Problem 1. *Let S be the torus of revolution defined in class. Is the Gauss Map for S injective? Surjective? Explain your answer*

Solution. We first recall the parameterization for the torus we've defined in class

$$\phi(t,\theta) = ((2+\cos t)\cos\theta, (2+\cos t)\sin\theta, \sin t)$$

For $t \in (0, 2\pi)$ and $\theta \in (0, 2\pi)$
We get the Gauss map by obtaining the cross product of the partials to find the normal vector at a point p on our torus. Thus we first find the partials to our parameterization

$$\phi_\theta = (-(2+\cos t)\sin\theta, (2+\cos t)\cos\theta, 0) \tag{1}$$
$$\phi_t = (-\cos\theta\sin t, -\sin\theta\sin t, \cos t) \tag{2}$$

Next, we calculate the normal vector, which forms the Gauss map

$$N = \frac{\phi_\theta \times \phi_t}{|\phi_\theta \times \phi_t|} \tag{3}$$
$$\phi_\theta \times \phi_t = ((2+\cos t)\cos\theta\cos t, (2+\cos t)\sin\theta\cos t, (2+\cos t)\sin t\cos^2\theta + (2+\cos t)\sin t\sin^2\theta) \tag{4}$$
$$= ((2+\cos t)\cos\theta\cos t, (2+\cos t)\sin\theta\cos t, -2+\cos t)\sin t) \tag{5}$$
$$|\phi_\theta \times \phi_t| = \sqrt{(2+\cos t)^2\cos^2\theta\cos^2 t + (2+\cos t)^2\sin^2\theta\cos^2 t + (2+\cos t)^2\sin^2 t} \tag{6}$$
$$= \sqrt{(2+\cos t)^2\cos^2 t + (2+\cos t)^2\sin^2 t} \tag{7}$$
$$= 2+\cos t \tag{8}$$
$$N = (\cos\theta\cos t, \sin\theta\cos t, \sin t) \tag{9}$$

Using this, we can prove that the Gauss map is not injective but surjective. To prove non-injectivity, we show that for 2 different (t,θ), the gauss map returns the same vector.

Consider the point at $(0,0)$, where $N(0,0) = (1,0,0)$. Next, consider the point (π,π), where $N(\pi,\pi) = (1,0,0)$ as well. This proves non-injectivity.

To observe surjectivity in our Gauss map, we note that our map resembles the parameterization of S^2, only with t being in the bounds $(0, 2\pi)$ instead of $(0,\pi)$. Because of this, we can see that our map covers every single point on S^2, so given a point $p \in S^2$, there exists (t,θ) such that $N(t,\theta) = p$

□

2. This proposition is strikingly false, as if α is a circle of radius r then it has curvature $\frac{1}{r}$; $c\alpha$ is then a circle of radius cr with curvature $\frac{1}{cr}$.

More generally, κ is the curvature of α, consider $c\alpha$ for constant c.

$$(c\alpha)' = c\alpha'$$

The major item of note here is that, if α is unit speed, then $c\alpha$ will not be unit speed. Regardless of whether $c\alpha$ is unit speed or not, we will be working with the general notion of curvature as derived in a homework.

$$(c\alpha)'' = c\alpha''$$

$$\kappa_{c\alpha} = \frac{|(c\alpha)' \times (c\alpha)''|}{|(c\alpha)'|^3}$$

$$\kappa_{c\alpha} = \frac{|c|^2|\alpha' \times \alpha''|}{|c|^3|\alpha'|^3}$$

$$\kappa_{c\alpha} = \frac{1}{|c|}\kappa$$

With this, the proposition is generally false.

Problem 3.

Solution. 3

Part a)

Begin by computing the partials for the coordinate chart:

$$\phi_t = (1, f'(t)\cos\theta, f'(t)\sin\theta)$$

$$\phi_\theta = (0, -f(t)\sin\theta, f(t)\cos\theta)$$

The first fundamental form is defined as $E = \phi_t \cdot \phi_t$, $F = \phi_t \cdot \phi_\theta$, $G = \phi_\theta \cdot \phi_\theta$

$E = (1, f'(t)\cos\theta, f'(t)\sin\theta) \cdot (1, f'(t)\cos\theta, f'(t)\sin\theta) = 1^2 + f'(t)^2\cos^2\theta + f'(t)^2\sin^2\theta = 1 + (f'(t))^2$

$F = (1, f'(t)\cos\theta, f'(t)\sin\theta) \cdot (0, -f(t)\sin\theta, f(t)\cos\theta) = 0 - f'(t)f(t)\cos\theta\sin\theta + f'(t)f(t)\sin\theta\cos\theta = 0$

$G = (0, -f(t)\sin\theta, f(t)\cos\theta) \cdot (0, -f(t)\sin\theta, f(t)\cos\theta) = 0 + f(t)^2\sin^2\theta + f(t)^2\cos^2\theta = f(t)^2$

