Problem 1. *Give a unit-speed parameterization of the logarithmic spiral.*

Solution. Let the parameterization of the logarithmic spiral be

$$\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}^2,\, t \mapsto \left(e^{-t}\cos t,\, e^{-t}\sin t\right).$$

The parameterization α is regular because

$$\alpha'(t) = e^{-t}\left(-\sin t - \cos t,\, -\sin t + \cos t\right) \implies \left|\alpha'(t)\right| = e^{-t}\sqrt{2}.$$

Follow the proof of the existence of unit-speed, there are three steps

- For each $x > 0$, define
$$\begin{aligned} g(x) &= \int_0^x \left|\alpha'(t)\right| dt \\ &= \int_0^x e^{-t}\sqrt{-2}dt \\ &= \int_0^x -\left(\sqrt{2}e^{-t}\right)' dt \\ &= \sqrt{2}\left(1 - e^{-x}\right). \end{aligned}$$

- $g : [0, +\infty) \to \left[0, \sqrt{2}\right)$ is a bijection, and its inverse f is given by
$$t \mapsto -\ln\left(1 - \frac{t}{\sqrt{2}}\right).$$

- Define $\beta = \alpha \circ f$ which maps t to $\left(1 - \frac{t}{\sqrt{2}}\right)\left(\cos\left(-\ln\left(1 - \frac{t}{\sqrt{2}}\right)\right), \sin\left(-\ln\left(1 - \frac{t}{\sqrt{2}}\right)\right)\right)$. Then β has unit speed.

□

Problem 2. *In this problem you work out a formula for curvature of a space curve that's not necessarily unit speed. Let* $\beta : [a, b] \to \mathbb{R}^3$ *be a curve (not necessarily unit speed!), and let* $g(t) = \int_a^t |\beta'(u)| du$ *be its arclength function (in particular* $g'(t) = |\beta'(t)|$*). From class, we can define a unit speed curve* α *so that* $\alpha \circ g = \beta$*. The curvature* $\kappa(t)$ *of* β *at time* t*, is by definition the curvature of* α *at time* $g(t)$*, which we will define in class as* $|\alpha''(g(t))|$*.*

(a) *Derive from this setup that the curvature of* β *is give by the formula*

$$\kappa(t) = \frac{|T'(t)|}{g'(t)},$$

where $T(t)$ *is defined as* $\beta'(t)/|\beta'(t)| = \beta'(t)/g'(t)$*.* [1]

(b) *Derive the formula*

$$\kappa(t) = \frac{\beta'(t) \times \beta''(t)}{|\beta'(t)|^3}$$

[2]

Solution. (a) If we set $\beta = \alpha \circ g$, this means that $\beta(t) = \alpha(g(t))$. Using chain rule we see

$$\beta'(t) = g'(t)\alpha'(g(t)) \Rightarrow \frac{\beta'(t)}{g'(t)} = \alpha'(g(t))$$

Let $T(t) = \dfrac{\beta'(t)}{g'(t)}$, so we now have $T(t) = \alpha'(g(t))$.

$$\begin{aligned} T'(t) &= g'(t)\alpha''(g(t)) \Rightarrow \frac{T'(t)}{g'(t)} = \alpha''(g(t)) \\ |\alpha''(g(t))| &= \left| \frac{T'(t)}{g'(t)} \right| \\ \kappa(t) &= \frac{|T'(t)|}{g'(t)} \end{aligned}$$

(b) Since we know $\beta' = g'T$, we can take the derivative to get $\beta'' = g''T + g'T'$. So

$$\begin{aligned} \beta' \times \beta'' &= (g'T) \times (g''T + g'T') \\ &= (g'T \times g''T) + (g'T \times g'T') \\ &= (g')^2 (T \times T') \end{aligned}$$

Since T and T' are orthogonal, we can say that $|T \times T'| = |T||T'|$ and $|T| = |\frac{\beta'(t)}{|\beta'(t)|}| = 1$, so we can now say

$$|T'| = \frac{|\beta' \times \beta''|}{|g'|^2}$$

Substitute that into the answer from part (a) to get

$$\kappa = \frac{|\beta' \times \beta''|}{|g'|^3}$$

□

[1] Hint: apply the first rule of differential geometry (twice).

[2] Hint: first differentiate $\beta' = g'T$ to get a formula for β''.

