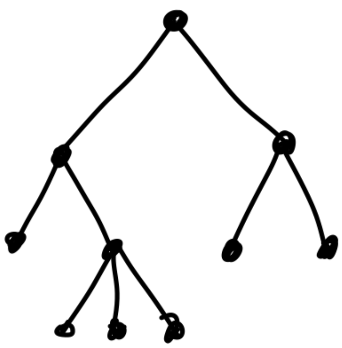
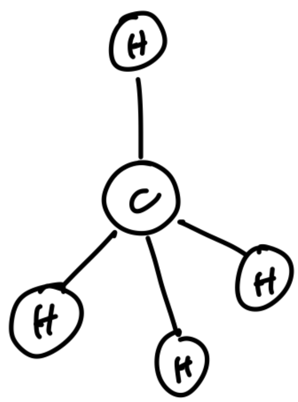
1. Introduction and fundamental concepts

1.1 What is a graph?

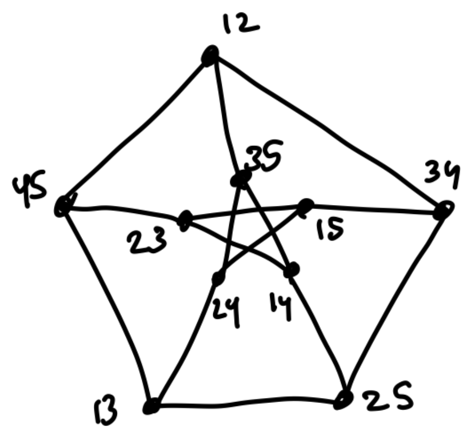
Graphs express relationships between objects.
Examples:



Family tree



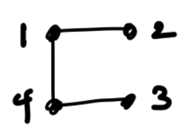
methane



Petersen graph
Subsets of {1,2,3,4,5}
connected by disjointness

Formal definition: a graph is a pair (V, E), where V is a set and $E \subset$ {two-element subsets of V}

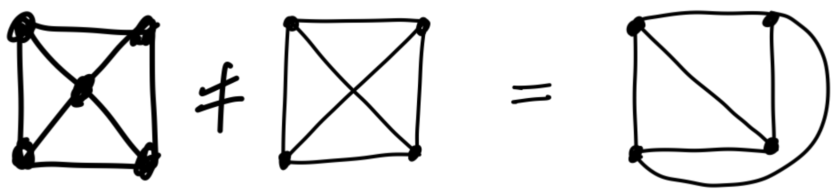
Example: $V = \{1,2,3,4\}$ and $E = \{ \{1,2\}, \{1,4\}, \{3,4\} \}$. Often easier to draw a picture.



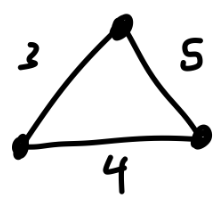
Remarks.

- Our definition excludes multiple edges and self loops (different from West!)

- Usually V is finite for us.
- When drawing graphs, not every place where lines cross is a vertex.



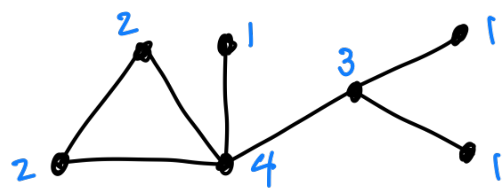
- Variations on the definition (not our focus, but will appear)
  - directed graphs
  - weighted graphs



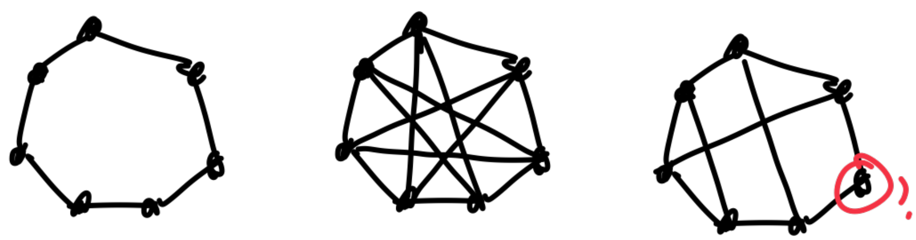
1.2 Vertex degrees

Definition. We say $v \in V$ and $e \in E$ are incident if $v \in e$, i.e. v is an endpoint of e. The degree of v, denoted $\deg(v)$, is the number of edges incident to v.