In matrix form, we have:

$$\begin{vmatrix} 1 + (f'(t))^2 & 0 \\ 0 & (f(t))^2 \end{vmatrix}$$

To compute the second fundamental form, we need the normal unit vector first:

$$N = \frac{\phi_t \times \phi_\theta}{|\phi_t \times \phi_\theta|}$$

$$\phi_t \times \phi_\theta = \begin{vmatrix} i & j & k \\ 1 & f'(t)\cos\theta & f'(t)\sin\theta \\ 0 & -f(t)\sin\theta & f(t)\cos\theta \end{vmatrix}$$

$= (f'(t)f(t)\cos^2\theta + f'(t)f(t)\sin^2\theta, -f(t)\cos\theta, -f(t)\sin\theta) = (f'(t)f(t), -f(t)\cos\theta, -f(t)\sin\theta)$

The magnitude is

$$\sqrt{f'(t)^2 f(t)^2 + f(t)^2\cos^2\theta + f(t)^2\sin^2\theta} = \sqrt{f(t)^2 f'(t)^2 + f(t)^2} = f(t)\sqrt{f'(t)^2 + 1}$$

Note we can pull out $f(t)$ from the square root without absolute value signs because we are given $f(t) > 0$

So we get

$$N = \frac{(f'(t)f(t), -f(t)\cos\theta, -f(t)\sin\theta)}{f(t)\sqrt{f'(t)^2 + 1}} = \frac{(f'(t), -\cos\theta, -\sin\theta)}{\sqrt{f'(t)^2 + 1}}$$

Next, we need the second partials:

$$\phi_{tt} = (0, f''(t)\cos\theta, f''(t)\sin\theta)$$

$$\phi_{t\theta} = (0, -f'(t)\sin\theta, f'(t)\cos\theta)$$

$$\phi_{\theta\theta} = (0, -f(t)\cos\theta, -f(t)\sin\theta)$$

Finally, to obtain the the second fundamental form:

$$e = \phi_{tt} \cdot N = \frac{1}{\sqrt{f'(t)^2+1}}(0 - f''(t)\cos^2\theta - f''(t)\sin^2\theta) = \frac{-f''(t)}{\sqrt{f'(t)^2+1}}$$

$$f = \phi_{t\theta} \cdot N = \frac{1}{\sqrt{f'(t)^2+1}}(0 + f'(t)\sin\theta\cos\theta - f(t)'\cos\theta\sin\theta) = 0$$

$$g = \phi_{\theta\theta} \cdot N = \frac{1}{\sqrt{f'(t)^2+1}}(0 + f(t)\cos^2\theta + f(t)\sin^2\theta) = \frac{f(t)}{\sqrt{f'(t)^2+1}}$$

Therefore the matrix for the second fundamental form is:

$$\begin{vmatrix} \frac{-f''(t)}{\sqrt{f'(t)^2+1}} & 0 \\ 0 & \frac{f(t)}{\sqrt{f'(t)^2+1}} \end{vmatrix}$$

To find the principal curvatures, we know from class and homework 5 that $k_1(p)$ and $k_2(p)$ are eigenvalues of DN_p such that form an eigenbasis for DN_p in the form of

$$\begin{vmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{vmatrix}$$

We also know from class that we can calculate DN_p with

$$DN_p = -\begin{vmatrix} e & f \\ f & g \end{vmatrix} \begin{vmatrix} E & F \\ F & G \end{vmatrix}^{-1}$$

If we substitute our answers from earlier, we have

$$DN_p = -\begin{vmatrix} \frac{-f''(t)}{\sqrt{f'(t)^2+1}} & 0 \\ 0 & \frac{f(t)}{\sqrt{f'(t)^2+1}} \end{vmatrix} \begin{vmatrix} 1+(f'(t))^2 & 0 \\ 0 & f^2(t) \end{vmatrix}^{-1}$$

Using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{-1} = \frac{1}{ad-bc}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

We get $DN_p = -\begin{vmatrix} \frac{-f''(t)}{\sqrt{f'(t)^2+1}} & 0 \\ 0 & \frac{f(t)}{\sqrt{f'(t)^2+1}} \end{vmatrix} \frac{1}{f^2(t)+f'(t)^2 f(t)^2} \begin{vmatrix} f^2(t) & 0 \\ 0 & 1+(f'(t))^2 \end{vmatrix}$

Which equals - $\begin{vmatrix} \frac{-f''(t)}{\sqrt{(f'(t))^2+1}} & 0 \\ 0 & \frac{f(t)}{\sqrt{f'(t)^2+1}} \end{vmatrix} \begin{vmatrix} \frac{1}{1+(f'(t))^2} & 0 \\ 0 & \frac{1}{f^2(t)} \end{vmatrix} = \begin{vmatrix} \frac{f''(t)}{((f'(t))^2+1)^{\frac{3}{2}}} & 0 \\ 0 & \frac{-1}{f(t)\sqrt{(f'(t))^2+1}} \end{vmatrix}$

