Homework 2

Math 126

Due September 24, 2021 by 5pm

Name:

Topics covered: complex exponential, logarithm, trig; holomorphic functions, Cauchy-Riemann equations

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *Determine all the solutions to* $z^3 + 1 = 0$ *and also* $z^4 + 1 = 0$. *Draw them in the complex plane.*

Solution. □

Problem 2. *Derive a formula for* $\sin(4\theta)$ *in terms of* $\sin(\theta)$ *and* $\cos(\theta)$.

Solution. □

Problem 3. *Define* z^w *where* $z, w \in \mathbb{C}$ *as* $e^{w \log(z)}$. *There is not well defined because* $\log(z)$ *is ambiguous. Show that we can resolve this issue if we assume that either* $z > 0$ *is real or* w *is an integer.*

Solution. □

Problem 4. *Derive the Cauchy–Riemann equations in polar coordinates.*[1]

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution. □

Problem 5. *Consider the function* $\tan(z) = \frac{\sin(z)}{\cos(z)}$.

(a) For which z *is* $\tan(z)$ *undefined?*

(b) What can you say about the value of $\tan(z)$ *when* z *has large positive/negative imaginary part?*

(c) Find a vertical line in the complex plane on which $1/\tan(z)$ *is bounded.* [2]

Solution. □

Problem 6. *Write* $z = re^{i\theta}$ *with* $\theta \in (0, 2\pi)$, *and define*

$$f(re^{i\theta}) = \sqrt{r}e^{i\theta/2} \quad \text{and} \quad g(re^{i\theta}) = \log(r) + i\theta.$$

Prove that f *and* g *are holomorphic functions defined on* $\mathbb{C} \setminus [0, \infty)$.

Solution. □

[1] Hint: This is an application of the chain rule.

[2] The following fact from analysis may be useful: a continuous function on a closed interval is bounded.