Example:



Exercise. Construct a graph with 7 vertices, each of degree 2; degree 3; degree 4.



Is there a graph with 7 vertices, each of degree 3? Remark a graph where every vertex has degree d is called d-regular.

Lemma (degree sum formula). For $G = (V, E)$,

$$\sum_{v \in V} deg(v) = 2|E|.$$

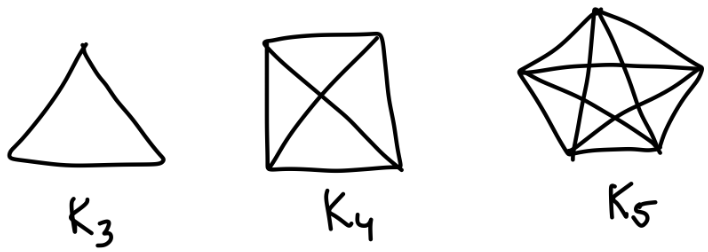
Corollary. In a graph, the number of vertices with odd degree is even. In particular, there is no 7-vertex, 3-regular graph.

Proof of Lemma. Each edge has two vertices, so counting each verticex degree counts each edge twice. More precisely,

$$\sum_{v \in V} deg(v) = \sum_{v \in V} \sum_{v \in e} 1 = \sum_{e \in E} \sum_{v \in e} 1 = \sum_{e \in E} 2 = 2|E|.$$

□

Example. The complete graph K_n has vertices $\{1,\ldots,n\}$ and all possible edges.



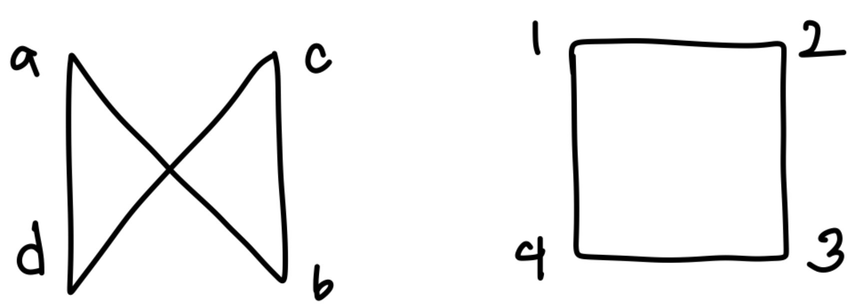
In K_n, each vertex has degree $n-1$, so $\sum deg(v) = n(n-1)$
Also $|E| = \binom{n}{2}$, the number of 2-element subsets of $\{1,\ldots,n\}$. Then the degree sum formula, gives the (perhaps familiar) identity

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

1.3 Isomorphic graphs

Definition. Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.

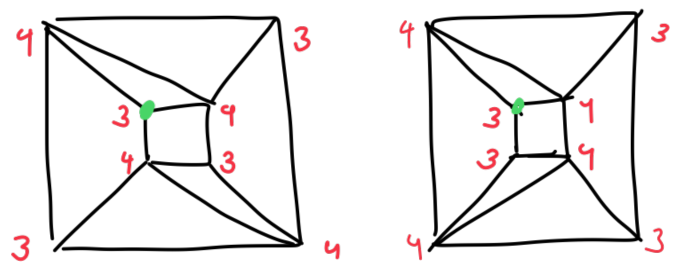
Example 1: The following graphs are isomorphic



An explicit isomorphism is given by $a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4$.

We think of isomorphic graphs as "the same".

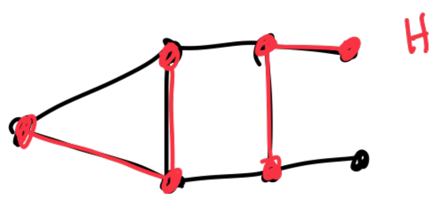
Example 2: Are these graphs isomorphic?



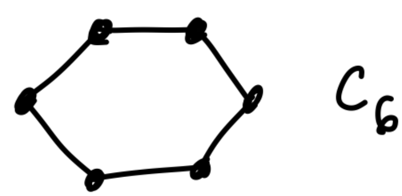
The isomorphism problem is hard in general. In this case we can look at vertex degrees to conclude $G_1 \neq G_2$.

