Homework 5

Math 126

Due October 23, 2021 by 5pm

Name:

Topics covered: analytic continuation, residue calculus

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *Give a detailed proof that the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is smooth.

Solution. □

Problem 2. *Compute $\int_0^\infty \frac{1}{1+x^3}\,dx$.* [1]

Solution. □

Problem 3. *We say that $f : (a,b) \to \mathbb{R}$ is analytic if f is given by a power series near each $x_0 \in (a,b)$.*

True or false: if $f : (a,b) \to \mathbb{R}$ is analytic and $Z = \{f(x) = 0\}$ contains a limit point, then $f = 0$. Give either a proof or a counterexample.

Solution. □

Problem 4. *Compute*

$$\int_0^\infty x^{-1/2} e^{-x}\,dx = \sqrt{\pi}.$$

[2]

Solution. □

Problem 5. *Let $P(z)$ and $Q(z)$ be polynomials, and assume that Q has no repeated roots, i.e. we can write*

$$Q(z) = (z - w_1) \cdots (z - w_n)$$

with $w_1, \ldots, w_n$ distinct. The partial fractions decomposition for $f(z) = P(z)/Q(z)$ says that it's possible to write

$$\frac{P(z)}{Q(z)} = \frac{a_1}{z - w_1} + \cdots + \frac{a_n}{z - w_n}.$$

Integrate both sides around an appropriate choice of curves to compute the coefficients $a_1, \ldots, a_n$. [3]

Solution. □

Problem 6. *Compute the partial fractions decomposition for*

$$f(z) = \frac{z^2 + 3}{z^4 - 1}.$$

Solution. □

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[1]Hint: divide the circle (of radius $R \gg 0$) into thirds (intelligently) and integrate over the boundary of one of these thirds...

[2]Hint: proceed in the following steps: (1) change variables convert the integrand to a function of exp only; (2) square everything to get a double integral; (3) change to polar coordinates to get an integral that can be treated with a simple u-substitution.

[3]The answer should be in terms of P and Q and possibly their derivatives.