Homework 3

Math 126

Due October 1, 2021 by 5pm

Name:

Topics covered: holomorphic and harmonic functions, complex integration

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *For a complex-valued function $f = f(z)$, define $\bar{f}$ by $\bar{f}(z) = \overline{f(z)}$.*

(a) Show that if f and $\bar{f}$ are both holomorphic, then f is constant.

(b) Show that if f is holomorphic and $|f|$ is constant, then f is constant. [1]

Solution. □

Problem 2. *Show that if D is a bounded region with differentiable boundary, then*

$$\int_{\partial D} \bar{z}\, dz = 2i\, area(D).$$

Solution. □

Problem 3. *Show that $u(re^{i\theta}) = \theta \log r$ is harmonic. Find a harmonic conjugate v for u. What is the holomorphic function $u + iv$?* [2]

Solution. □

Problem 4. *Consider the set $D = \{a < |z| < b\} \setminus (a, b)$, which is an annulus with a slit.*

(a) Give a proof, similar to the one in class, showing that any harmonic function on D has a harmonic conjugate on D.

(b) Give a different proof using the logarithm function to reduce the problem to a more familiar one.[3]

Solution. □

Problem 5. *Fix a curve $C \subset \mathbb{C}$ and fix a function $f(z)$ that is continuous on a curve C. Show that*

$$F(a) = \int_C \frac{f(z)}{z - a}\, dz$$

is holomorphic on $\mathbb{C} \setminus C$. Find $F'(a)$.

Solution. □

Problem 6. *Evaluate $\int_C e^z\, dz$ where C is the concatenation of straight line segments from -1 to $-1 + 10i$ to $1 + 10i$ to 1.*

Solution. □

Problem 7 (Bonus)**.** *Learn about the integraph. Write a short paragraph explaining what it is, its connection to the course, and how it is similar to/different than the planimeter.*

Solution. □

[1] Hint: use part (a).
[2] Hint: it is a function of $\log(z)$.
[3] Make sure to explicitly explain why solving the more familiar problem solves the problem at hand.