1.4 Subgraphs

Definition. Say $H = (V(H), E(H))$ is a subgraph of $G = (V(G), E(G))$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.



A subgraph isomorphic to P_n is called a path. A subgraph isomorphic to C_n is called a cycle.

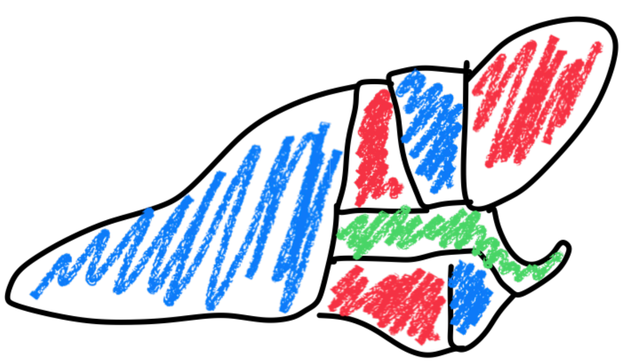


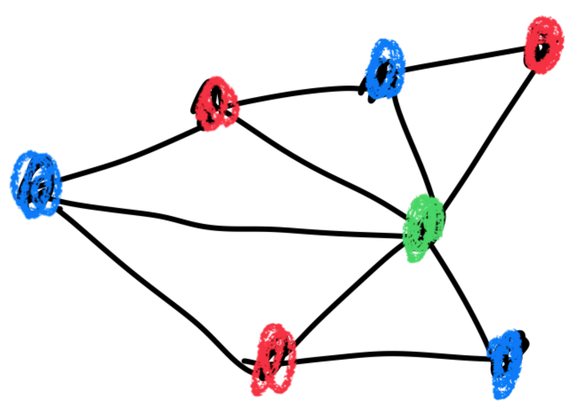
A (connected) graph that does not contain any cycle is called a tree.

1.5 What is graph theory?

In short, graph theory studies properties of graphs and solves problems using graphs. It's a big subject, so this doesn't really capture the richness of the subject. For now we will use graph coloring to illustrate some of the kinds of problems we'll consider and tools we'll use.

Given a map, try to color the regions so that adjacent regions have different colors. How many colors are needed?





This problem translates to a graph theory problem. A <u>coloring</u> of a graph is a coloring of the vertices so that adjacent vertices have different colors.

<u>Extremal problem</u>: Given a graph G, what is the minimum number of colors needed to color G ? This is called the <u>chromatic number</u> χ(G). Later we will show (using a greedy algorithm) that

$$\chi(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ is the maximum vertex degree. (Applied to the example above gives an upper bound of 7, which is not optimal.)

<u>Classification problem</u>: Which graphs can be colored with 2 colors? We'll prove: A graph can be 2-colored if and only if it has no odd cycle.

<u>Counting/combinatorics</u>: How many different ways are there to color G with t colors? Let this number be $\chi(G,t)$, which we view as a function of t. Surprisingly, $\chi(G,t)$ is a polynomial(!) whose degree is $|V|$. This makes this function somewhat computable (for any fixed graph). Furthermore, we can use facts about polynomials/algebra to study graphs (e.g. what do the roots of χ tell us about G? If $\chi(G,t) = \chi(G',t)$, does that mean G, G' are isomorphic?)

The function $\chi(G,t)$ was introduced by Birkhoff to study the map coloring problem.

<u>Application</u>. Often there are surprising applications of graph theory problems to the "real world". Consider the problem of exam scheduling for Spring 2023 courses. We want to assign each course an exam day/time so that there are no conflicts (a student with two exams at the same time).

To formulate this graph theoretically, let V be the set of Spring 2023 courses. Form a graph G where two courses are connected by an edge if there is a student enrolled in both. Viewing colors as exam times, a coloring of G gives a way to schedule the exams without conflicts. In particular, $\chi(G)$ is the minimum number of different exam times needed to schedule without conflict.

Summary. There are many different kinds of problems studied in graph theory. Corrensponding ly there are many different techniques that can be used to study graphs. In this course, this will include combinatorics, algebra, linear algebra, topology, and probability.

Consequently, this course, which doesn't assume you are already familiar with all these subjects, is a good opportunity to see different parts of math "in action".