claim the only 2-colorings of G are those which assigns all vertices in X to color 1 and those in Y to color 2, or vice versa. These are in fact valid colorings, as a monochromatic edge in these colorings would mean that two vertices of the same partite set are adjacent, and G wouldn't be bipartite. We note that these are distinct colorings for any number of vertices except 0.

To see that these are the only two colorings, assume, for the sake of contradiction, that there is a 3rd coloring that is not one of the two above. In this coloring, we see that there must be two vertices within the same partite set that have different colors (otherwise it is the same as the other 2). Without loss of generality, assume these vertices are in X and denote them x_1 and x_2. Since G is connected there exists a x_1, x_2-path. Since G is bipartite, this path alternates between vertices in X and vertices in Y. For both the endpoints to be in X, this path must have an even number of edges. We also note that there are no monochromatic edges and the vertices in the path also alternate in color. For the endpoints to have different colors, the path must have an odd number of edges. This is a contradiction, and shows that this 3rd coloring cannot exist.

Thus, we see we can construct 2 2-colorings for any connected, bipartite graph and that the number of 2-colorings is less than 3. Thus, $\chi(G, 2) = 2$ for connected, bipartite G. $\square$

2

The most edges possible is 40.

We can write $n = 11 = 3 \cdot 3 + 2 = 3r + 2$. Therefore, according what we proved in class, $M_{4,4,3}$ has the most number of edges (since this is the only way to write 11 into sum of numbers that differ by no more than 1, otherwise we will be able to find a graph with larger edge count). It has $4 \cdot 4 + 4 \cdot 3 + 4 \cdot 3 = 40$ edges.

Problem 3. *Prove that $M_{2,2,2,2}$ is not planar.*

Solution. We proceed by contradiction: suppose there was embedding of $G(V, E) = M_{2,2,2,2}$ in $\mathbb{R}^2$. As the graph is connected and noting that it has 24 edges, by Euler's formula, we have

$$\begin{aligned} F &= 2 - |V| + |E| \\ &= 2 - 8 + 24 \\ &= 18 \end{aligned}$$

On the other hand, each edge is part of two faces, and each face has at least 3 sides, which is true for any planar graph except in a few cases with graphs having one or fewer edges, such as K_2. Thus, we have

$$2|E| \geq 3F$$

giving

$$\begin{aligned} F &\leq \frac{2}{3}|E| \\ &\leq \frac{2}{3}24 \\ F &\leq 16 \end{aligned}$$

This contradicts that $F = 18$, proving the claim. □

. Prove that $M_{2,2,2,2}$ is not planar.

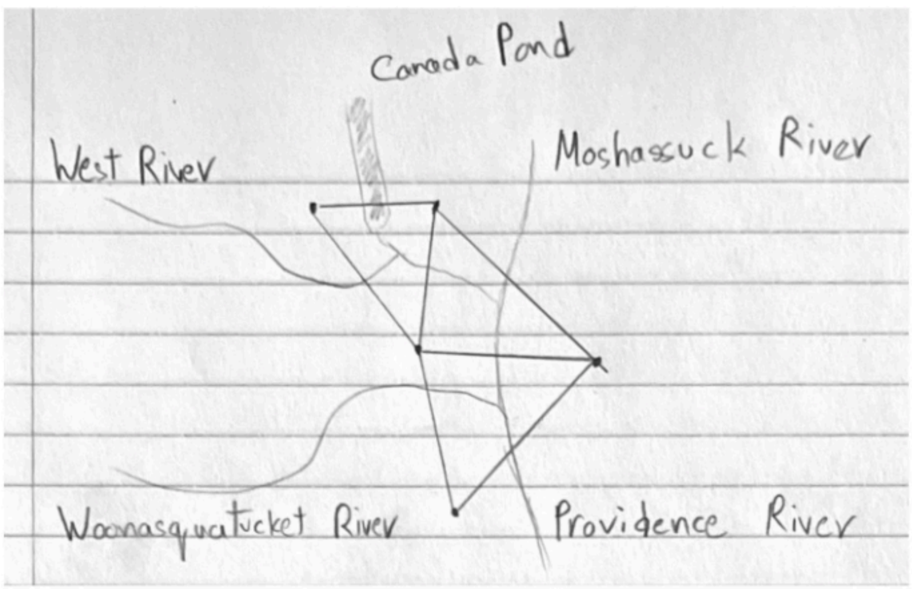
Proof.

