Homework 4

Math 126

Due October 8, 2021 by 5pm

Name:

Topics covered: complex integration, Cauchy's theorem, Cauchy integral formula

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *Evaluate the following integrals.*

(a) $\int_{|z|=1} \frac{e^z}{z^m}\, dz$ *for each* $-\infty < m < \infty$.

(b) $\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$.

Solution. □

Problem 2. *Let f be a complex-valued function, and define $g(z) = f(1/z)$. We say f is "continuous at infinity" if $\lim_{z\to 0} g(z)$ exists; if g is also holomorphic at 0, then we say that f is "holomorphic at infinity".*

(a) *Prove that* $f(z) = \frac{z^2+3}{z^4-2z+3}$ *is holomorphic at infinity.*

(b) *Prove that if f is holomorphic at infinity, then $\int_C f = 0$ for any circle of sufficiently large radius.*

Solution. □

Problem 3. *Fix a complex number a with nonzero imaginary part, and assume that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic function and*

$$f(z+1) = f(z) \quad \text{and} \quad f(z+a) = f(z)$$

for all $z \in \mathbb{C}$. [1] *Prove that f is constant.* [2] [3]

Solution. □

Problem 4. *Integrate $e^{-z^2/2}$ around a rectangle with vertices $\pm R, it \pm R$ and take $R \to \infty$ to show that*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\, dx = e^{-t^2/2}.$$

You may use that $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$, which is a famous computation.[4]

Solution. □

Problem 5. *Assume $f : W \to \mathbb{C}$ is holomorphic. Prove that f has an antiderivative if and only if $\int_C f(z)\, dz = 0$ for every closed curve $C \subset W$.* [5]

Solution. □

Problem 6. *True or false:*

(a) $\lim_{x\to 0} x \sin(1/x) = 0$.

[1] Remark: What do these equations mean geometrically?

[2] Hint: You may want to use a fact from real analysis, similar to the one on HW2#5.

[3] Hint: your solution should be short.

[4] Aside: this problem shows that $e^{-x^2/2}$ is an eigenfunction for the Fourier transform with eigenvalue 1.

[5] Hint: for one direction, you need to carefully figure out how to use the Fundamental Theorem of Calculus. Warning: there is not "Fundamental Theorem of Complex Analysis."

(b) $\lim_{z\to 0} z\sin(1/z) = 0$.

Be sure to explain your answer.[6]

Solution. □

[6] You may want to use L'Hoptial's rule.