Therefore the principal curvatures are $\frac{f''(t)}{((f'(t))^2+1)^{\frac{3}{2}}}$ and $\frac{-1}{f(t)\sqrt{(f'(t))^2+1}}$

Part b)

From class, the formula for the Gauss curvature is $K = \frac{eg-f^2}{EG-F^2}$

Plugging in our results from we before, we have

$$K = \frac{\frac{-f''(t)}{\sqrt{(f'(t))^2+1}}\frac{f(t)}{\sqrt{(f'(t))^2+1}}}{(1+(f'(t))^2)f^2(t)} = \frac{-f''(t)f(t)}{((f'(t))^2+1)^2 f^2(t)} = \frac{-f''(t)}{((f'(t))^2+1)f(t)}$$

We are given that $f(t) > 0 \quad \forall t$, so we conclude that the Gauss curvature is positive for t where $f''(t) < 0$ and negative for t where $f''(t) > 0$

$\square$

Problem 4.

Solution. (a) We have that

$$\alpha'(t) = (-r/c\sin(t/c), r/c\cos(t/c), h/c),$$

so

$$|\alpha'(t)|^2 = r^2/c^2\sin(t/c)^2 + r^2/c^2\cos(t/c)^2 + h^2/c^2 = \frac{r^2+h^2}{c^2}$$

. Setting this equal to 1 gives $r^2 + h^2 = c^2$.

(b) First, we will calculate the curvature κ and torsion τ of α. We will assume that $r \geq 0$, because negating r just rotates the helix by π. We have the following calculation:

$$T = \alpha'(t) = (-r/c\sin(t/c), r/c\cos(t/c), h/c)$$
$$N = T'/|T'| = (-r/c^2\cos(t/c), -r/c^2\sin(t/c), 0)/(r/c^2) = (-\cos(t/c), -\sin(t/c), 0)$$
$$B = T \times N = (h/c\sin(t/c), -h/c\cos(t/c), r/c)$$

Now, we have that $\kappa = |T'| = r/c^2$ from a step in our calculation, and this is equal to $r/(r^2+h^2)$ because we are assuming α is a unit-speed helix. By the Frenet equations we have that $B' = -\tau N$, so $\tau = -B' \cdot N$. We have that $B' = (h/c^2\cos(t/c), h/c^2\sin(t/c), 0)$, so $\tau = h/c^2 = \frac{h}{h^2+r^2}$. Thus, curvature and torsion are constant.

Now, suppose that some curve α with constant curvature κ and torsion τ. If $\kappa = 0$, then α is a line and is thus isometric to the helix with $r = 0$ and $h = c$. Therefore, we can assume that $\kappa \neq 0$. The curvature is positive by definition since it is the absolute value of α''. We want to find r and h such that the unit speed helix with parameters $r = r, h = h$, and $c = \sqrt{r^2+h^2}$ has curvature and torsion equal to κ and τ. By the fundamental theorem of space curves this implies that they are isometric, and we are done.

Using our previous calculation, we want to find $r \geq 0$ and h such that $\frac{r}{r^2+h^2} = \kappa$ and $\frac{h}{r^2+h^2} = \tau$. This is the formula for circular inversion about a circle of radius 1 centered at the origin, and circular inversion is its own inverse, so we can check that $r = \frac{\kappa}{\kappa^2+\tau^2}$ and $h = \frac{\tau}{\kappa^2+\tau^2}$ satisfy the requisite equations. Note that $\kappa > 0$ so we can divide by $\kappa^2 + \tau^2$. Because $\kappa > 0$, we have that $r = \frac{\kappa}{\kappa^2+\tau^2} > 0$ as well, so we are done. □

Problem 5. *The url below takes you to an image of a curve. Let T, N, B denote the Frenet frame at the specified point. Rotate the image so that you are looking down at the plane spanned by T, N. Do the same with T, B, and with N, B. Submit screenshots of your answer, and draw and label the T, N, B axes. Please explain your answer.*

`https://www.wolframcloud.com/obj/077c82ab-4b22-4588-8936-b76a5e2698a9`

Solution. In coordinates of the Frenet frame, the third-degree Taylor approximation of a curve is

$$\alpha(t) \approx (\alpha_T(t), \alpha_N(t), \alpha_B(t)) = \left(t, \frac{1}{2}\kappa t^2, -\frac{1}{6}\kappa\tau t^3\right).$$

We can use this approximation to reason about how the curve looks projected onto each plane:

- $\alpha_N = \frac{1}{2}\kappa\alpha_T^2$, so in the T, N-plane, the curve looks parabolic around $\alpha(t)$.
- $\alpha_B = -\frac{1}{6}\kappa\tau\alpha_T^3$, so in the T, B-plane, the curve looks cubic with an inflection point at $\alpha(t)$.
- $\alpha_N = (\alpha_B)^{2/3}$ times some constant, so in the N, B-plane, the curve has a cusp at $\alpha(t)$. □

great