We proved in class that every planar graph must have a vertex of degree ≤ 5. But every vertex in $M_{2,2,2,2}$ is connected to the 6 other vertices that don't share it's color. Every vertex in $M_{2,2,2,2}$ has degree 6 so there exists no such vertex with degree ≤ 5. Therefore, $M_{2,2,2,2}$ cannot be planar.

$\square$

Problem 4. *Is it possible to walk through Providence starting and ending at the same place and crossing every bridge exactly once? (You're walking, so ignore the highways.)*

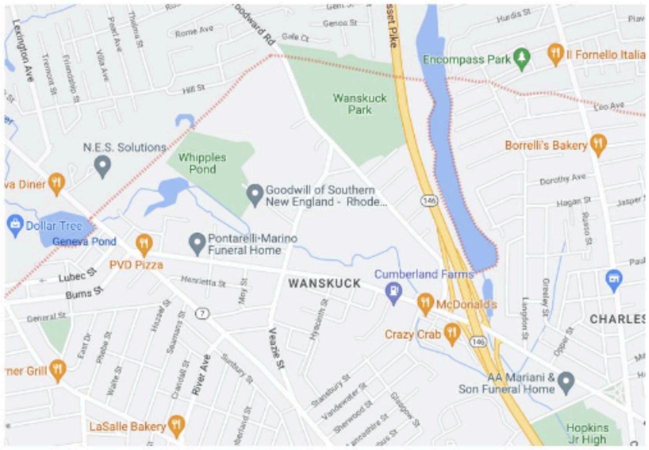
Solution. We must first convert Providence into a graph, where vertices represent landmasses and the number of edges between two vertices represents the number of bridges between those landmasses (we can add a dummy vertex on each edge to avoid multi-edges without affecting the problem). The Providence River clearly divides Providence into two landmasses, but once it breaks off, things get murkier. The Woonasquatucket River outlines all of South Providence, so we assign one vertex to this area, including Federal Hill, West End, Silver Lake, and so on. College Hill gets its own vertex, separated by the Moshassuck River. The area north of Woonasquatucket and south of the West River, including Smith Hill and the Providence College area, gets a vertex. Finally, delineate the areas east and west of Canada Pond, each with one vertex. Now, add edges for each bridge as appropriate between landmasses. Ignore any bridges that start and end on the same landmass (maybe due to a river tributary ending in the landmass or so). These bridges have no bearing on the problem since the landmass can be used to return to either side of the bridge. This results in the graph below, with only one edge drawn between landmasses for clarity.



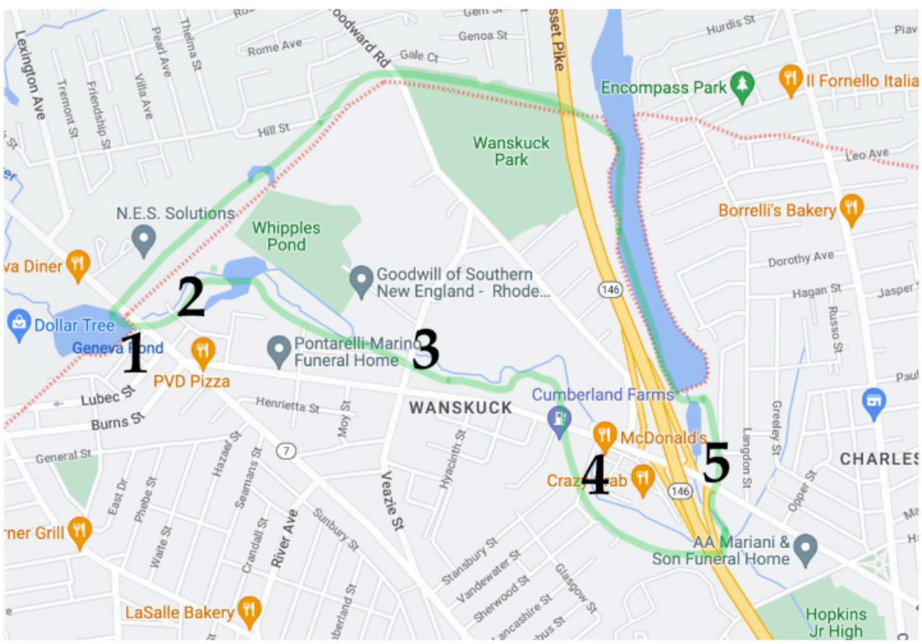
We now have to check if the graph is Eulerian; in other words, if there is a closed walk visiting each edge exactly once. As Providence is clearly connected, we know this is the case if and only if every vertex has even degree. However, the northwestern vertex (unfortunately) has degree 5, an odd degree. It has two neighbours, one to the south and one to the east. There are four bridges going from that vertex to its southern neighbour, namely Douglas Ave, Caxton St - it counts even if it's a dead end, since we can walk across the landmass beyond the West River - Veazie St, and Branch Ave. A later bridge on Branch Ave is the only access to its other neighbour, the eastern one. Thus, since there is a vertex with odd degree, the graph is not Eulerian, and the walk is not possible. □

