Homework 1

Math 123

Due February 3, 2023 by 5pm

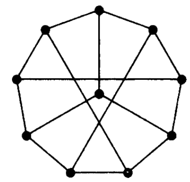
Name:

Topics covered: graph, subgraph, cycle, path, vertex degrees,

Instructions:

- This assignment must be submitted on Gradescope by the due date. Gradescope Entry Code: RZ277D.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck, please ask for help (from me, a TA, a classmate).

Problem 1. *Prove that the graph below is isomorphic to the Petersen graph.*[1]



Solution. □

Problem 2. *How many cycles of length n are there in the complete graph K_n?*

Solution. □

Problem 3. *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \ldots, a_k)$ with coordinates $a_i \in \{0, 1\}$ and with an edge between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if they differ in exactly one coordinate.*[2]

(a) *Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.*

(b) *Let $K_{2,3}$ be the complete bipartite graph with 2 red vertices, 3 blue vertices, and all possible edges between red and blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.*

Solution. □

Problem 4. *For a graph $G = (V, E)$, the complement of G is the graph $\bar{G} = (V, \bar{E})$, where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$. Prove or disprove: If G and H are isomorphic, then the complements $\bar{G}$ and $\bar{H}$ are also isomorphic.*

Solution. □

Problem 5.

(a) *Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices. It has $n - 1$ edges.)*

(b) *We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n is divisible by 4 or $n - 1$ is divisible by 4.* [3]

In fact, whenever n or $n - 1$ is divisible by 4, there is a self-complementary graph with n vertices – see the bonus problem below.

Solution. □

[1] Hint: label the graph.
[2] Suggestion: Draw Q_k for $k = 2$ and $k = 3$.
[3] Hint: count edges

Problem 6. *Prove that the Petersen graph has no cycles of length* 3 *or* 4. [4]

Solution. □

Problem 7 (Bonus)**.** *Let* G, H *be a self-complementary graphs, and assume* G *has with* $4k$ *vertices. Construct a self-complementary graph obtained by taking the union of* G *and* H *and adding some edges.*[5] *Deduce that if either* n *or* $n-1$ *is divisible by* 4*, then there is a self-complementary graph with* n *vertices.*

Solution. □

[4] Hint: use the definition of Petersen graph given in class.

[5] Hint: How does the degree of even/odd vertices of G change after taking the complement?