Problem 3 (B, 1.3.4). *Let $f : I \to \mathbb{R}$ be a smooth function, and define $\alpha(t) = (t, f(t))$ (the trace of α is the graph of f). Compute the curvature of α.*

Solution. Consider α as a three-dimensional curve in the xy-plane, so the third component of α is zero. The first and second derivatives of α are

$$\alpha'(t) = \big(1, f'(t), 0\big),$$
$$\alpha''(t) = \big(0, f''(t), 0\big).$$

By Problem 2(b), the curvature of α as a three-dimensional curve is

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \frac{\big|\big(1, f'(t), 0\big) \times \big(0, f''(t), 0\big)\big|}{\big|\big(1, f'(t), 0\big)\big|^3}$$
$$= \frac{\big|\big(0, 0, f''(t)\big)\big|}{\big|\big(1, f'(t), 0\big)\big|^3} = \frac{|f''(t)|}{\left(\sqrt{1 + \big(f'(t)\big)^2}\right)^3}.$$

One caveat is that the two-dimensional curvature can be negative depending on the direction of curvature. We can solve this by using the z-component of the cross product instead of its magnitude. That is, the curvature of $\alpha(t)$ is really

$$\kappa(t) = \frac{f''(t)}{\left(\sqrt{1 + \big(f'(t)\big)^2}\right)^3}.$$

We justify this as follows. Let $T(t)$ be the unit tangent vector to α at t, and let $N(t)$ be the unit normal vector, defined as a $90°$ counterclockwise rotation of $T(t)$ in the xy-plane. Mirroring Problem 2(b), we obtain that $\alpha'(t) \times \alpha''(t)$ is a positive multiple of $T(t) \times T'(t)$. We know that $T'(t)$ is a multiple of $N(t)$. If it is a positive multiple, then $T(t) \times T'(t)$ points in the positive z-direction, and so does $\alpha'(t) \times \alpha''(t)$, and the curvature is positive. Likewise, if $T'(t)$ is a negative multiple of $N(t)$, then the curvature is negative. This is the desired behavior. □

Problem 4. *Consider a bike traveling in the plane. Let $\alpha(t)$ and $\beta(t)$ be the positions of the front and back wheels at time t, respectively.*[4] *Assume α is unit speed and that the distance between the wheels is one unit. If α is known, how do we determine β? (That's what you'll figure out here.)*

(i) Since $\alpha'(t)$ and $N(t)$ are an orthonormal basis for each t, we can write

$$\alpha - \beta = (\cos\theta)\alpha' + (\sin\theta)N$$

for some function θ. [5] *Use this to express β' as a linear combination of α' and N.* [6]

(ii) Since the rear wheel always points in the direction $\alpha - \beta$, we also know $\beta' = \lambda(\alpha - \beta)$ for some function λ. Compute λ. [7]

(iii) Combine the previous two parts to write a differential equation[8] *satisfied by θ and κ.* [9]

Solution. (i) Let's differentiate $\alpha - \beta$ to get an expression for β:

$$(\alpha-\beta)' = \alpha' - \beta' = \Big((-\sin\theta\cdot\theta')\alpha' + (\cos\theta)\alpha''\Big) + \Big((\cos\theta\cdot\theta')N + (\sin\theta)N'\Big)$$

by the product and chain rules, where $\theta(t)$ is a function of t. But $\alpha'' := \kappa N$ where $\kappa(t)$ is the curvature of α, and $N' = -\kappa\alpha'$ from lecture. Substituting these gives

$$\begin{aligned}\alpha' - \beta' &= \Big((-\sin\theta\cdot\theta')\alpha' + (\kappa\cos\theta)N\Big) + \Big((\cos\theta\cdot\theta')N + (-\kappa\sin\theta)\alpha'\Big) \\ &= \Big(-\sin\theta\cdot(\theta'+\kappa)\Big)\alpha' + \Big(\cos\theta\cdot(\theta'+\kappa)\Big)N \\ \implies \beta' &= \Big(1 + (\theta'+\kappa)\sin\theta\Big)\alpha' - \Big((\theta'+\kappa)\cos\theta\Big)N\end{aligned}$$

(ii) Since $\alpha(t)$ and $N(t)$ form an orthonormal basis and $\alpha - \beta = (\cos\theta)\alpha' + (\sin\theta)N$, we have

$$\begin{aligned}\lambda = \beta'\cdot(\alpha-\beta) &= (1 + (\theta'+\kappa)\sin\theta)(\cos\theta) - ((\theta'+\kappa)\cos\theta)(\sin\theta) \\ &= \cos\theta + \sin\theta\cos\theta((\theta'+\kappa) - (\theta'+\kappa)) \\ &= \cos\theta\end{aligned}$$