4. We can relate our question of walking through Providence, crossing each bridge exactly once, and returning to our original point, as attempting to construct an Eulerian tour of the following graph:

A graph in which each bridge is an edge, and each vertex is an island as shown above (with intermediate vertices added in between bridges and islands if multiple bridges connect to the same island). By definition, being able to construct an Eulerian tour of this graph is exactly equivalent to our desired condition.
Consider the following portion of the Providence map, which we note is cut off above by the Providence border, below by the West River, and to the right by the Canada Island, making it a sort of "island."



We outline this region a bit more carefully, labeling all the bridges were one can cross over into this region.



Considering our aforementioned graph representing the islands and rivers of Providence, the vertex corresponding to the circled region will have exactly 5 edges connecting into it, as we have but 5 (labeled) ways of traversing into this region. Therefore we can invoke the Theorem given from class that, for a graph G, we can construct an Eulerian tour (as we desire to do for Providence) if and only if every vertex is of even degree. As the vertex corresponding to the above region has degree 5, which is odd, this disqualifies us from constructing an Eulerian tour as desired and thus it is not possible to walk through Providence, crossing every bridge exactly once.
□

Problem 4

Things I did not consider walkable Providence bridges:

- Bridges that only have highways on them
- Bridges that are train bridges
- This one road that went over a river but was still connected to land on one side in addition to both of its ends.
- Bridges that begin in Providence and dump you out in a different city
- Buildings that go across rivers
- Bridges that do not cross rivers
- Places where the river goes underground for a bit
- Places where the river does not come back out from under a road

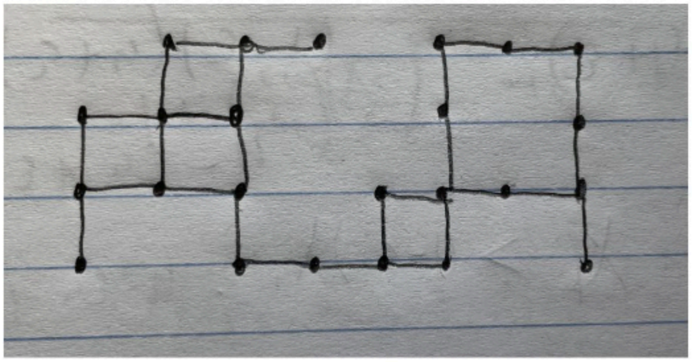
Thing that made labeling all 49 walkable bridges in Providence pointless:

This can be found in Roger Williams Park.

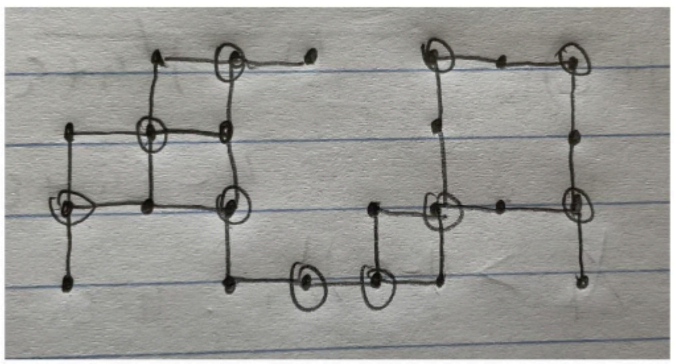
If we represent this map as a graph, with discrete lands being vertices and bridges being edges, the rightmost island is a degree-1 vertex. We either start our path at it and can never return, or we start somewhere else, go to that island, and get stuck. (Alternately, it is an odd-degree vertex, and graphs with any odd-degree vertices do not have Eulerian circuits.)

Problem 5.

Solution. We consider the following dual graph $G = (V, E)$ where each vertex corresponds to a white square, and two vertices are adjacent in G if they are neighbors on the board.



We notice that G is in fact bipartite because it only has even cycles. Also, we there exists a vertex cover Q of G with $|Q| = 10$ (see figure below).



By Kőnig's Theorem, any matching M must satisfy

$$|M| \leq \max|M| \leq \min|Q| \leq 10.$$

However, G has 24 vertices. This implies that there exists no perfect matching of G, so the board cannot be tiled by 1×2 and 2×1 tiles.

$\square$