(iii) Since $\beta' = \lambda(\alpha-\beta) = \cos\theta(\alpha-\beta)$,

$$\Big(1 + (\theta'+\kappa)\sin\theta\Big)\alpha' + \Big(-(\theta'+\kappa)\cos\theta\Big)N = \Big(\cos\theta\cos\theta\Big)\alpha' + \Big(\cos\theta\sin\theta\Big)N$$

Now, as we have an orthonormal basis, we can equate the components of each basis vector to get a system of two equations.

$$\begin{cases} 1 + (\theta'+\kappa)\sin\theta = \cos^2\theta \\ (\theta'+\kappa)\cos\theta = -\cos\theta\sin\theta \end{cases}$$

From the second equation we get $\theta' + \kappa = -\sin\theta$. Substituting this into the first equation gives $-\sin^2\theta = \cos^2 - 1$ which is always true. Therefore $\theta' + \kappa = -\sin\theta$

□

[4] More precisely, think of the position that each wheel touches the ground.

[5] Physically, $\theta(t)$ is the angle that the front wheel is turned at time t.

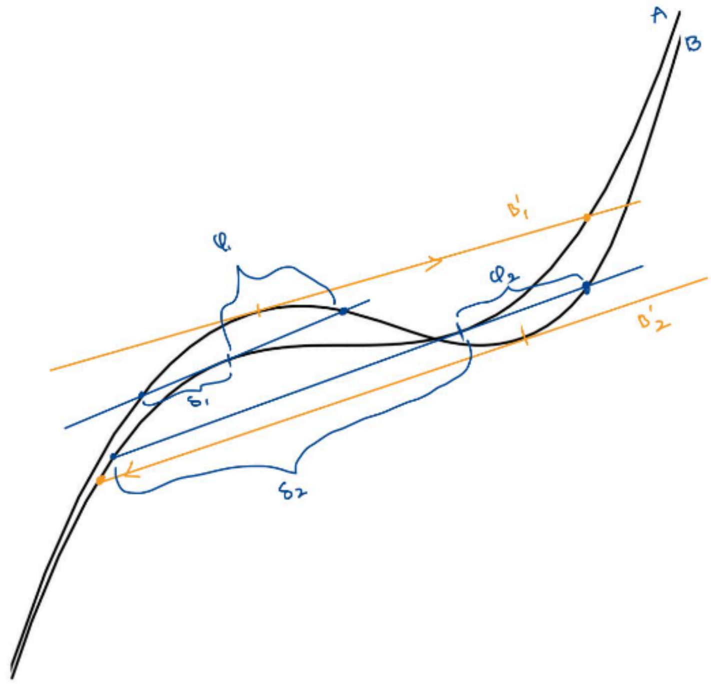
[6] The coefficients will involve θ and the curvature κ of α.

[7] Observe that $\lambda = \beta'\cdot(\alpha-\beta)$.

[8] "Differential equation" just means an equation satisfied by functions that also involves their derivatives.

[9] This differential equation can be used to simplify the equation for β' in part (i).

Problem 5. *Below is the tire tracks of a bike.*



Which is the front/back wheel? Which direction was the bike traveling? [10]

Solution. (i) We know that the back wheel must always be pointing in the direction of the front wheel. The direction that the back wheel is traveling is denoted by the tangent of its tire track. If it points towards the front wheel, the tangent should always intersect the path of the front wheel in the same direction.

Consider the hypothesis that curve B is the back wheel and curve A is the front wheel, and the two tangent lines of B at different points, B_1' and B_2'. While B_1' intersects A on the right side, B_2' intersects B on the left side. Then, if curve B denotes the back wheel, it does not always point towards the front wheel in the same direction. Then, curve B cannot denote the back wheel; **curve A is the back wheel and curve B is the front wheel.**

So now consider the tangents of curve A, the back wheel's tire tracks. Since the back wheel and the front wheel are at a constant distance apart (i.e., the bike frame), the segment length between any point on curve B and the intersection between the point's corresponding tangent line and curve A should be constant in the direction of travel. Comparing the blue tangent lines at different points on curve A, we observe that the segments δ_1 and δ_2 have very different lengths, while the segments ϕ_1 and ϕ_2 have roughly the same length. Then, the direction of travel cannot be left/downwards, in the direction of the δ segments; **the direction of travel is right/upwards.** □

[10]Hint: The tangent line through one of the curves always intersects the other curve...